Blet Vienna Nov 6, 2018

Particle moving in $\mathbb{Z}^2$. Two variables:

position : q , momentum : p

Start: $q = \begin{pmatrix}1\\0\end{pmatrix}$ $p = \begin{pmatrix}0\\1\end{pmatrix}$ $\qquad M_0 := \begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2$

Two possible moves

$$A: \begin{cases} q \mapsto q+p \\ p \mapsto p. \end{cases} \qquad B: \begin{cases} q \mapsto q \\ p \mapsto p-q \end{cases}$$

$$A = \begin{pmatrix}1&1\\0&1\end{pmatrix}, \qquad B = \begin{pmatrix}1&0\\-1&1\end{pmatrix}$$

multiplication on the right on $M(q,p) = (q\ p)$

(vectors q, p as columns)

For a sequence of A's and B's corresponds a sequence: $M_0, M_1, M_2, \dots$ All M_k's have $\det = +1$

We associate to this a polygonal path in $\mathbb{R}^2 \setminus \{0\}$ by joining q_k to q_{k+1} with a line segment

Similarly there is a dual path joining p_k to p_{k+1} by a line segment.

These paths γ, γ^* are projections of a path Γ in $SL_2(\mathbb{R})$

Let

$$\sigma_A : [0,1] \longrightarrow SL_2(\mathbb{R})$$
$$t \longmapsto \begin{pmatrix}1&t\\0&1\end{pmatrix}$$

$$\sigma_B : [0,1] \longrightarrow SL_2(\mathbb{R})$$
$$t \longmapsto \begin{pmatrix}1&0\\-t&1\end{pmatrix}$$

Given a sequence of A's, B's we consider the corresp-
onding sequence of product of σ_A's and σ_B's
Projection to first and second column gives the two paths γ, γ^*.

Conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ exchanges σ_A and σ_B. In general takes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

and γ, γ^* are interchanged and rotated by $\pi/2$ in opposite directions

If the sequence is $\underbrace{A \cdots A}_{a_1} \underbrace{B \cdots B}_{b_1} \cdots\cdots \underbrace{A \cdots A}_{a_N} \underbrace{B \cdots B}_{b_N}$, $a_k, b_k \in \mathbb{Z}_{\geq 0}$ $\;b_1, \dots, b_{N-1} > 0$, $a_2, \dots, a_N > 0$

$= A^{a_1} B^{b_1} \cdots A^{a_N} B^{b_N}$,

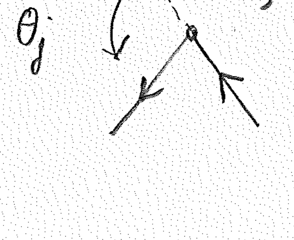
then for Γ is closed (γ, γ^* closed)

$$\sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k = 12\, W(\Gamma),$$

where W = winding number of γ or γ^* wrt origin. $\pi_1(SL_2(\mathbb{R}), 1) \simeq \mathbb{Z}$ By Iwasawa decomposition $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Let $v_1, \dots, v_r$ be the vertices of γ. These corres-pond in S to the blocks of B's. when we do not change position in γ.

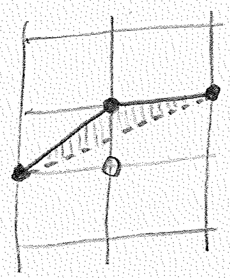
Let $\theta_1, \dots, \theta_r$ be the exterior angles at these vertices $\quad 0 < \theta_j < \pi$



Then from the winding number formula

$$\sum_{j=1}^{r} \theta_j = \frac{\pi}{6}\left[\sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k\right]$$

If we replace ABA by BAB in S we change the path γ at the vertex corresponding to B as follows:



This does not change the winding number (nor the sum (of a_j's and b_j's).

Conversely, changing BAB by ABA adds a vertex to γ.

<u>Prop</u> $\quad r > \frac{\ell}{6}$, $\quad r = \#$ vertices of γ

<u>Pf</u> $\quad \sum_{j=1}^{r} \theta_j = \frac{\pi}{6}\ell \quad$ hence $\quad \frac{\pi}{6}\ell < \pi r \quad \square$

<u>Cor</u> $\quad \ell_A, \ell_B > \frac{1}{6}\ell$

<u>Pf</u> $\quad \ell_B \geqslant r \quad$ and similarly for $\gamma^* \quad \square$

<u>Cor</u> $\quad \frac{1}{6} < \frac{\ell_A}{\ell}, \frac{\ell_B}{\ell} < \frac{5}{6}$

<u>Pf</u> $\quad$ Since $\ell_A + \ell_B = \ell$ the right inequality follows from the previous corollary $\square$

Blet

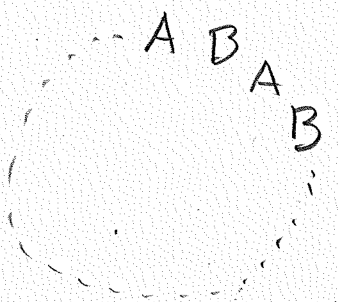
Remarks

1) The operation of chopping off a vertex makes sense for reflexive polytopes in any dimension? This is the analogue of the Blet moves and gives a way to define the game in any dimension.

2) Graph of 16 reflexive polygons via chopping or adding vertex is connected and very symmetrical (extra symmetry besides duality).
Is it connected for higher dimensions?

3) We can define a graph for closed paths with any given winding number. Playing Blet is moving in that graph and looking to get to a minimum.
The graph organized by number of vertices is a poset with rank?
The local minima should be visible from the graph

The puzzle consists of starting with a configuration



of n A's, B's alternating on a circle. By using the moves

$$ABA \longleftrightarrow BAB$$

we should the maximum possible number of A's.

In the TCL implementation $n=28$ and we can achieve the maximum of 23 A's. If we only use the greedy move

$$BAB \rightarrow ABA$$

we only achieve a total of 21 A's.

This puzzle is somewhat analogous to peg solitaire except there the moves are irreversible

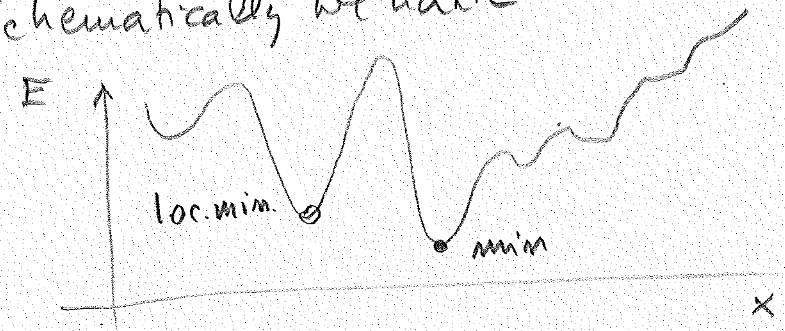
$$AAB \rightarrow BBA$$

and we want to minimize the number of A's.

In both cases we are trying to minimize a function on a discrete space. We can't use calculus!

we could call these: replacement puzzles
(lights out is another case actually)

Schematically we have



We can start an arbitrary point in space x and then make an allowable move to say x'. We compute $E(x')$ and compare it with $E(x)$.

If $E(x') < E(x)$ we go to x'.

Doing only moves of this kind is a greedy procedure.

If E has local minima we are likely to get stuck in those, which could be much higher than the actual absolute minimum.

One approach is to use simulated annealing.

Choose x' if $E(x') > E(x)$ with probability

$$p = e^{-\frac{1}{T}(E(x') - E(x))}$$

where T is a non-negative continuous decreasing function of time which approaches zero.

Here T represents temperature: when large the probability p is close to 1 and the movement is chaotic; when small the probability p is also small and we are close to greedy.

Popular choice: $T = \frac{d}{\log t}$ for some positive constant d.

$$p(t) = t^{-\frac{1}{d}(E(x') - E(x))}$$

Markov process

To any word W on A and B we associate a path in $\mathbb{Z}^2$ as follows:

Two variables

q = position

p = momentum

start at $q = \begin{pmatrix}1\\0\end{pmatrix}, \ p = \begin{pmatrix}0\\1\end{pmatrix}$

$$A: \begin{cases} q \longmapsto q+p \\ p \longmapsto p \end{cases}$$

$$B: \begin{cases} q \longmapsto q \\ p \longmapsto p-q \end{cases}$$

Writing the current state as the matrix

$M_K := \begin{pmatrix} q_K \\ p_K \end{pmatrix}$ (row vectors)

$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

we have A is multiplication on the left by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and B by $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

We actually have a path in $SL_2(\mathbb{Z})$

$\in SL_2(\mathbb{Z}) \subseteq SL_2(\mathbb{R})$

$M_0, M_1, \dots$

[using

$\tilde{A} := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $\quad 0 \leq t \leq 1$]

$\tilde{B} := \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$

Let $\rho(W)$ be the final state matrix.

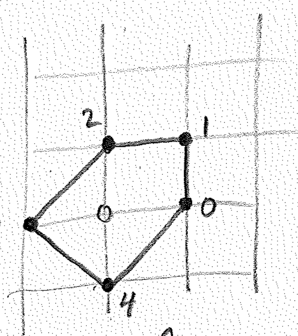
Projecting to first row gives a path but so does projection to the second row. We call these dual paths.

For example (read left to right)

ABABABBABBAB

gives the pentagon

γ

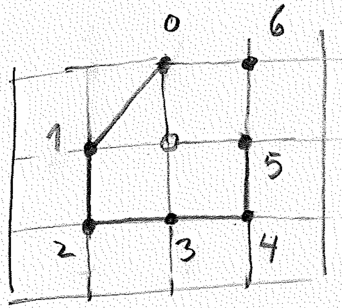


1st row

and the dual

same as record of tangent vectors on γ

γ^*



2nd row

Note that the total number of dots is 12. This happens in all cases!

We get the same path up to a rotation if we swap A & B. Conjugation by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ exchanges A and B and in general takes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

so 2nd row becomes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 1st row

i.e. rotated clockwise by $\pi/2$.

THM (w/ B. Poonen)

$$\ell(\gamma) + \ell(\gamma^*) = 12w,$$

where $w = w(\gamma) = w(\gamma^*)$ is the winding number of either path around the origin.

In terms of sequences of A's and B's: the total number of letters is $12w$ (for a closed path).

Key property

$$\boxed{ABA = BAB}$$

$\rho(W)^k = I_2$ for some $k>0$.

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THM

Suppose γ is eventually closed then

$$\frac{1}{6} < \frac{\ell_A}{\ell} < \frac{5}{6}, \quad \frac{1}{6} < \frac{\ell_B}{\ell} < \frac{5}{6}.$$

Suppose γ is closed.

Pf Let $v_1, \dots, v_r$ be the vertices of γ. These correspond precisely to blocks of B in our sequence.

θ_j = exterior angle at v_j

Then

$$\sum_{j=1}^{r} \theta_j = 2\pi w(\gamma)$$

Hence
$$\sum_{j=1}^{r} \theta_j = \frac{\pi}{6}\, \ell(\gamma)$$

Now $0 \le \theta_j < \pi$. Hence

$$\frac{\pi}{6}\, \ell < \pi r \le \pi \ell_B$$

Similarly

$$\frac{\pi}{6}\, \ell < \pi \ell_A$$

For an eventually closed path repeat it to get a closed path and apply to it the previous argument $\square$

<u>Claim</u> Any Blet configuration is eventually closed.

<u>Pf</u> The original Blet configuration is eventually closed as $(AB)^6$ is closed. Indeed,
$$((AB)^6)^{n/2} = ((AB)^{n/2})^6$$

Given a word W in A & B

$$W = w_1 w_2 \cdots w_\ell \qquad w_i \in \{A, B\}$$

applying the Blet rule to $w_2, \ldots, w_{\ell-1}$ does not change the final state. Applying it to w_1 has the effect of conjugating

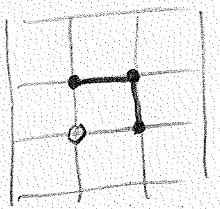
E.g.

$$W = A W_1 A B \mapsto B W_1 B A = (BA^{-1}) W (BA^{-1})^{-1}$$

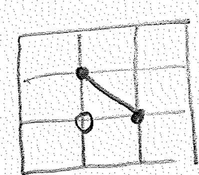
So $\rho(W)^k = I_2 \Leftrightarrow \rho(W')^k = I_2$ where W' is Blet equivalent to W.

In terms of paths

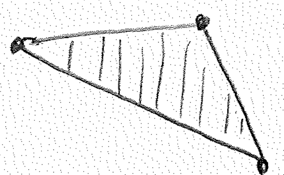
ABA



BAB



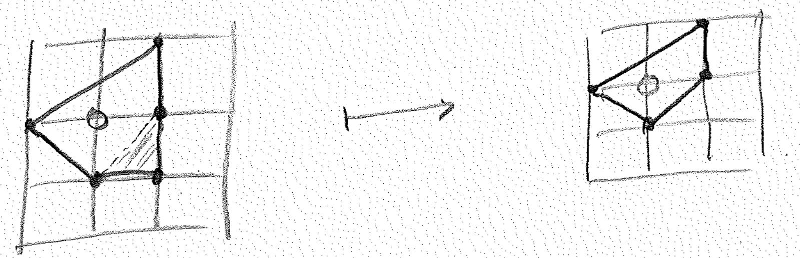
So $ABA \longrightarrow BAB$ corresponds to cutting a corner of γ



(no other lattice points in the triangle

Example

$ABBABBBAB\underline{ABA} \mapsto ABBABBBAB\underline{BAB}$



Playing Blet w/ n pairs of A & B the largest number of A's possible is $\left[\frac{5n-1}{6}\right]$. For $n=28$ this gives 23.