Course at the Univ of Bologna
March 6 – May 23
F. Rodriguez Villegas
(Notes by Elisa Vitale)

Counting representations of quivers over finite fields [1] – Villegas

Jordan's theorem (1870) K field, $\bar{K}$ = alg closure

$A \in K^{n\times n}$ has a Jordan form, i.e. $\exists\, U \in \bar{K}^{n\times n}$ invertible s.t.

$$UAU^{-1} = \bigoplus_{\xi\in\bar{K}} J_\xi$$

$$J_\xi = \underset{\bar{K}^{m_\xi \times m_\xi}}{\underset{\cap}{\begin{bmatrix} \xi & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \xi \end{bmatrix}}}$$ Jordan block with eigenvalue ξ

$\{\cancel{m}\xi, m_\xi\} \longleftrightarrow$ conjugacy class of $n\times n$ matrices in $\bar{K}$

$\Lambda : \bar{K} \longrightarrow \mathcal{P} = \{\text{partitions}\}$

$\xi \longmapsto \lambda_\xi$ $\qquad \lambda = (\lambda_1, \lambda_2, \dots)$ $\quad \lambda_i \geq 0 \quad \lambda_1 \geq \lambda_2 \geq \dots$

$|\lambda| = \lambda_1 + \lambda_2 + \dots$

$\lambda_\xi :=$ size of the Jordan blocks of ~~given~~ eigenvalue ξ.

$$|\Lambda| := \sum_{\xi\in\bar{K}} |\lambda_\xi| = n$$

$\Lambda \longleftrightarrow$ conjugacy class in $\bar{K}^{n\times n}$.

If $\xi \in \bar{K}$ is an eigenvalue of $A \in K^{n\times n}$ and $\sigma : \bar{K} \to \bar{K}$ is an automorphism, then $\xi^\sigma \in \bar{K}$ is also an eigenvalue of A.

The conjugacy class of $A \in K^{n\times n}$ is indexed by $\Lambda : \bar{K} \to \mathcal{P}$ such that

- $|\Lambda| = n$
- $\Lambda(\xi^\sigma) = \Lambda(\xi)$ "the blocks with eigenvalue ξ and ξ^σ have the same size..".

E.g. $K = \mathbb{R} \Rightarrow K = \mathbb{C}$

$\cup$ ↓ ↓ Jordan block.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underset{\overline{K}}{\cong} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \iff \Lambda : \begin{matrix} i \longmapsto 1 \\ -i \longmapsto 1 \end{matrix}$$

In the language of quivers:

$\dim V = n$ $\quad V$ vsp over K

J Jordan quiver

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\psi"] \\ V \arrow[r, "\varphi'"'] & V \end{tikzcd}$$

$\iff \varphi \sim \varphi'$ conjugate, i.e. $\psi^{-1} \varphi' \psi = \varphi$

enough to study indecomposable repr. ~~in the absolute sense~~

Jordan's theorem $\longleftrightarrow$ description of representations of J up to isom over $\overline{K}$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

↳ indecomposable / $\mathbb{R}$

decomposable / $\mathbb{C}$ $\rightsquigarrow$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = (i) \oplus (-i)$

We're interested in absolutely indecomposable repr., i.e. indecomposable over K and still indec. over an algebraic closure of K.

For J: absolutely indec $\iff$ conjugate to $\begin{bmatrix} \xi & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \xi \end{bmatrix}$ with $\underline{\underline{\xi \in K}}$.

§ 2. V. Kac

Count absolutely indec. repr of a quiver of given dimension up to isom

The number is a polynomial in q where $q = \# \mathbb{F}_q$. size of the field.

It's called Kac polynomial.

Recall: $\mathbb{F}_q \quad q = p^r \quad p$ prime

$$\begin{tikzcd} \mathbb{F}_p \simeq \mathbb{Z}_p \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & \mathbb{F}_{p^r} \subset \overline{\mathbb{F}_p} \\ & \mathbb{F}_{p^s} \arrow[u, phantom, "\subset"] \end{tikzcd}$$

↖ organized by divisibility.

Example: The Kac polynomial for the Jordan quiver J is: q.
(because there are q possible eigenvalues).

Weil '40s $f \in \mathbb{Z}[x]$ irred poly. $f \equiv f_1 \cdots f_r \mod p$ $\deg f_i = n_i$

$X := \{f = 0\}$ $\underset{N_X(q)}{\underset{\shortparallel}{\#X(\mathbb{F}_q)}} = ?$ (assuming $p \nmid \underset{\text{discrimina}}{\Delta(f)}$)

$\#X(\mathbb{F}_p) = \#\{n_i = 1\}$

Example: $f(x) = x^2 + 1$ $N_X(p) = \begin{cases} 0 & p \equiv 3 \mod 4 \\ 1 & p = 2 \\ 2 & p \equiv 1 \mod 4. \end{cases}$

$$N_X(q) = \sum_{n_i | n} n_i$$

Example: $f(x) = x^2 - a$ $a \in \mathbb{Z}$

$\#X(\mathbb{F}_p) =$ when is a a square mod p?

$\Updownarrow$

answer: QRL = Quadratic Reciprocity Law.

i.e. answer depends on p mod N.

Conj: Kac polynomials have coeffs in $\mathbb{Z}_{\geq 0}$. (!)

Counting quiver representations over finite fields [2] - Villegas.

$N_X(q) = ?$

1) $\dim X = 0$ $\quad f(x) = 0 \quad f \in \mathbb{Z}[x]$ irred

For $p \nmid \mathrm{discr}(f) \quad f \equiv f_1 \cdots f_r \mod p$

$\Rightarrow N_X(q) = \dots ? \quad q = p^s$

For $\deg f = 2$ describing $N_X(q)$ as a function of q <=> QRL.

2) $\dim X = 1$ curve of genus $g = 1$ (=> elliptic)

$y^2 = x^3 + ax + b \quad a, b \in \mathbb{Z}$

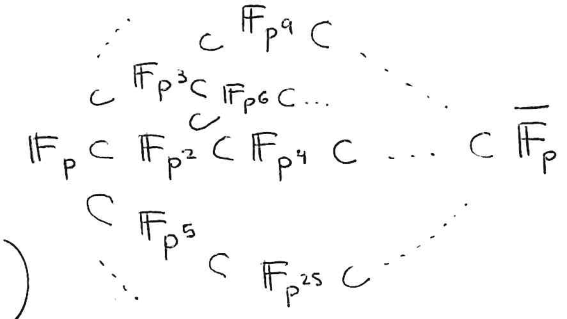
$N_X(q)$ depends on modular forms. [Shimura-Taniyama]
↓
(Wiles)

3) In general: describing $N_X(q)$ <=> Langlands program.

For J 

$$\# \begin{matrix}\text{abs indec.}\\ \text{of dim } n \text{ on } \mathbb{F}_q\end{matrix} = q.$$

Weil zeta function



Fix $p \rightsquigarrow q = p^n$

$$Z(X,T) := \exp\left(\sum_{n \geq 1} \# X(\mathbb{F}_{p^n}) \frac{T^n}{n}\right)$$

E.g. $X = \mathbb{A}^k \Rightarrow \# \mathbb{A}^k(\mathbb{F}_q) = q^k$

$$Z(\mathbb{A}^k, T) = \exp\Big(\underbrace{\sum_{n \geq 1} p^{nk} \frac{T^n}{n}}_{-\log(1 - p^k T)}\Big) = (1 - p^k T)^{-1}.$$

$X = X_1 \sqcup X_2 \Rightarrow Z(X,T) = Z(X_1,T) \cdot Z(X_2,T)$

E.g. $X = \mathbb{P}^k = \mathbb{A}^k \sqcup \mathbb{A}^{k-1} \sqcup \dots \sqcup \mathbb{A}^0$

$\# \mathbb{P}^k(\mathbb{F}_q) = q^k + \dots + q + 1 \quad \Rightarrow Z(\mathbb{P}^k, T) = (1-T)^{-1} \cdots (1 - p^k T)^{-1}.$

Conj [Weil]: $Z(X,T)$ is the power series expansion of a rational function

$$\# X(\mathbb{F}_q) = \sum_{i=0}^{N} a_i q^i \qquad Z(X,T) = \prod_{i=0}^{N} (1-q^iT)^{-a_i}$$

Could we have $N_X(q) = \frac{1}{2}q(q-1)$?

$$X = (\underset{\substack{\downarrow \\ (x,y)}}{\mathbb{A}^1\times\mathbb{A}^1} \setminus \text{diag}) / x \leftrightarrow y$$

$X \ni \{x,y\}$ we don't core about ordering.

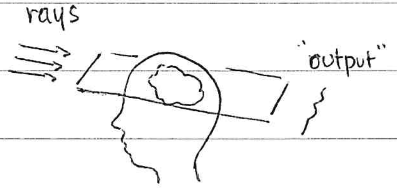
$$X(\mathbb{F}_q) = \{\{x,y\} \mid x,y \in \mathbb{F}_q,\ x \neq y\}$$
$$\cup$$
$$\{\{z, z^\sigma\} \mid z \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q\} \qquad \langle\sigma\rangle = \text{gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$$

$$\begin{tikzcd} \mathbb{F}_{q^2} \arrow[d, dash, "2"' , "\text{deg } 2 \text{ exten}"] \\ \mathbb{F}_q \end{tikzcd}$$

defined over $\mathbb{F}_q$ <=> fixed by gal

$$\{z^\sigma, z\} = \{z, z^\sigma\}$$

$$\Rightarrow \# X(\mathbb{F}_q) = \frac{1}{2}q(q-1) + \frac{1}{2}(q^2-q) = q^2 - q.$$



Radon transform

"p-plane" ⇝ where we "throw" the rays
$\# X(\mathbb{F}_p)$ outcome } ⇝ what can we tell about X?

next page...

Weil conjecture   Thm Suppose $X/\mathbb{C}$ polynomial count smooth & projective

$$\# X(\mathbb{F}_q) = \overset{N_X(q)}{\sum_{i=0}^{N} a_i q^i}. \qquad \text{(some } p \text{ excluded)}$$

$$\text{Then } b_k(X) = \begin{cases} a_{\frac{k}{2}} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

$X/\mathbb{C}$ ⇝ spreading out: choice of a scheme $\mathcal{X}/\mathbb{R}$ where $R \hookrightarrow \mathbb{C}$ finitely gen $\mathbb{Z}$-alg such that

$$\mathcal{X} \otimes_R \mathbb{C} = X.$$

$$\phi : R \longrightarrow \mathbb{F}_q$$

$\mathcal{X} \otimes_R \mathbb{F}_q \ (\mathbb{F}_{q^n})$ ⇜ makes sense to consider this

E.g. $X : \{x^2+1 = 0\}$

$$\# X(\mathbb{F}_p) = \begin{cases} 0 & p \equiv 3 \mod 4 \\ 1 & p = 2 \\ 2 & p \equiv 1 \mod 4 \end{cases}$$

This is polynomial count:

$R = \mathbb{Z}\left[\frac{1}{2}, i\right]$ $\quad \phi : R \longrightarrow \mathbb{F}_q$.

a polynomial count is a scheme $X/\mathbb{C}$ st. $\exists$ spreading out $\mathcal{X}/R$ s.t. $\# \mathcal{X} \otimes_R \mathbb{F}_q = \sum_{i=0}^{N} a_i q^{n_i}$ $\forall\ \phi : R \to \mathbb{F}_q$.

Cubic surface $S/\mathbb{Q}$

$\# S(\mathbb{F}_q) = q^2 + tq + 1$ $\quad 1 \leq t \leq 7$.

This is polynomial count.

Example: $\# X(\mathbb{F}_q) = q^{n^2-n}$ What can we say about X?

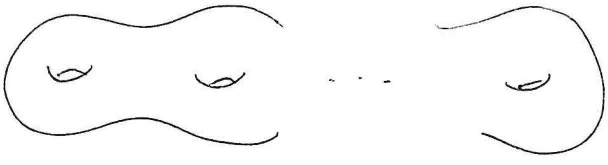
$X = \{ A \in \mathbb{C}^{n \times n} \mid A^n = 0 \}$

A priori these varieties can be pretty complicated.

Grassmannian: $G(n,k)$

$X/\mathbb{C}$ $\quad b_k(X) = \dim H^k(X, \mathbb{C})$

$\dim X = 1$



$g =$ genus

$\dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 2g$

$g = 0$ $\quad X = \mathbb{P}^1$ $\quad \# X(\mathbb{F}_q) = q + 1$

$H^1(X, \mathbb{C}) = 0$

$H^2 = H^0 \simeq \mathbb{C}$

negative! Here X is not cpt...
[Weil] fails in this case.

$X = \mathbb{C}^* \qquad \#X(\mathbb{F}_q) = \#\mathbb{F}_q^* = q-1$

$\{xy=1\}$

What can we say about X if it is...

- polynomial count
- NOT smooth and projective ? (can fail one or the other or both)

N. Katz: If X is polynomial count, in any case there is a polynomial

$$\underbrace{E(X;x,y)}_{\text{"E-polynomial"}} = N_X(xy)$$

purely geometric — related to counting

$H^k(X,\mathbb{Q})$ mixed Hodge structure.

pure Hodge structure: X sm proj $\Rightarrow H^k(X,\mathbb{Q})\otimes\mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$.

Two filtrations:

$0 = W_{-1} \subseteq W_0 \subseteq \dots \subseteq W_{2k} = H^k$

W_r/W_{r-1} has a Hodge structure

↳ $h^{i,j,k}$ = dim of the (i,j)-th piece of $i+j=r$.

smooth projective: $\#X(\mathbb{F}_q) = \sum_{i=0}^{N} a_i q^i \qquad a_i = \dim H^{2i}(X,\mathbb{Q}) = b_{2i}(X)$

→ can do the same w/ cpt support cohom $h_c^{i,j,k}$.

$H(X;x,y,t) = \sum h^{ijk} x^i y^j t^k$

Mixed Hodge poly

$$H_c(X; x, y, t) = \sum h_c^{ijk} x^i y^j t^k .$$

$$E(X; x, y) = H_c(X; x, y, -1)$$

$$\chi(X) = \sum_k (-1)^k b_k(X)$$

May 25th 2023

Counting quiver representations over finite fields [3] - Villegas

Weight filtration: $0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \dots \subseteq W_{2k} = H^k(X, \mathbb{Q})$

W_i / W_{i-1} are a pure Hodge structure ~~which~~ of weight i which "looks like H^i of a sm proj var".

$$h^{ijk} := \dim\left(Gr^{W}_{i+j}(H^k(X,\mathbb{Q}))\right)^{i,j}$$

MH polynomial: $H(X; x,y,t) := \sum_{i,j,k} h^{ijk} x^i y^j t^k$

$$E(X; x,y) = H_c(X; x,y,-1) = \\ \overset{!}{=} \sum_{i,j} \left(\sum_k (-1)^k h_c^{i,j,k} \right) x^i y^j .$$

$$\chi(X) = \sum (-1)^k \dim H^k(X,\mathbb{Q}) = H(X; 1,1,-1)$$

$U \subseteq X$ open $\quad Y = X \setminus U \quad E(U) + E(Y) = E(X)$

$E(X \times Y) = E(X) E(Y)$

Katz: X polynomial count => $E(X; x,y) = N_X(xy)$

Topology $f: X \to X$ cpt conn mfld

$\Lambda_f := \sum_{k \geq 0} (-1)^k tr(f_* | H_k(X, \mathbb{Q}))$ Lefschetz number

Assuming f has finitely many (isolated) fixed pts, then

$\Lambda_f = \sum_{x = f(x)} i(f,x) \qquad i(f,x) = \dots$

Frobenious automorphism $\quad \overline{\mathbb{F}_q} \xrightarrow{\text{Frob}_q} \overline{\mathbb{F}_q} : x \longmapsto x^q$

If X is defined over $\mathbb{F}_q$, then Frob_q acts on $X(\overline{\mathbb{F}_q})$.

Fixed points? Fixed points of $\text{Frob}_q = \mathbb{F}_q$

$$\Downarrow$$

$$\text{fixed pts} = X(\mathbb{F}_q)$$

<u>Key fact:</u> $\xi =$ eigenvalue of Frob_q on $H^k_c(X, \overset{\ell\text{-adic numbers}\ldots}{\mathbb{Q}_\ell})$. Then

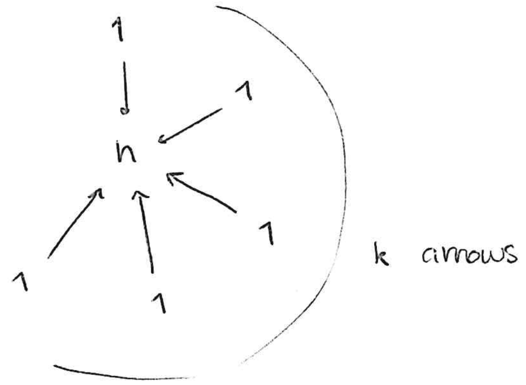
*) ξ is an algebraic integer

*) $|\xi^\sigma| = q^{\frac{1}{2}w} \quad \forall\, \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

for some $\underset{w(\xi)}{\overset{\|}{w}} \geq 0$ integer "weight of ξ".

(X sm proj $\Rightarrow$ $w = k$.)

<u>Want:</u> # absolutely indec. repr of a quiver $/\mathbb{F}_q$.



a representation is:

$$GL_n(\mathbb{F}_q) \curvearrowright \left[\begin{array}{c|c|c} V_1 & \cdots & V_k \end{array}\right] \in \mathbb{F}_q^{n\times k}$$

$$\underset{GL_1\mathbb{F}_q}{\circlearrowleft} \qquad \underset{GL_1\mathbb{F}_q}{\circlearrowleft}$$

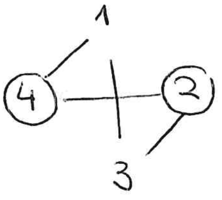
$\in G(n,k)$ $k \geq n$.

$\searrow$ Schubert cells

Under the $GL_n(\mathbb{F}_q)$ we can classify by <u>row echelon form</u> $\nearrow$ forma a gradini

<u>Example:</u> $n=2, k=4$

$$\begin{array}{c|cccc} & 1 & 2 & \textcircled{3} & \textcircled{4} \\ \hline \textcircled{4} & * & * & 0 & \textcircled{1} \\ \textcircled{3} & * & * & \textcircled{1} & 0 \end{array}$$

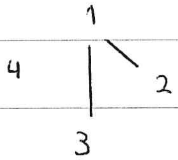


one edge for each nonzero entry $*$.

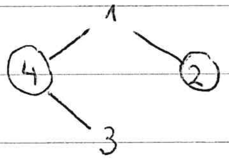
<u>claim:</u> repr is indec. (abs. indec) $\Leftrightarrow$ graph is connected.

$\rightsquigarrow$ only worry about Schubert cells where graph is connected.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 3 & * & * & 1 & 0 \\ 2 & * & 1 & 0 & 0 \end{array}$$



$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 4 & * & 0 & * & 1 \\ 2 & * & 1 & 0 & 0 \end{array}$$



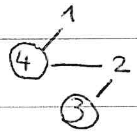
⊙ = local maxima

· = local minima.

Fix the alternating graph with n local maxima.

Look at subgraphs: $A \subseteq E = \{\text{edges of } \Gamma\} \rightsquigarrow \Gamma_A =$ corresponding graph

e.g. $\begin{pmatrix} * & * & 0 & 1 \\ 0 & * & 1 & 0 \end{pmatrix} \rightsquigarrow$



$\big\downarrow$ act by GL_1 on columns

$$\# \begin{pmatrix} 1 & * & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = q-1$$

there are q-1 choices for $*$ to be $\neq 0$.

$\Gamma_A \rightsquigarrow (q-1)^{n_A}$ in general

"relgability polynomial"

$$\sum_{\substack{A \subseteq E \\ \Gamma_A \text{ connected}}} (q-1)^{n_A} = T_\Gamma(1,q) =: R_\Gamma(q) \qquad n_A = \#A - \#V + 1 = \overset{\text{Betti }\#}{b_1}(\Gamma_A)$$

Tutte polynomial: $$T_\Gamma(x,y) = \sum_{A \subseteq E} (x-1)^{\alpha(A)} (y-1)^{\beta(A)}$$

$\alpha(A) = \#$ conn comp of Γ_A $-$ $\#$ conn comp of Γ

$\beta(A) = b_1(\Gamma_A)$

Second expression: $$T_\Gamma(x,y) = \sum_{T \subseteq \Gamma} x^{i(T)} y^{j(T)}$$

$i(T)$ = internal activity of

$j(T)$ = external activity of

$\downarrow$

require edges labeling, but not depend on

T = spanning tree

= tree w/o cycles that touches every vertex.

$x=1 \Rightarrow \alpha(A)=0 \Rightarrow \Gamma_A$ connected.

$\boxed{n=2,\ k=5}$

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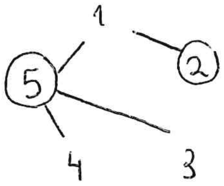

edge = smaller than local maxima.

⇝ # = 1

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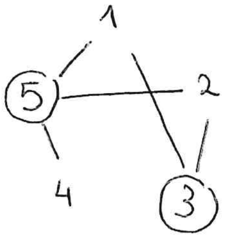

= $q+3$

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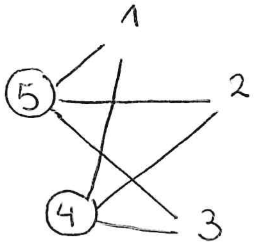
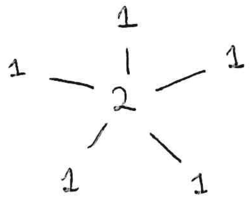


= q^2+4q+7

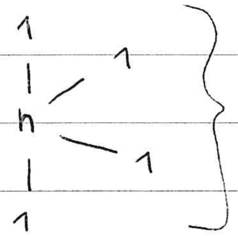
total # $\overset{\text{abs}}{\text{indec}}$ $= 1+q+3+q^2+4q+7 = q^2+5q+11$

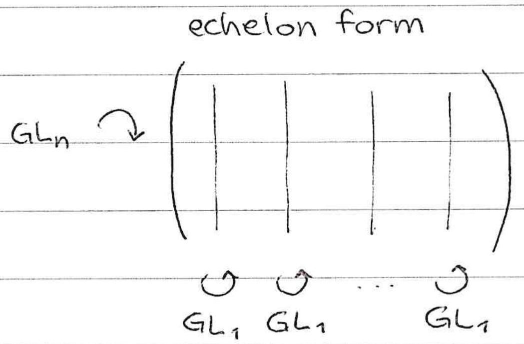
of the quiver



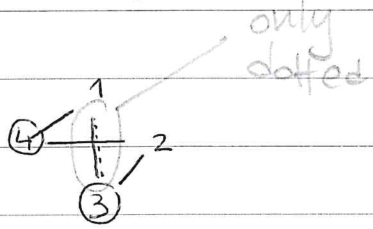
May 26th, 2023

Counting quiver representations over finite fields [4] - Villegas.





$$\begin{pmatrix} a & b & 0 & 1 \\ 0 & c & 1 & 0 \end{pmatrix} \quad a, b, c \neq 0 \quad \rightsquigarrow \quad \Gamma_A :$$



$A \subseteq E$

$$n\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} a & b & 0 & 1 \\ 0 & c & 1 & 0 \end{pmatrix} \qquad \begin{cases} \alpha_1 \beta_4 = 1 \\ \alpha_2 \beta_3 = 1 \end{cases}$$

$$\underbrace{\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{array}}_{k}$$

$\# A \qquad \mathbb{G}_m^{n+k} / \mathbb{G}_m^{n}$

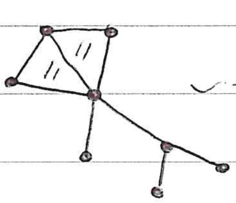
overall stabilizer: $\mathbb{G}_m$ $\qquad \alpha_1 = \alpha_2 = \alpha$

$\beta_1 = \beta_2 = \beta_3 = \beta_4 = \alpha^{-1}$

$\downarrow$

$\# A - \# V + 1 = b_1(\Gamma_A)$

$\downarrow$



$\rightsquigarrow$ # cycles.

$$T_\Gamma(x,y) = \sum_{A\subseteq E} (x-1) \qquad \cdot (y-1)$$

$K(\Gamma_A) - K(\Gamma) \geqslant 0$ $\qquad$ $K(\Gamma) = \#$ components of Γ.

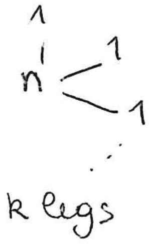
At $x=1$ the only nontrivial contribution corresponds to the case

$K(\Gamma_A) = K(\Gamma)$ $\quad (\Rightarrow \Gamma_A$ connected because Γ is conn$)$.

So :

$$T_\Gamma(x,y)\Big|_{x=1} = T_\Gamma(1,q) = \sum_{\substack{A\subseteq E \\ \Gamma_A \text{ conn.}}} (q-1)^{b_1(\Gamma_A)} =$$

$$= \# \left\{ \begin{matrix} \text{indec. repr} \\ \text{with graph } \Gamma_A \end{matrix} \right\} / \sim$$

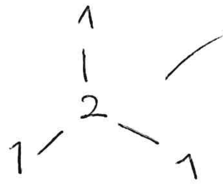


$\boxed{n=2}$

$$A(q) = \sum_{j=0}^{k-3} \sum_{r=j+2}^{k-1} \binom{k-1}{r} q^j$$

closed formula for $n=2$, any $k \geqslant 4$

E.g.



n	k	A(q)
$n=2$	$k=3$	$\rightsquigarrow A(q) = ①$
2	$k=4$	$\rightsquigarrow A(q) = q + ④$
2	$k=5$	$q^2 + 5q + ⑪$
2	$k=6$	$q^3 + 6q^2 + 16q + ㉖$

Eulerian numbers.

General quivers:

$M(q) = \#\{\text{repr}\}/\sim \quad = \quad \#$ orbits of $G(\mathbb{F}_q)$ on $V = \bigoplus_{i\to j} \mathrm{Hom}(V_i, V_j)$

$I(q) = \#\{\text{indec. repr}\}/\sim$

$A(q) = \#\{\text{absolutely indec. repr}\}/\sim$.

Fix dim vector $n = (n_i)_{i\in Q_0}$ $\quad n_i = \dim(V_i)$

$Q_0 = \{1,\dots,r\}$ vertices of Q.

$V \circlearrowleft G(\mathbb{F}_q)$

vsp on $\mathbb{F}_q$

$A \sim UAU^{-1}$ $\quad U \in GL_n(\mathbb{F}_q)$.

$\downarrow$

E.g. $\quad \circlearrowleft n \quad M(q) = \#$ similarity classes of $n \times n$ matrices $/\mathbb{F}_q$.

Counting orbits: $\underset{\text{finite grp}}{G} \curvearrowright \underset{\text{finite set.}}{X}$

Burnside's formula: $\underbrace{\# X/G}_{\text{set of orbits}} = \frac{1}{|G|} \sum_{g \in G} \# \mathrm{Fix}(g)$ "# orbits = average # fixed

$$\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$$

e.g. $g = 1 \Rightarrow \frac{\# X}{|G|}$ is a summand.

Fix(g) only depends on g up to conjugacy.

$\hookrightarrow \# \mathrm{Fix}(g) = \# \mathrm{Fix}(hgh^{-1}) \quad h \in G^*$. (?)

$G = GL_{n_1} \times \dots \times GL_{n_r}$

Conjugacy class in $GL_n(\mathbb{F}_q)$:

$\Lambda : \overline{\mathbb{F}_q} \longrightarrow \mathcal{P} \qquad \lambda = (\lambda_1, \lambda_2, \dots) \quad |\lambda| = n$.

$\xi \longmapsto \bigoplus_{i \geq 1} J_{\lambda_i}(\xi) \qquad \overset{\shortparallel}{\Lambda}(\xi)$

Conditions: $\bullet \sum_{\xi} |\Lambda(\xi)| = n$

$\bullet \Lambda(\xi^\sigma) = \Lambda(\xi) \quad \sigma \in gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ same block structure up to conjugacy.

$$\begin{tikzcd} V_i \arrow[r, "\varphi_{ij}"] \arrow[d, "\psi_i"'] & V_j \arrow[d, "\psi_j"] \\ V_i \arrow[r, "\tilde{\varphi}_{ij}"'] & V_j \end{tikzcd}$$

$\psi_j \circ \varphi_{ij} \circ \psi_i^{-1} = \varphi_{ij}$ fixed by GL_n-action.

$\Updownarrow$

$\psi_j \varphi_{ij} = \varphi_{ij} \psi_i$

$\gamma := \{\xi^{\sigma}\}$ $\quad \sigma \in Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ orbit

$J_\lambda(\gamma) := \underset{\substack{\sigma \\ \uparrow \\ gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)}}{\bigoplus} J_\lambda(\xi^{\sigma})$ $\leftsquigarrow$ size: $(d\cdot|\lambda|)\times(d\cdot|\lambda|)$

$d = \#\gamma .$

$J_\lambda = \bigoplus_{i \geq 1} J_{\lambda_i}(\xi)$

<u>Key calculation:</u>

$$W := \left\{ M \in \mathbb{F}_q^{dm \times dm} \;\middle|\; J_\lambda(\gamma) M = M J_\mu(\delta) \right\}$$

(λ,γ)

(μ,δ)

$\# W = q^{\dim W}$ because it is a linear space...

$$\dim W = \begin{cases} d\langle \lambda,\mu\rangle & \text{if } \gamma = \delta, \; d = \#\gamma \\ 0 & \text{otherwise} \end{cases}$$

identical orbits...

$$\langle \lambda,\mu\rangle := \sum_{i,j} \min(i,j)\, m_i(\lambda)\, m_j(\mu)$$

$\lambda = (\lambda_1, \lambda_2, \dots) = 1^{m_1(\lambda)}\, 2^{m_2(\lambda)} \dots$

$\mu = (\mu_1, \mu_2, \dots) = 1^{m_1(\mu)}\, 2^{m_2(\mu)} \dots$

$m_i(\lambda) :=$ multiplicity of i in λ.

If we ~~put~~ arrange $m_i(\lambda)$, $m_j(\mu)$ into vectors and $\min(i,j)$ into a matrix, we get that $\langle\lambda,\mu\rangle$ can be written as:

$$(m_1(\lambda), m_2(\lambda) \dots) \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & \cdots \\ 1 & 2 & 3 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{=(\min(i,j))_{ij}} \cdot \begin{pmatrix} m_1(\mu) \\ m_2(\mu) \\ \vdots \end{pmatrix} = \langle \lambda,\mu\rangle .$$

$\Downarrow$

<u>exe</u> find inverse!

If $(\lambda, \gamma) = (\mu, \delta)$, $|\lambda| = m$, $\overset{d=1}{\vee}$ then

$$\dim \mathrm{Cent}_{\mathbb{F}_q^{m\times m}} \left(J_\lambda(\xi) \right) = \langle \lambda, \lambda \rangle . \qquad \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}^{\lambda_1}$$

$\xi \in \mathbb{F}_q$

"centralizer"

$$\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix} \lambda_2$$

$$\ddots$$

$$\# \{ M \in \mathrm{Rep}_n \mid gM = M \} = \prod_{i,j} q^{d \cdot a_{ij} \langle \lambda^i, \lambda^j \rangle}$$

$a_{ij} = \# \{ i \longrightarrow j \}$

where $g \sim \bigoplus_{\substack{i=1 \\ \#\gamma = d}}^{r} J_{\lambda^i}(\gamma)$?

$$\left[\frac{1}{|G|} \sum_{g \in G} \# \mathrm{Fix}(g) = \sum_{\substack{\text{conj classes} \\ C}} \frac{1}{\left| \mathrm{Cent}_G(C) \right|} \# \mathrm{Fix}(C) \right]$$

$$\Rightarrow \quad M(q) = \sum_{\substack{d, \lambda^1, \dots, \lambda^r \\ |\lambda^i| = n_i}} \frac{\prod_{i,j} q^{a_{ij} d \langle \lambda^i, \lambda^j \rangle}}{\prod_i a_{\lambda^i}(q^d)}$$

Krull-Schmidt:

$$\boxed{\begin{aligned} &\sum_{n \neq 0} M_n(q) T^n = \prod_{n \geq 1} (1 - T^n)^{-I_n(q)} \\ &T^n = T_1^{n_1} \cdots T_r^{n_r} \end{aligned}}$$

$$I_n(q) = \sum_{d \mid n} \frac{1}{d} \sum_{e \mid d} \mu\left(\frac{d}{e} \right) A_{\frac{n}{d}}(q^e) .$$

$d, e \geq 1$ integers
$n =$ vector $= (n_i)$
$d \mid n \Rightarrow d$ divides all n_i

$I_n(q) = A_n(q)$ if $n_i = 1$ for some i.

Galois descent.

$A, B \in K^{n\times n}$ $\quad B = UAU^{-1}$ $\quad$ conjugate over $F \subset \overline{K}$

$F \supseteq K$ finite extension.

Q: Is B conjugate to A over K?

↳ yes!

$B = UAU^{-1}$ $\quad \sigma \in \mathrm{Gal}(F/K)$ $\quad B = U^\sigma A U^{-\sigma} =$

$\overset{!}{=} UAU^{-1}$

$U^\sigma A U^{-\sigma} = UAU^{-1}$

$\Rightarrow U^{-1}U^\sigma A U^{-\sigma} U$

$\Rightarrow U^{-1}U^\sigma A U^{-\sigma} U = A$

$\Rightarrow \sigma \longmapsto U^{-1}U^\sigma \in \mathrm{Cent}_{Gl_n F}(A)$

1-cocycle.

Field definition / Field of moduli

varieties X/K $\quad \sigma \in \mathrm{gal}(\overline{K}/K)$

$H := \{\sigma \mid X^\sigma \xrightarrow{\sim} X$ over $\overline{K}\}$

$F := \mathrm{Fix}(H)$

$$\begin{tikzcd} F \arrow[d] \\ K \end{tikzcd}$$

Shimura: $X : y^2 = x^6 + ax^5 + x^3 - \bar{a}x + 1$ $\quad a \in \mathbb{C}$

conjugate cplx coeffs: $\overline{X} : y^2 = x^6 + \bar{a}x^5 + x^3 - ax + 1$.

$X \cong \overline{X}$ isom as curves.

$\mu : X \simeq X^\sigma$

$(x, y) \longmapsto (-x^{-1}, ix^{-3}y)$

For generic a, $\mathrm{Aut}(X) = \{id, i\}$

$(x, y) \longmapsto (x, \pm y)$

$\nu = \mu \quad \mu \circ i$

$\nu \circ \nu^\sigma = i$

X not isom to curve$/\mathbb{R}$