Jan 18, 2006

cplx analysis very different from real analysis.

Rigid. crossroads of algebra, geometry, topology.

$$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}$$

(Fresnel)

—|||—

$i^2 = -1$ "imaginary" number

$z = a + bi$ $a, b \in \mathbb{R}$

Usual algebraic operations

$\mathbb{C}$ cplx numbers

a <u>field</u>.

Every $z \neq 0$ has an inverse.

$\overline{z} := a - bi$ conjugate of z

$$|z|^2 := z \cdot \overline{z} = (a+bi)(a-bi)$$
$$= a^2 - abi + bai - b^2 i^2$$
$$= a^2 + b^2$$
$$\geq 0$$

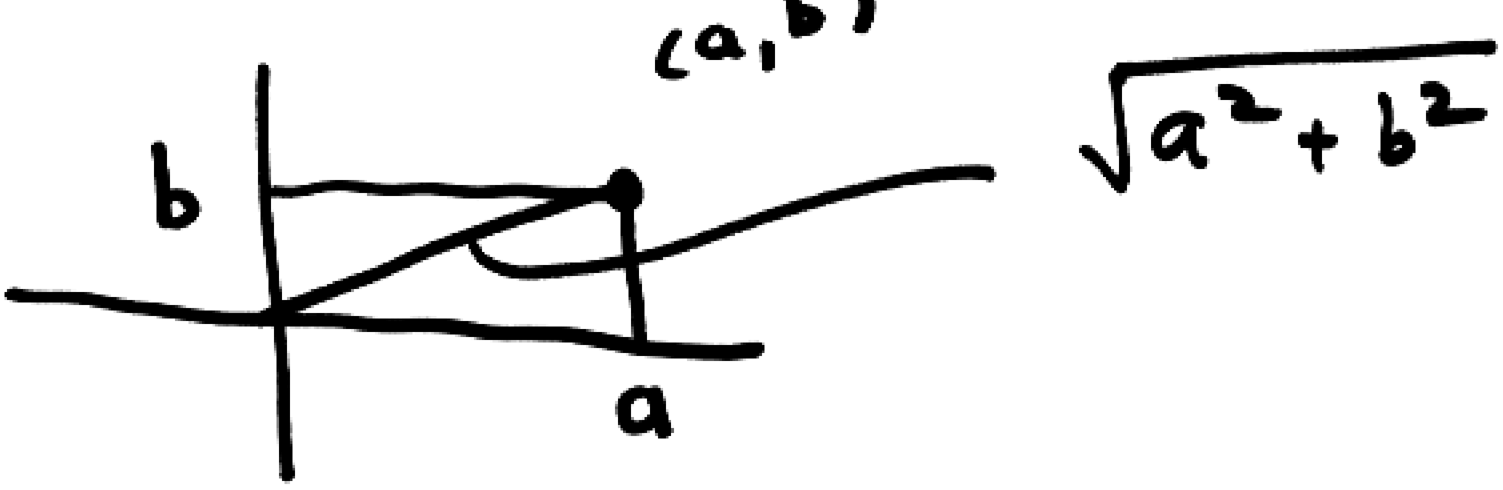
equality only if $a = b = 0$

If $z \neq 0$ then $|z| = \sqrt{a^2+b^2} \neq 0$

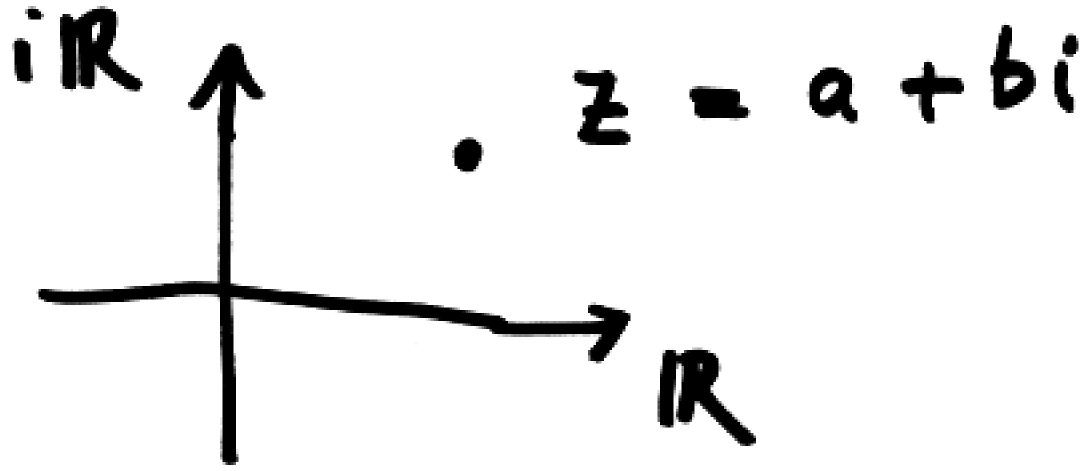
Hence $z \cdot \frac{\overline{z}}{|z|^2} = 1$

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

in $\mathbb{R}^2$



Gauss in his thesis introduced geometric way of thinking of cplx numbers. ③



The original motivation for introducing i was to solve equations.

I.e. $x^2 + 1 = 0$

solutions : $i, -i$

<u>**Miracle**</u> : once we have i we have <u>all</u> solutions to <u>any</u> equation.

<u>**FTA**</u> $p \in \mathbb{C}[X]$

$p = a_n x^n + \dots + a_0$ p has a root in $\mathbb{C}$

$a_i \in \mathbb{C}$

In other words

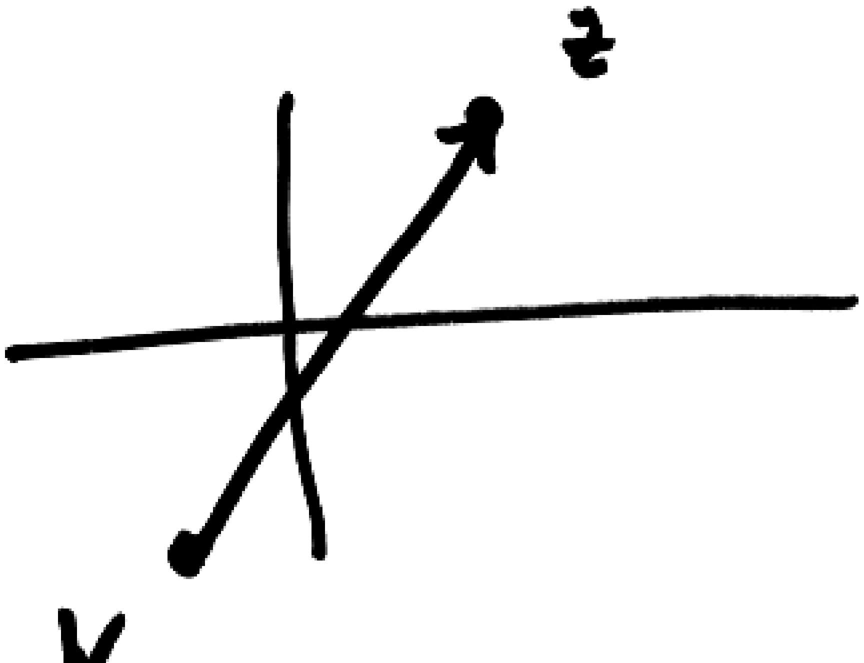
$\mathbb{C}$ is algebraically closed

Formally $\mathbb{C} = \mathbb{R}[x] / (x^2+1)$

$$|z| = \sqrt{a^2+b^2}$$

= distance of (a,b) to the origin

$|z-w|$ = distance from z to w



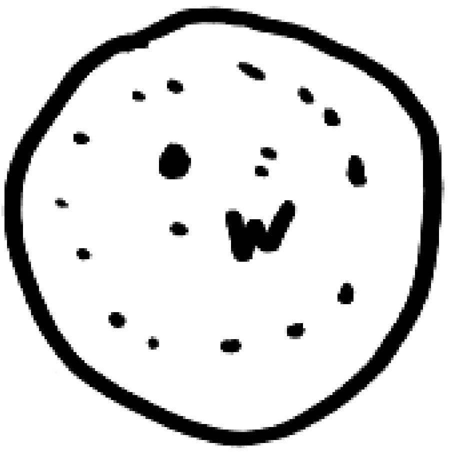
Triangle inequality

$$|z+w| \leq |z| + |w|$$

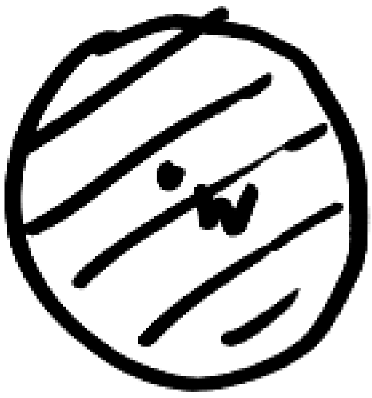
This distance makes $\mathbb{C}$ into a metric space. this is non other than $\mathbb{R}^2$ with euclidean norm.

Disks

open $|z-w|<R$

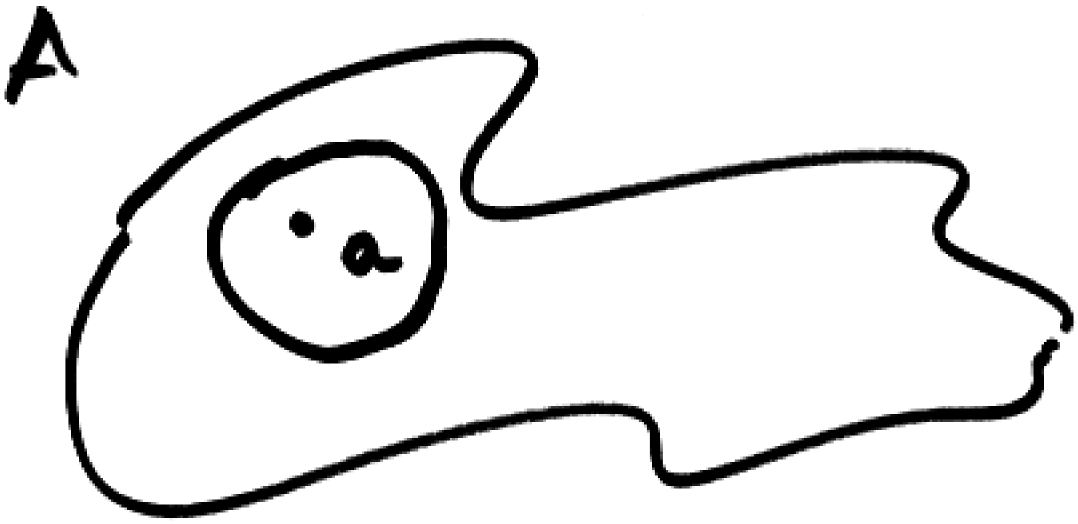


closed $|z-w| \leq R$



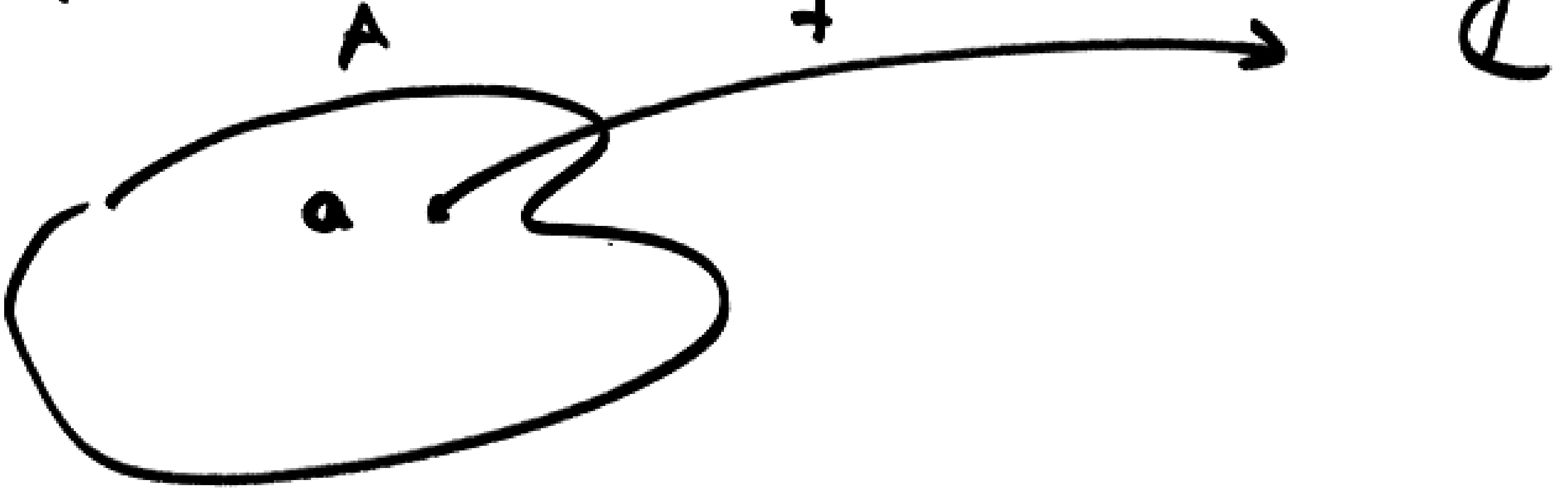
$A \subseteq \mathbb{C}$ is open iff every $a \in A$ has a disk

$$D(a, \varepsilon) \subseteq A$$
$$= \{z \mid |z-a|<\varepsilon\}$$

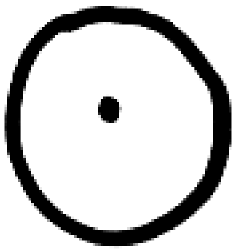


Differentiation

$f : A \subseteq \mathbb{C} \longrightarrow \mathbb{C}$

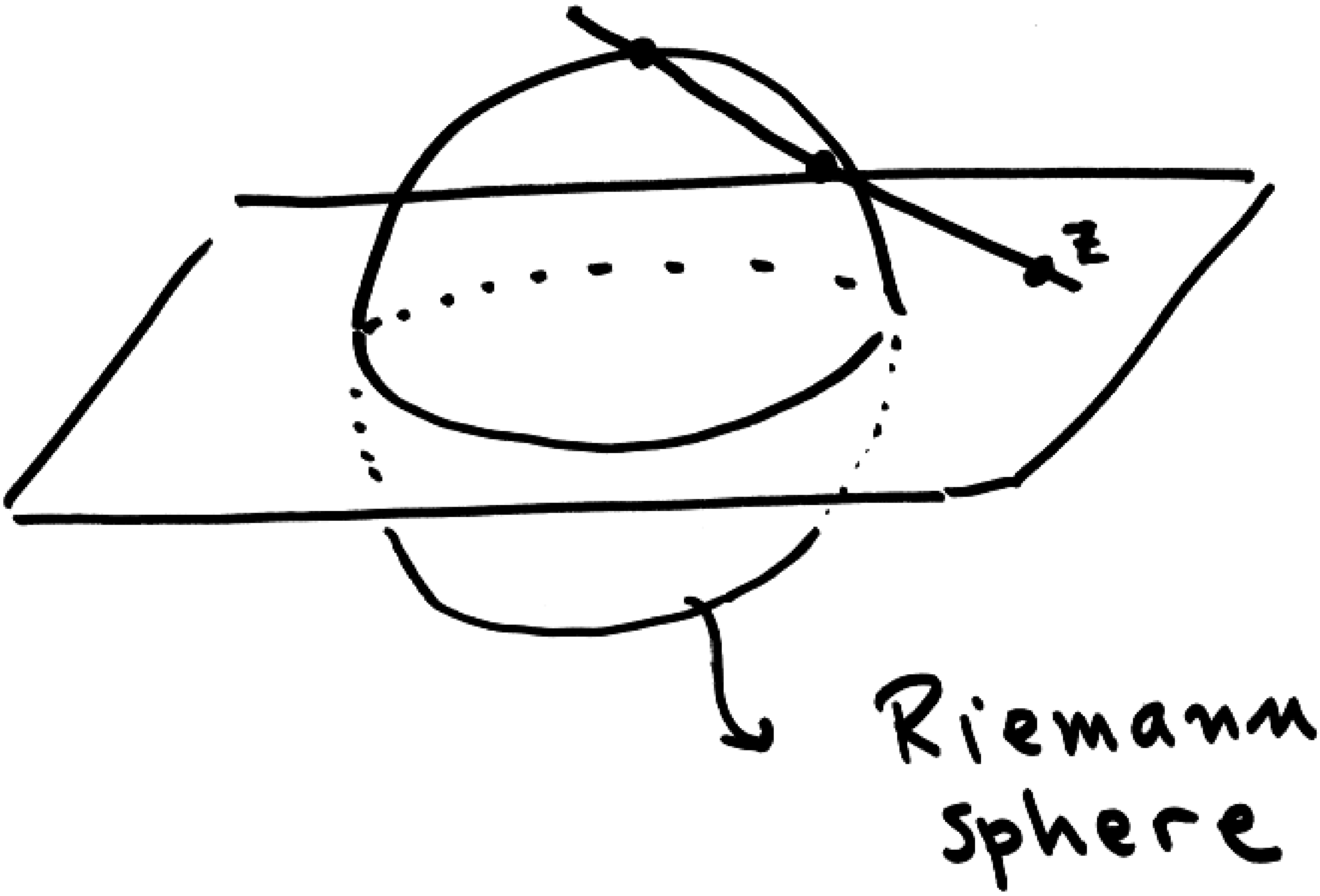


$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$|h| \to 0$

say f is <u>differentiable at a</u> if limit exists.


z
Riemann sphere

Jan 20, 2006

$U \subseteq \mathbb{C}$ open set

$f : U \to \mathbb{C}$

continuity, limits, etc. (topology) is that of $\mathbb{R}^2$.

$\lim_{z \to a} f(z) = L$ For each $\varepsilon > 0$

$|f(z) - L| < \varepsilon$

$\exists \delta > 0$ s.t. for all $|z-a| < \delta$

$D(a, \delta) = \{ |z-a| < \delta \}$

In particular

$\lim_{z \text{ along line}} f(z) = L$

It's not true that

if $\lim_{z \text{ along line}} f(z) = L$

for all lines then

$$\lim_{z \to a} f(z) = L.$$

Example $f(x,y) = \dfrac{x+y}{x-y} : U \to \mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y}$$ Does not exist

$\swarrow$

$y = a x$

$(0,0)$

$$f(x,y) = \frac{x + a x}{x - a x} = \frac{a+1}{a-1}$$

$x \neq 0$ is constant hence has limit, namely the constant.

$$g(x,y) = \frac{(x+y)^2}{x-y} \quad \text{on} \quad y = ax$$

$$g(x,y) = \frac{a+1}{a-1}\, x$$

has limit 0 as $(x,y) \to (0,0)$

$$f : U \to \mathbb{C}$$

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

f is analytic (or holomorphic

on U if $f'(a)$ exists for $a \in U$.

$$f = u + iv$$

$$v, u : U \longrightarrow \mathbb{R}$$

$$z = x + iy$$

If $h \in \mathbb{R}$

$$\frac{f(z+h)-f(z)}{h} = \frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{v(x+h,y)-v(x,y)}{h}$$

$h \to 0$ in $\mathbb{R}$

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$h \in \mathbb{R}$

$$\frac{f(z+ih)-f(z)}{ih} = -i\,\frac{u(x,y+h)-u(x,y}{h} + \frac{v(x,y+h)-v(x,y)}{h}$$

$$f'(z) = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \left[\begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = - \dfrac{\partial u}{\partial y} \end{array}\right.$$

<u>Cauchy - Riemann</u>

equations.

<u>THM</u> Suppose $u, v : U \to \mathbb{R}$ satisfy CR equation. and

$\dfrac{\partial u}{\partial x}$, $\dfrac{\partial v}{\partial y}$ are continuous.

$\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$

Then $f = u + iv$ is analytic.

<u>Pf</u>. $u(x+a, y+b) - u(x,y)$

$$= \frac{\partial u}{\partial x} a + \frac{\partial u}{\partial y} b + \varepsilon$$

$h = a + ib$

$\varepsilon / h \to 0 \qquad$ as $h \to 0$.

similarly for v ... ε'

$$f(z+h) = f(z) + \frac{\partial u}{\partial x} a + \frac{\partial u}{\partial y} b$$

$$+ i \left(\frac{\partial v}{\partial x} a + \frac{\partial v}{\partial y} b \right)$$

$$+ \; \varepsilon + i\varepsilon'$$

$$\frac{\partial u}{\partial x} a - \frac{\partial v}{\partial x} b + i \left(\frac{\partial v}{\partial x} a + \frac{\partial u}{\partial x} b \right)$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) a + \left(i \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right) b$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (a + ib)$$

$$f(z+h) - f(z) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) h + \varepsilon + i\varepsilon'$$

$$\frac{f(z+h)-f(z)}{h} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \underbrace{\frac{\varepsilon + i\varepsilon'}{h}}_{\substack{\searrow \\ h\to 0 \quad 0}}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \square$$

<u>Example</u> $f(z) = z$

$$u(x,y) = x$$
$$v(x,y) = y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 1$$

$U = \mathbb{C}$, $f'(z) = 1$

Sums, products of analytic functions are analytic.

$\frac{1}{f(z)}$ is analytic as long as $f(z) \neq 0$ on U.

In particular, any polynomial is analytic.

$$f(z) = \bar{z}$$

Is <u>not</u> analytic.

$u = x$
$v = -y$

$$\bar{z} = x - iy$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

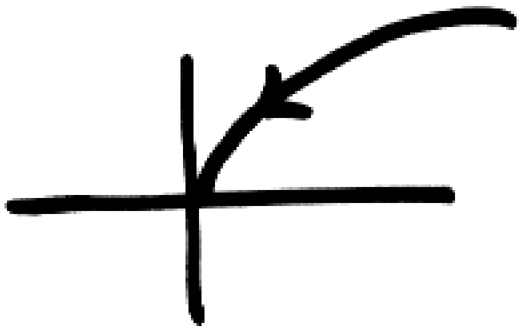
We can think of z & $\bar{z}$ as indep. variables (instead of x, y)

CR $\quad \frac{\partial f}{\partial \bar{z}} = 0$

Jan 23, 2006

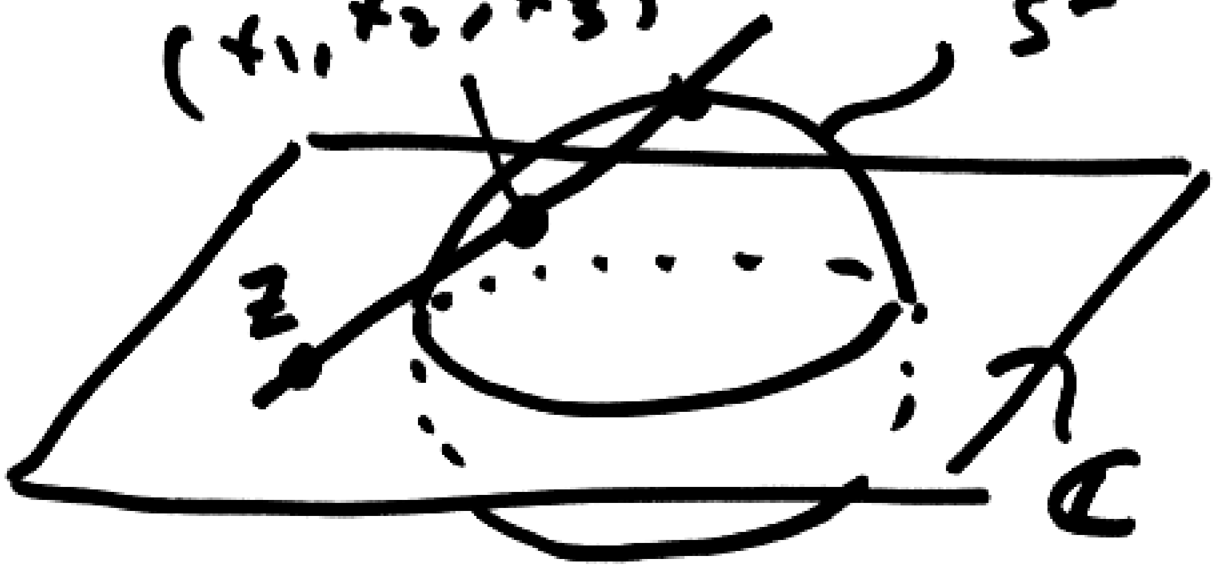
$f(x,y) = y^2/x$ $\qquad x \neq 0$

$\lim\limits_{y=ax} f(x,y) = 0$



$y^2 = x$

lim = 1



Stereographic projection

S^2 = unit sphere in $\mathbb{R}^3$

$$\mathbb{C} \xleftrightarrow{1-1} S^2 \setminus \{(0,0,1)\}$$

$$(x_1, x_2, x_3 - 1) = \lambda(x, y, -1)$$

$\underline{\lambda = 0}$ $\qquad (x_1, x_2, x_3) = (0, 0, 1)$

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$0 = \lambda\left(\lambda\left(|z|^2 + 1\right) - 2\right)$$

$$z = x + iy$$

$\underline{\lambda \neq 0}$ $\qquad \lambda = \dfrac{2}{|z|^2 + 1}$

$$(x_1, x_2, x_3) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

$$z = \frac{x_1 + i x_2}{1 - x_3}, \quad x_3 = 1$$

$$|z| = 1 \leftrightarrow x_3 = 0$$

$$|z| > 1 \leftrightarrow x_3 > 0$$

$$|z| < 1 \leftrightarrow x_3 < 0$$

$$d(z, z') = \frac{2|z - z'|}{\sqrt{|z|^2+1} \cdot \sqrt{|z'|^2+1}}$$

($\shortparallel$
distance in $\mathbb{R}^3$
between (x_1, x_2, x_3),
(x_1', x_2', x_3'))

Angles are preserved
conformal map.

Circle on S^2

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = d$$

If $a_3 = d$ then plane

goes through $(0,0,1)$.

$$a_1(2x) + a_2(2y) + a_3(|z|^2-1) - d(|z|^2+1) = 0$$

If $a_3 = d$ then the equation is linear and we get a line.

otherwise, divide through by $a_3 - d$

$$x^2+y^2-2\alpha x-2\beta y+\gamma=0$$

$$(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2-\gamma$$

circle in x-y plane.

Topologically chordal distance defines the same open sets as the usual distance.

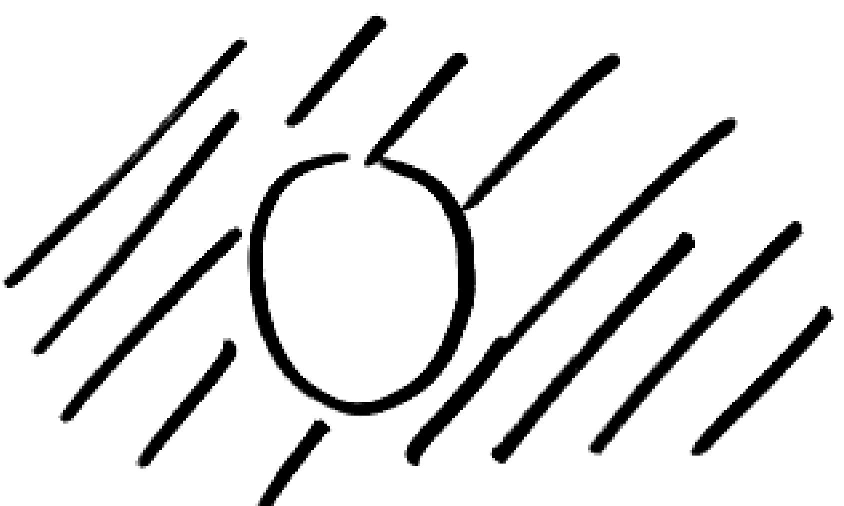
$$d(z, \infty) = \frac{2}{\sqrt{|z|^2+1}}$$

$$(x_1, x_2, x_3 - 1) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{-2}{|z|^2+1} \right)$$

$$\| \cdots \| = 2 \left(\frac{x^2+y^2+1}{(|z|^2+1)^2} \right)^{1/2}$$

Open Disk ~~[illegible]~~ centered at ∞

$$\{ |z| > R \}$$



(Alexandrov compactification of $\mathbb{C}$)

Advantage of Riemann sphere, extended cplx plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

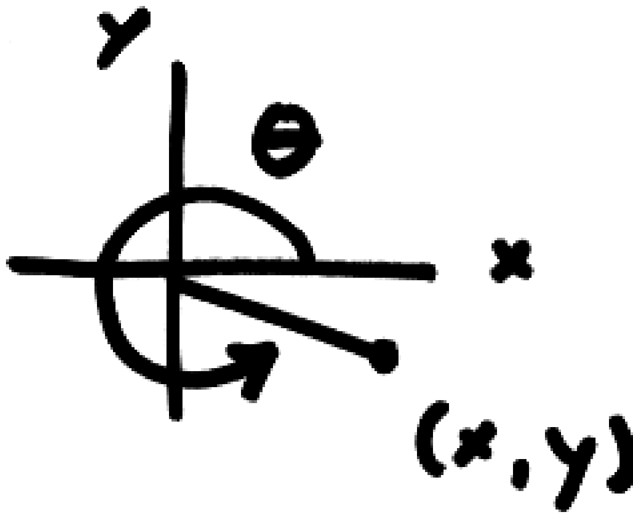
- compact
- ∞ is like everybody else.

In this format $+, \cdot$ of cplx numbers is pretty weird...

We want $z^n \xrightarrow[n \to \infty]{} \infty$ if $|z|>1$.

Polar form

$z = x + iy$



$r = |z| = \sqrt{z \cdot \bar{z}}$

$e^{i\theta} := \cos\theta + i\sin\theta$

$\theta \in \mathbb{R}$

$z = r \cdot e^{i\theta}$

$z^n = r^n \cdot e^{in\theta}$

de Moivre formulas

$(\cos(n\theta) , \sin(n\theta))$

z^n

- Another point of view: we may view S^2, $\hat{\mathbb{C}}$ as $\mathbb{P}^1(\mathbb{C})$

<u>projective line</u>

$$\{ (z_1 : z_2) \}$$

$z_1, z_2 \in \mathbb{C}$ not both 0

modulo scalars.

$$(z_1, z_2) \sim (z_1', z_2')$$

if $(z_1, z_2) = \lambda (z_1', z_2')$

some $\lambda \in \mathbb{C}$.

If $z_2 \neq 0$ then

$$(z_1 : z_2) \sim (z_1/z_2 : 1)$$

If $z_2 = 0$

$$(z_1 : z_2) \sim (1 : 0)$$

- $(z_1/z_2 : 1) = (z : 1)$

  $z \in \mathbb{C}$

- $(1 : 0)$

To deal with ∞ we bring it to $\mathbb{C}$.

$f(z)$

$z \longmapsto z^{-1}$

$g(z) := f(z^{-1})$

$\infty \longleftrightarrow 0$

Jan 25, 2006

Review X metric space

- $x_n \in X$ is a **Cauchy sequence** iff For all $\varepsilon > 0$ there exists an N

$$d(x_n, x_m) < \varepsilon$$

all $n, m \geq N$

- X is **complete** if every Cauchy sequence has a limit in X.

E.g. $\mathbb{R}$, $\mathbb{C}$, closed subset of $\mathbb{R}$, $\mathbb{C}$ is complete

- **Uniform convergence**

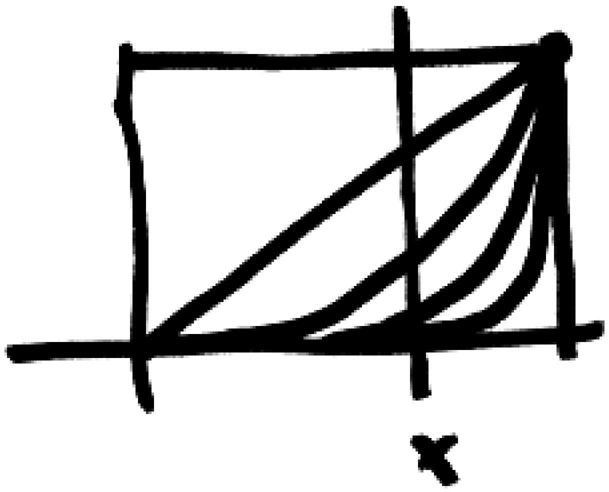
$f_n : X \longrightarrow Y$ sequence of functions

$f_n \to f$ **uniformly**
$n \to \infty$ **on X**

$d(f_n(x), f(x)) < \varepsilon$
$n \geq N$, all $x \in X$.

Example non-uniform convergence ?

$f_n(x) := x^n : [0,1] \to [0,1]$



$$x^n \to \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases} \overset{!!}{=} f(x).$$

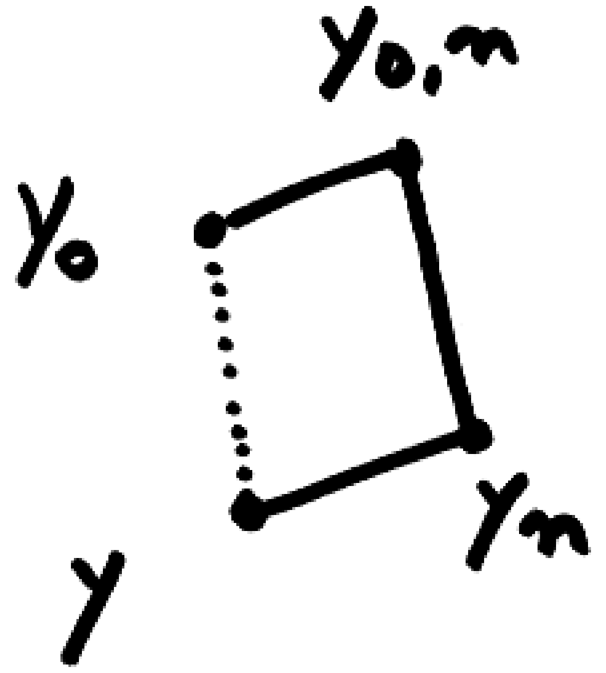
Thm

$f_n \to f$ uniformly
f_n continuous $\Rightarrow f$ continuous.

Pf $x_0 \in X$, $x \in X$

$y_0 := f(x_0) \qquad y := f(x)$

$y_{0,n} := f_n(x_0) \qquad y_n := f_n(x)$



$$d(y, y_0) \leq d(y, y_n) + d(y_n, y_{0,n}) + d(y_{0,n}, y_0)$$

choose N s.t.

$d(y, y_n) < \varepsilon$ for all $n \geq N$

$d(y_0, y_{0,n}) < \varepsilon$

choose δ s.t. $d(y_n, y_{0,n}) < \varepsilon$

if $d(x_0, x) < \delta$

$$d(y, y_0) < 3\varepsilon \qquad \square$$

Weierstrass M - test

$$u_n : X \longrightarrow \mathbb{C}$$

Suppose

- $$|u_n(x)| \leq M a_n$$ all $x \in X$ $\quad M, a_n \in \mathbb{R}_{>0}$

- $$\sum_{n \geqslant 0} a_n < \infty$$

then

$$\sum_{n \geqslant 0} u_n(x)$$

converges uniformly on X.

(I.e. $\sum_{k=0}^{n} u_k(x) =: f_n(x)$ converges uniformly on X)

Pf $n > m$

$$|f_n(x) - f_m(x)| \leq M \sum_{k=m+1}^{n} a_k$$

$$\overset{\|}{}$$

$$\sum_{k=m+1}^{n} u_k(x)$$

Sequence $\sum_{k=0}^{n} a_k$ is Cauchy

$$|f_n(x) - f_m(x)| < \varepsilon$$

$n \geq N$ all $x \in X$

$$f(x) := \lim_{n \to \infty} f_n(x)$$

$$|f(x) - f_m(x)| \leq M \sum_{k \geq m+1} a_k$$

$< \varepsilon$ $m \geq N$

all $x \in X$ □

Example

$u(z) = 1 + z + z^2 + \cdots\cdot$

$u_n(z) = z^n,$ $\quad X = \overline{D(0,r)}$

$0 < r < 1$

$|z|^n \leq r^n$

$$\sum_{n \geq 0} r^n < \infty$$

u exists, is continuous on X.

$$u(z) = \frac{1}{1-z}$$

Analytic in $\mathbb{C} \setminus \{1\}$.

- Cauchy-Riemann Eqns

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\Delta(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(if $\frac{\partial^2}{\partial x \partial y}$ continuous)

→ u, v are <u>harmonic</u>.

- $f(z) = \frac{1}{z}$ analytic $\mathbb{C} \setminus \{0\}$

$f'(z) = -\frac{1}{z^2}$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$\left\lceil u = \frac{x}{x^2+y^2}\right.$$

$$v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2 - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{1}{z^2} = \frac{\bar{z}^2}{|z|^4}}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2}$$

$$= -1/z^2$$

Jan 27, 2006

$$R(z) = \frac{P(z)}{Q(z)}$$

$P, Q \in \mathbb{C}[z]$ polynomials with coeff. in $\mathbb{C}$.

P, Q have no common zeros.

If $z = a$ is a zero of Q then we call it a <u>pole</u> of R.

In the extended plane sense

$$R(a) = \infty$$

$$\lim_{z \to a} R(z) = \infty$$

$$R(z) = R_1(z) \cdot \frac{1}{(z-a)^k}$$

R_1 continuous at a

$R_1(a) \neq 0$

$$|R_1(z)| > c$$

$$|z-a| < \delta$$

$$|R(z)| = |R_1(z)| \frac{1}{|z-a|^k} > \frac{c}{|z-a|^k}$$

Letting $z \to a$ we see

$$|R(z)| \to \infty$$

$$R : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$$

k = order of the pole

$$R = \frac{a_n z^n + \cdots + a_0}{b_m z^m + \cdots + b_0}$$

$$a_n, b_m \neq 0$$

Replace z by z^{-1}

$$R_1(z) = R(z^{-1})$$

$$= \frac{a_n z^{-n} + \cdots + a_0}{b_m z^{-m} + \cdots + b_0}$$

$$= z^{m-n} \frac{a_0 z^n + \dots + a_n}{b_0 z^m + \dots + b_m}$$

R at ∞

zero $m > n$ $R_1(0) = 0$

$m = n$ $R_1(0) = a_n / b_m \neq 0, \infty$

pole $m < n$ $R_1(0) = \infty$

$$R : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

order of zero / pole

zeros of R in $\hat{\mathbb{C}}$

in $\mathbb{C}$ R has n zeros
m poles

∞	$m > n$	zero	$m - n$
	$m = n$	—	
	$m < n$	pole	$n - m$

	zeros	poles
$\underline{m > n}$	$n + m - n = m$	m
$\underline{m = n}$	n	m
$\underline{m < n}$	n	$m + n - m$ $= n$

$$\# \text{zeros} = \# \text{poles} =: \underline{\text{order}} \text{ of } R$$

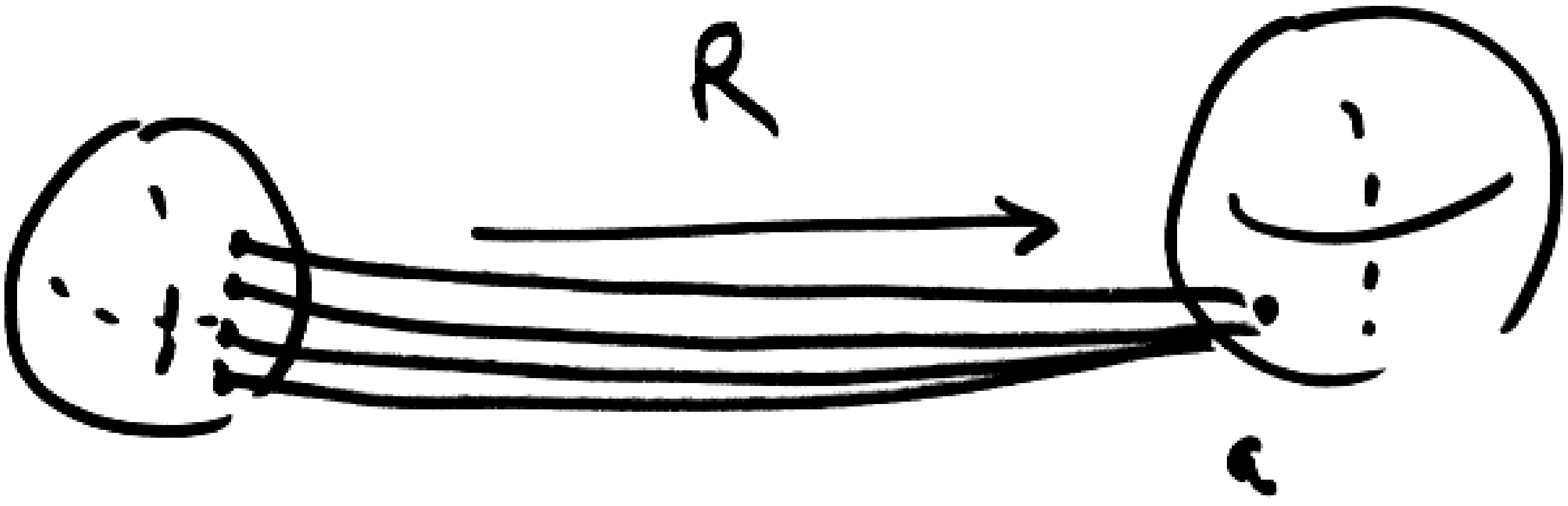
$$= \max\{m, n\} \qquad \underline{\text{degree}} \text{ of } R$$

$R(z) = a$

zeros of $R(z) - a$

same degree as R.

\# preimages of $a \in \hat{\mathbb{C}}$



deg = 1 ?

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \frac{\alpha z + \beta}{\gamma z + \delta} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$\alpha\delta - \beta\gamma \neq 0$$

Möbius or linear transformation

Translations

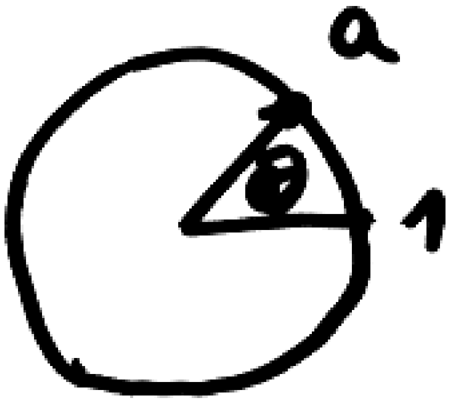
$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad z \longmapsto z + a \quad , \quad a \in \mathbb{C}$$

Rotations

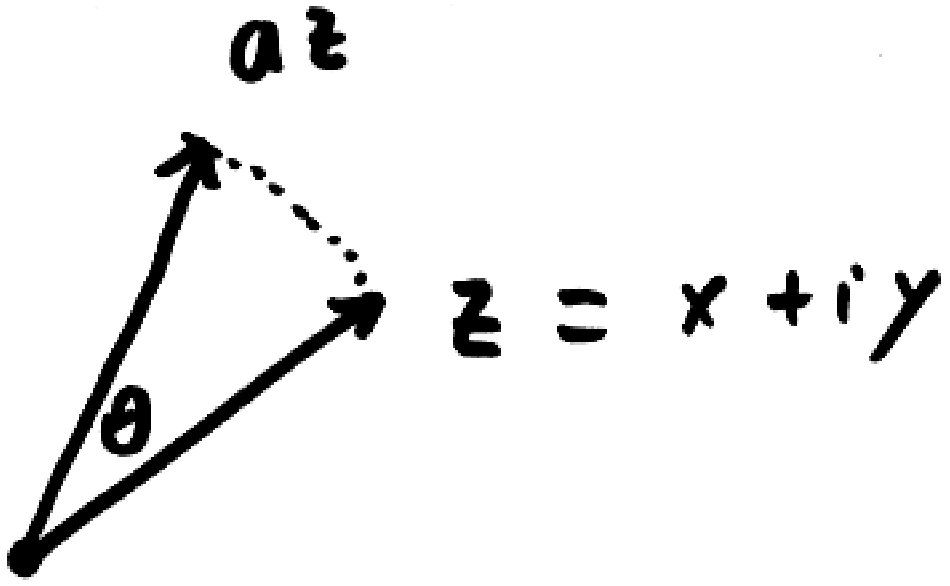
$$z \longmapsto a z \quad , \quad |a| = 1$$

$$a = \cos\theta + i \sin\theta =$$

$$= e^{i\theta}$$



$$\mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longmapsto a z$$



$$(x+iy)((\cos\theta + i\sin\theta)$$

$$= x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)$$

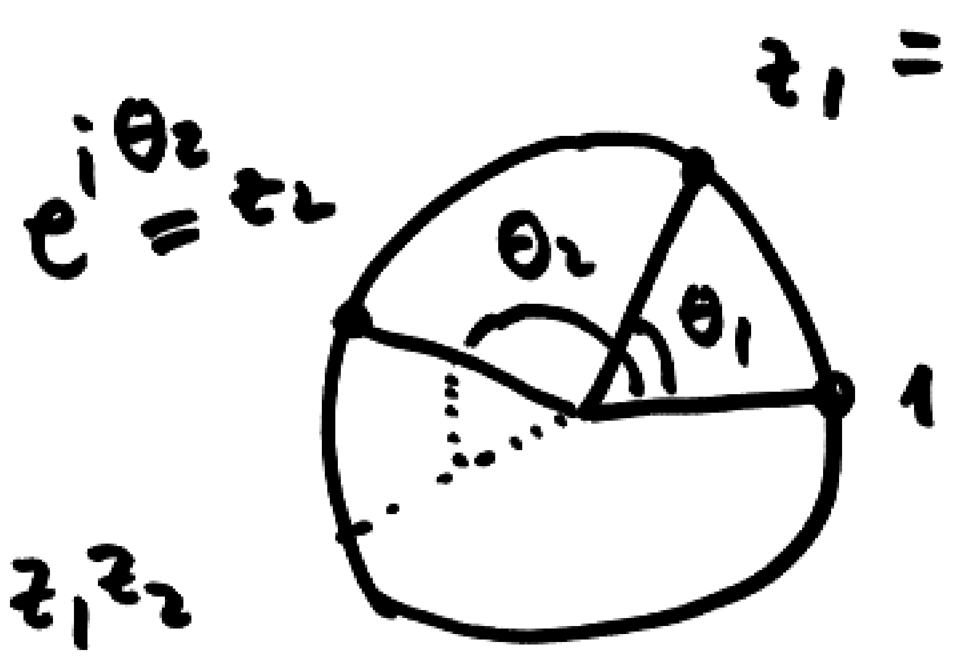
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

↑ rotation counterclockwise angle θ

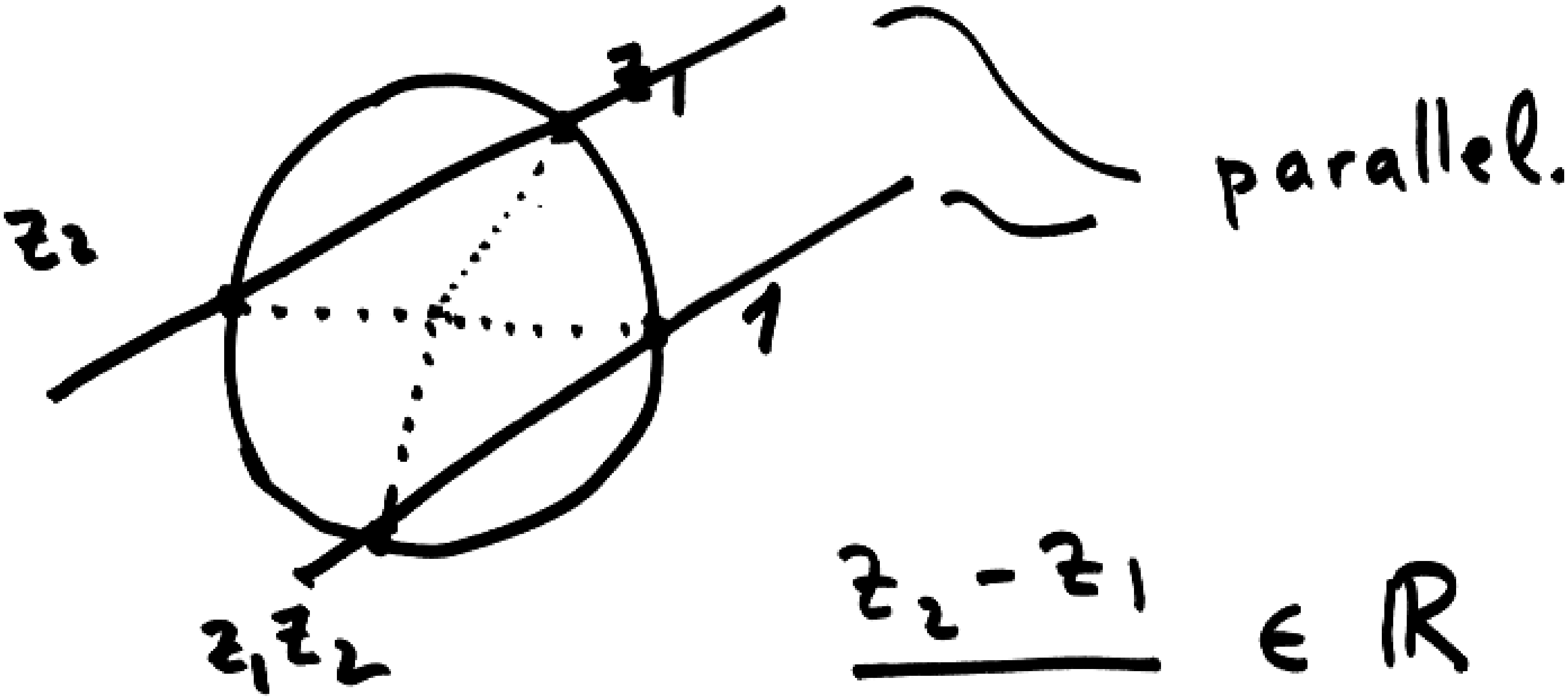
$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$



$$e^{i\theta_1} \bullet e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$



$$\frac{z_2 - z_1}{z_1 z_2 - 1} \in \mathbb{R}$$

$$\overline{\left(\frac{z_2 - z_1}{z_1 z_2 - 1}\right)} = \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_1 \bar{z}_2 - 1}$$

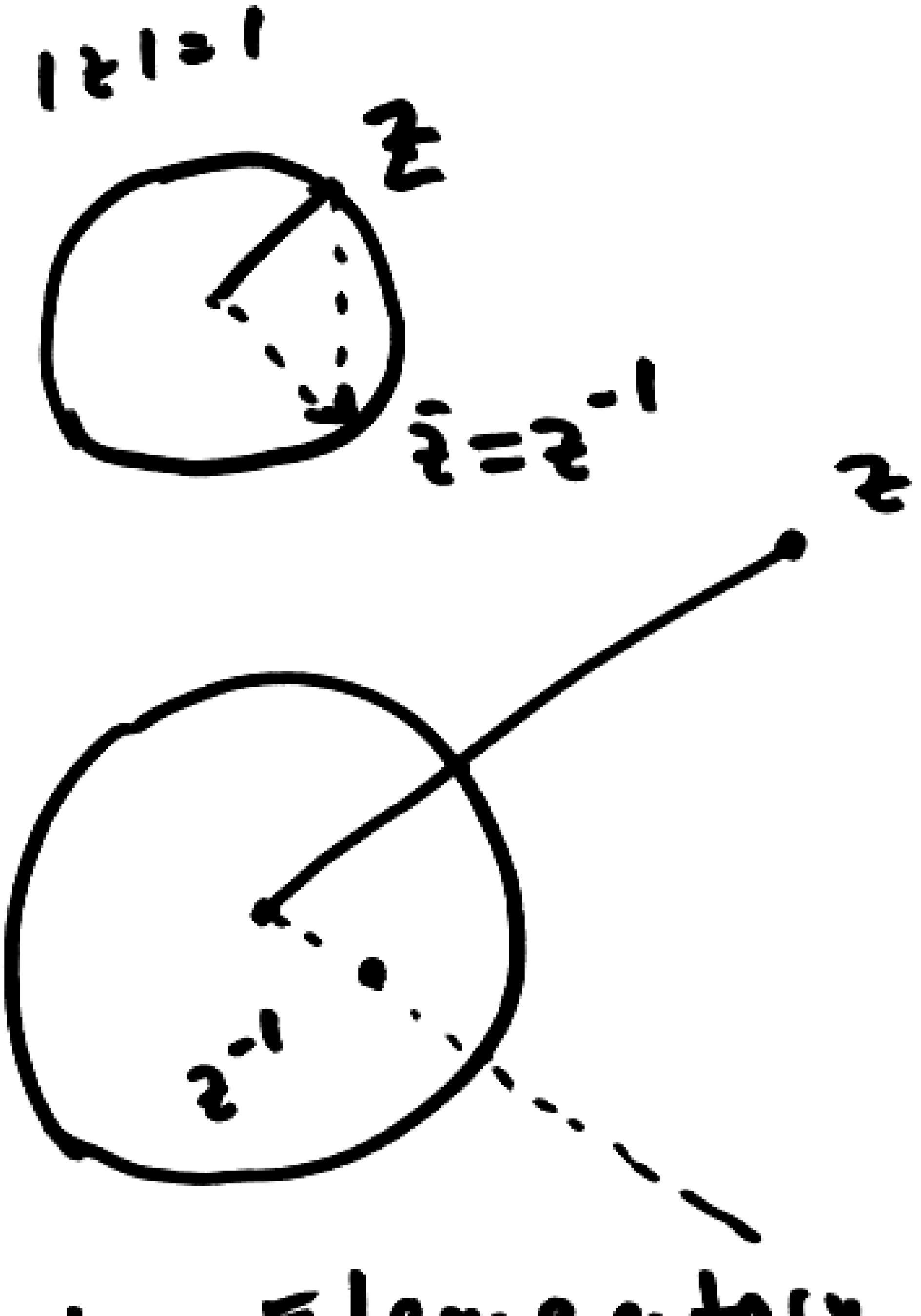
$|z| = 1$ $\quad$ $\bar{z} = z^{-1}$

$$= \frac{z_2^{-1} - z_1^{-1}}{z_1^{-1} z_2^{-1} - 1} = \frac{z_1 - z_2}{1 - z_1 z_2}$$

Dilations $a > 0$

$$z \longmapsto az$$

Inversion $z \longmapsto \frac{1}{z}$



Fact Elementary transformations generate the group of all transformations.

Jan 30, 2006

Power series

$$f(z) = \sum_{n \geq 0} a_n (z-a)^n$$

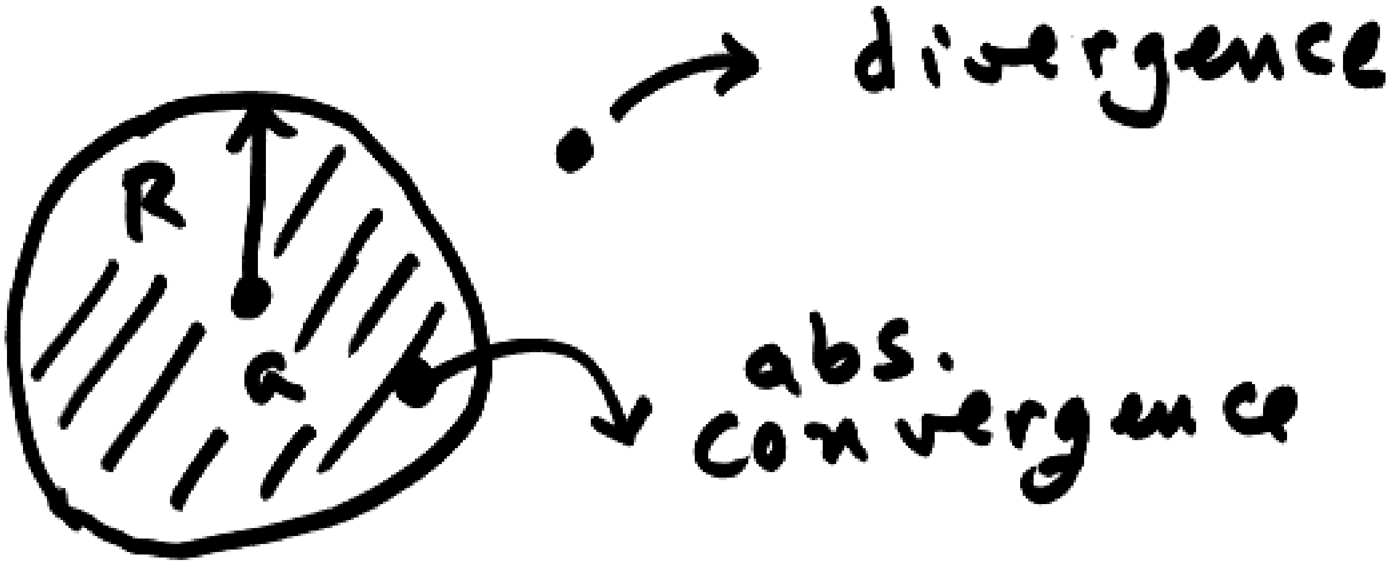
$a_n \in \mathbb{C}$, $a \in \mathbb{C}$.

THM There exists a unique $R = [0, \infty]$ such that

1) If $|z-a| < R$ series converges absolutely.

1') Convergence is uniform on $|z-a| \leq r < R$

2) If $|z-a| > R$ the series diverges



3) $$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

Compare with geometric series

$$\sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1} \qquad (z\neq 1)$$

$$\sum_{k=0}^{n} |z|^k = \frac{|z|^{n+1}-1}{|z|-1} \qquad (|z|\neq 1)$$

$$|z|<1 \qquad |z|^{n+1}\to 0$$

$$\sum_{k=0}^{n} |z|^k \longrightarrow \frac{1}{1-|z|}$$

If $|z| \leq r < 1$ then

$\sum_{k\geq 0} z^k$ converges uniformly by M-test

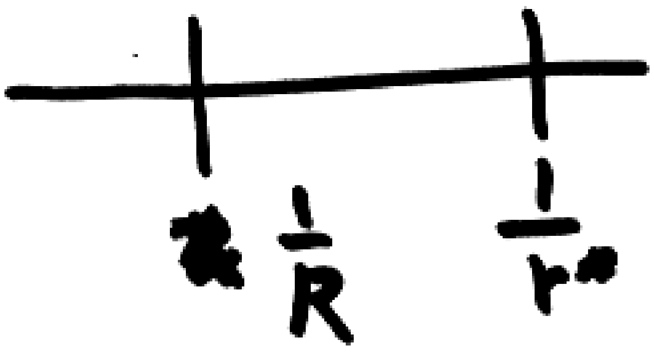
If $|z| > 1$ then

$z^K \to \infty$ terms become unbounded. series diverges

$a=0$ wlog

Pf $|z| \le r' \overset{<r}{<} R$

then $\frac{1}{r} > \frac{1}{R} = \limsup\limits_{n\to\infty} |a_n|^{1/n}$



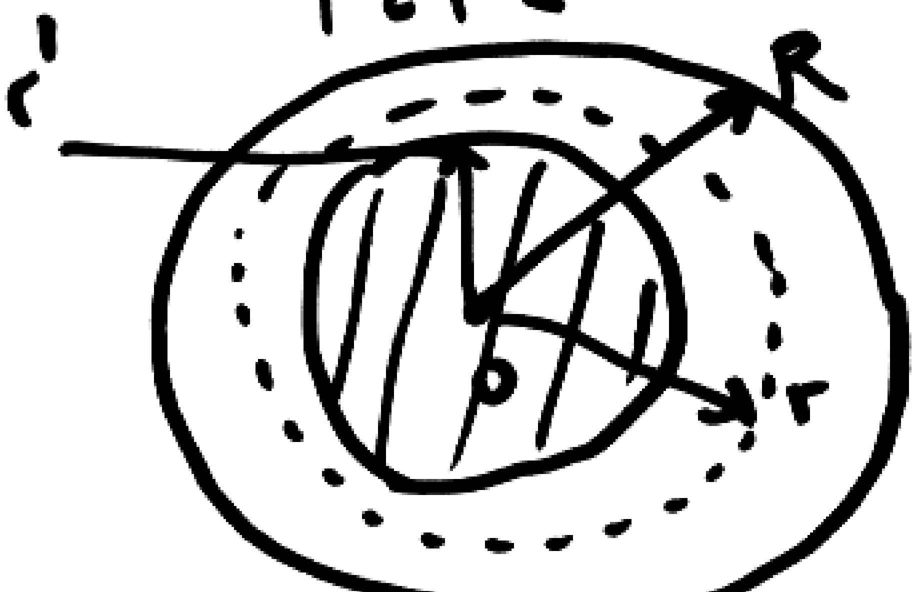
- for all $n \geq N$ $|a_n|^{1/n} < \frac{1}{r}$

  for some N

$|a_n z^n| < \left(\frac{|z|}{r}\right)^n < \left(\frac{r'}{r}\right)^n$

$|z| \le r' < r$, $\frac{r'}{r} < 1$

$n \geq N$



By M-test

$$\sum_{n \geqslant N} a_n z^n$$

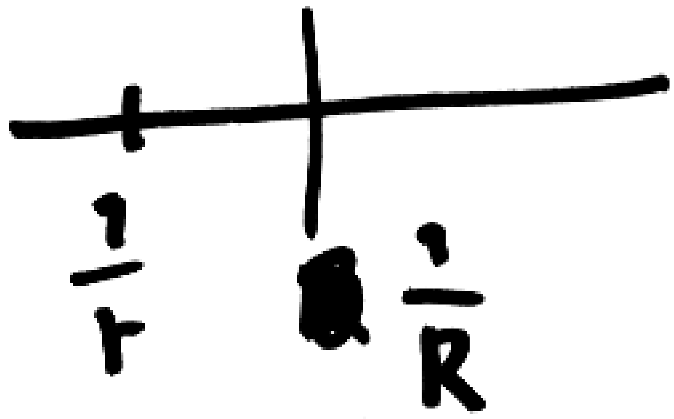
converges absolutely and uniformly on $|z| \leqslant r' < R$

→ same for $\sum_{n \geqslant 0} a_n z^n$

This assumed $R > 0$

If $R = 0$ convergence at 0 is obvious.

2) $|z| > R$



There are arbitrarily large n s.t. $|a_n|^{1/n} > \frac{1}{r}$

$$|a_n| > \frac{1}{r^n}$$

$$|a_n z^n| > \left(\frac{|z|}{r}\right)^n$$

$$\frac{|z|}{r} > 1$$

The terms in our series are unbounded ⇒ series diverges

3) unique R $\square$

geometric series $\sum_{n\geq 0} c^n z^n$

$$R = \frac{1}{|c|}$$

$$\limsup_{n\to\infty} |c^n|^{1/n} = |c|$$

$$\frac{c^{n+1}}{c^n} = c$$

If

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists it equals R.

Example

$$\sum_{n\geq 0} \frac{z^n}{n!} =: e^z$$

$$a_n = \frac{1}{n!}$$

$$\frac{|a_n|}{|a_{n+1}|} = \frac{(n+1)!}{n!} = n+1 \to \underset{R}{\overset{\infty}{\shortparallel}}$$

Exponential function

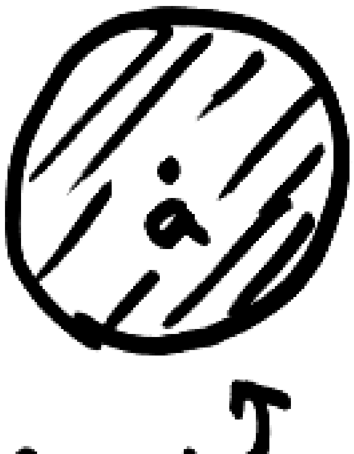
$$\limsup_{n\to\infty} (n!)^{1/n} = \infty$$

Stirling's fmla.

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Prop $f(z)=\sum_{n\geq 0} a_n (z-a)^n$

$R > 0$

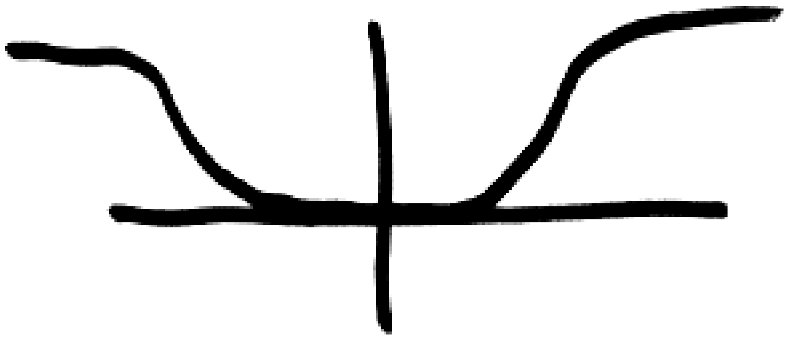


1) f analytic in
infinitely differentiable

2) $\frac{f^{(n)}(a)}{n!} = a_n$

- a_n are uniquely determined by f.

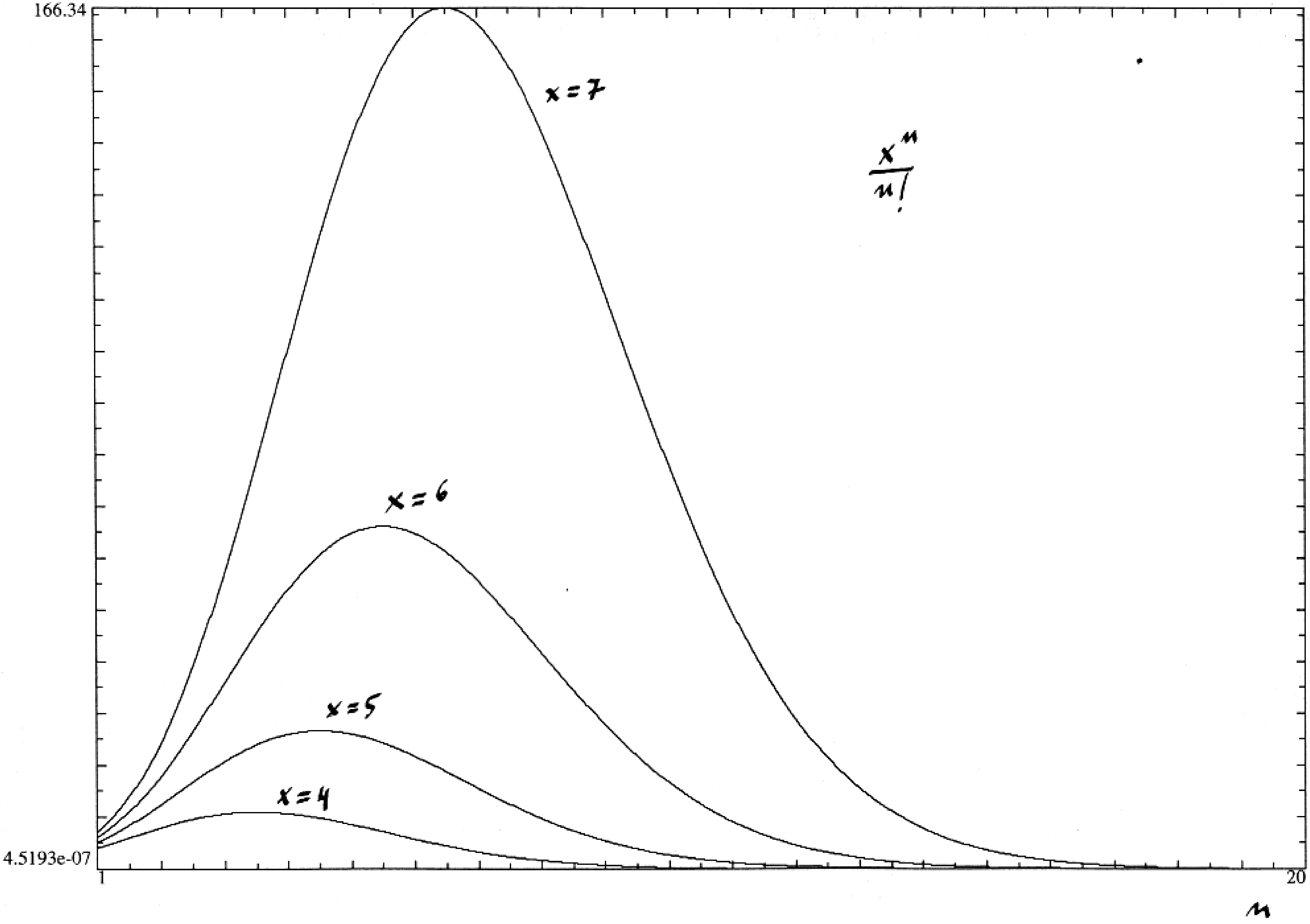
Example $f(x)=\begin{cases} e^{-1/x^2} & x\neq 0 \\ 0 & x=0 \end{cases}$



$f^{(n)}(0)=0$

C^∞ and

$x=it$ $t\in\mathbb{R}$, $f(it)=e^{1/t^2} \underset{t\to 0}{\longrightarrow} \infty$
$t\neq 0$


166.34
4.5193e-07
1
20
n
x = 7
x = 6
x = 5
x = 4
$\frac{x^n}{n!}$

Feb 1, 2006

Product of series

$$U = \sum_{n\geqslant 0} u_n \quad , \quad V = \sum_{n\geqslant 0} v_n$$

$$U \cdot V = \sum_{n\geqslant 0} w_n =: W$$

Cauchy product

$$w_n := \sum_{k=0}^{n} u_k \, v_{n-k}$$

If $\quad u_n = a_n z^n$
$\quad\quad v_n = b_n z^n$

$$\rightarrow \quad w_n = c_n z^n$$

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}$$

Suppose U converges

and V converges absolutely
then W converges

Abel summation
(Partial)

$$U_n := \sum_{k=0}^{n} u_k$$

$$U_0 = u_0$$

$$U_n \to U \qquad , \; U_1 - U_0 = u_1$$
$$\vdots$$

$$u_0 v_0 + u_0 v_1 + u_1 v_0 + u_0 v_2 + u_1 v_1 + u_2 v_0 + \cdots$$

$$U_0 v_0 + U_0 v_1 + (U_1 - U_0) v_0 = U_0 v_1 + U_1 v_0$$

$$U_0 v_0 + U_0 v_1 + (U_1 - U_0) v_0 + (U_0 v_2 + (U_1 - U_0) v_1 + (U_2 - U_1) v_0$$

$$= U_0 v_2 + U_1 v_1 + U_2 v_0$$

$$W_n := \sum_{k=0}^{n} w_k = \sum_{k=0}^{n} U_{n-k} v_k$$

Define $U_n := 0$ if $n < 0$

converges absolutely $\longleftarrow$ $$= \sum_{k \geqslant 0} U_{n-k} v_k$$

want to show $W_n \to U \cdot V$

$$|W_n - UV| = \Big|\sum_{k \geqslant 0} (U_{n-k} - U) v_k\Big|$$

Pick K s.t.

$$\Big|\sum_{k \geqslant K} (U_{n-k} - U) v_k\Big| \leq 2M \sum_{k \geqslant K} |v_k| < \varepsilon$$

U convergent $\Rightarrow |U_k| \leq M$ all k.

$$|W_n - UV| \leq \sum_{k \geqslant 0}^{K-1} |U_{n-k} - U||v_k| + \varepsilon$$

Pick N s.t.

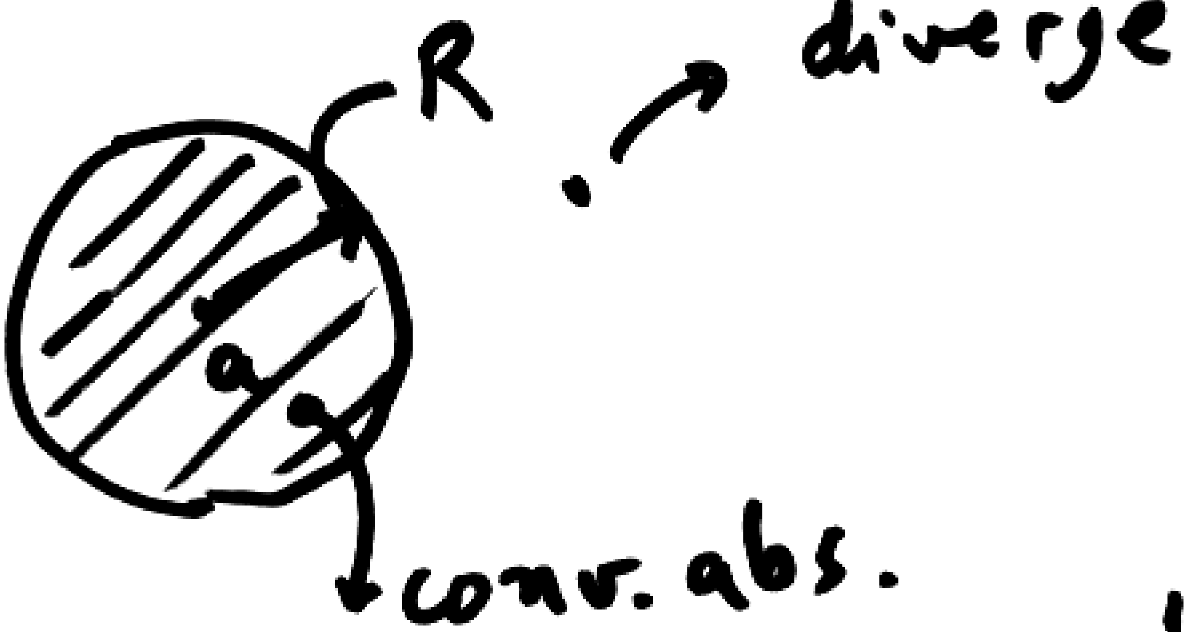
$$|U_n - U| < \varepsilon$$

all $n \geqslant N$.

Hence for all $n \geqslant N + K$

$$\leq \varepsilon \sum_{K \geqslant 0}^{K-1} |V_K| + \varepsilon \qquad \square$$

$$f(z) = \sum_{n \geqslant 0} a_n (z-a)^n$$



- $R^{-1} = \limsup_{n \to \infty} |a_n|^{1/n}$

If $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists

it equals R.

Example $$\sum_{n\geq 0} \frac{z^n}{n!} =: e^z$$
$$(\exp(z))$$

$R = \infty$ $\qquad \frac{|a_n|}{|a_{n+1}|} = n+1 \to \infty$

$R > 0$

Prop
1) analytic in $D(a, R)$
2) infinitely diff. term by term
3) $\frac{f^{(n)}(a)}{n!} = a_n$

Pf. $a = 0$

$$g_1(z) = \sum_{n\geq 0} n\, a_n z^n$$
$$(z f'(z))$$

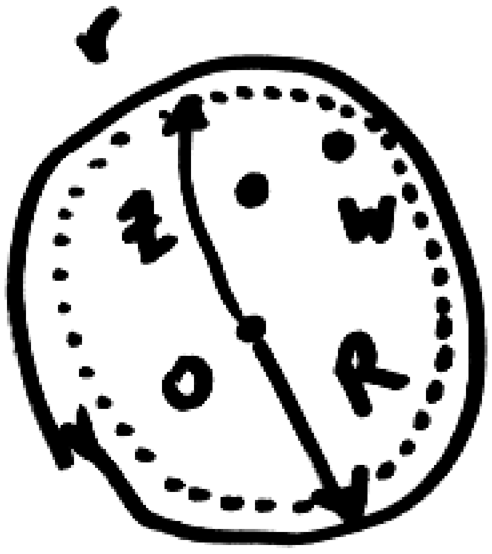
$\sum_{n\geq 0} n!\, z^n$
has $R = 0$
$\downarrow$
asymptotic expansions

$$\frac{f(z)-f(w)}{z-w}-g(z)=\left[\frac{f_n(z)-f_n(w)}{z-w}-f_n'(z)\right]$$
$$+\left(f_n'(z)-g(z)\right)$$
$$+\left|\frac{R_n(z)-R_n(w)}{z-w}\right]$$

~~[illegible]~~

last term

$$\sum_{k\geqslant n} a_k\left(\frac{z^k-w^k}{z-w}\right)$$



$$\frac{z^k-w^k}{z-w}=z^{k-1}+z^{k-2}w$$
$$+z^{k-3}w^2$$
$$+\dots+w^{k-1}$$

$$|\dots|\leq k\,r^{k-1}$$

What's the radius of convergence of g_1 ?

$$\limsup_{n\to\infty} |n a_n|^{1/n}$$

$$n^{1/n} |a_n|^{1/n}$$

$$\left(\lim_{n\to\infty} n^{1/n} = 1 \right)$$

$$g(z) = \sum_{n\geq 0} n a_n z^{n-1}$$

check also has radius of convergence R

Claim $f'(z) = g(z)$

Pf

$$\frac{f(z) - f(w)}{z - w} - g(z)$$

$$f = f_n + R_n$$

$$= \sum_{k=0}^{n-1} a_k z^k + \sum_{k\geq n} a_k z^k$$

$$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| \leq \sum_{k \geq n} |a_k| \cdot k\, r^{k-1}$$

$\sum_{k \geq 0} a_k k z^{k-1}$ is absolutely convergent on $D(a, R)$

Make two last terms $< \varepsilon/3$ for all $n \geq N$.

Pick such n.

Now for $|z - w| < \delta$ we have first term $< \varepsilon/3$ $\square$

Proves claim: $f' = g$

$\Rightarrow$ f inf. diff. term by term

$f^{(n)}(0) = a_n \cdot n!$

Feb 3, 2006

Hmwk

chap 2

3.2 # 2 p. 44

3.4 # 6, 8, 10 p. 47

chap 3

2.2 # 1, 2 p. 72

3.1 # 1, 4 p. 78

3.2 # 1, 4 p. 80

$$U = \sum_{n \geq 0} u_n \quad , V = \sum_{n \geq 0} v_n$$

THM If U, V converge absolutely then so does W <u>and</u> $W = U \cdot V$

$$W = \sum_{n \geq 0} w_n$$

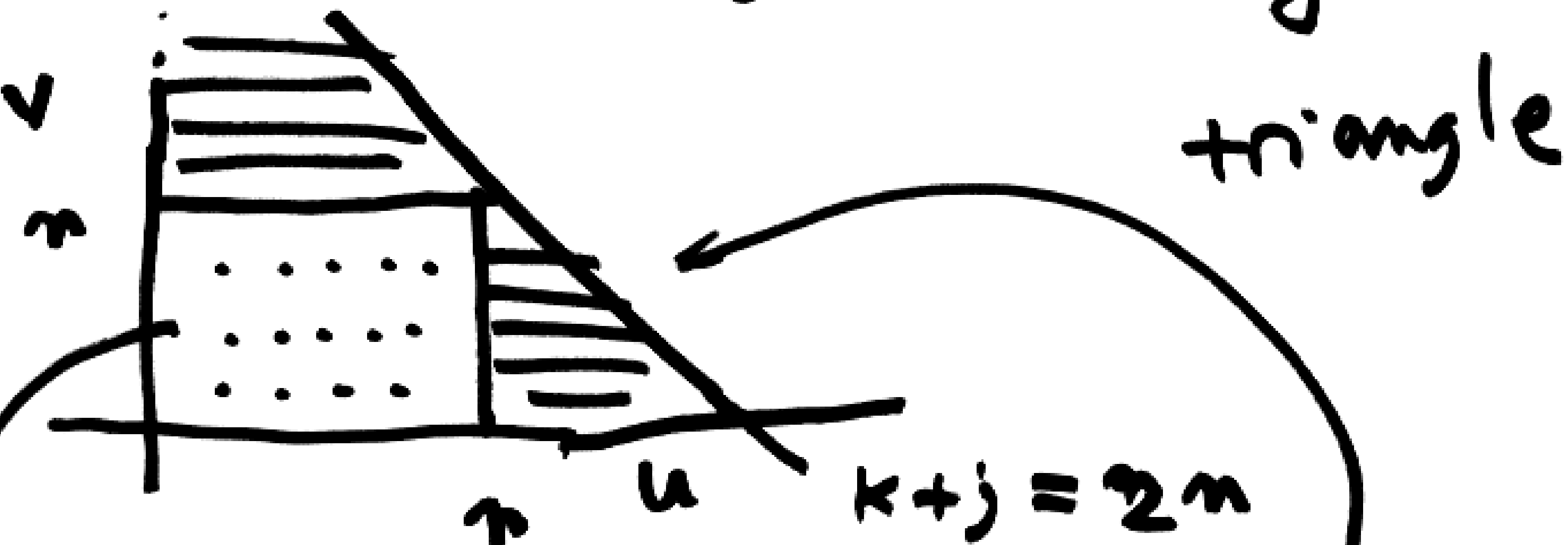
$$w_n : \sum_{k=0}^{n} u_k \cdot v_{n-k}$$

<u>Pf</u>

$$W_n := \sum_{k=0}^{n} |w_k| \leq \sum_{k,j\geq 0} |u_k||v_j|$$

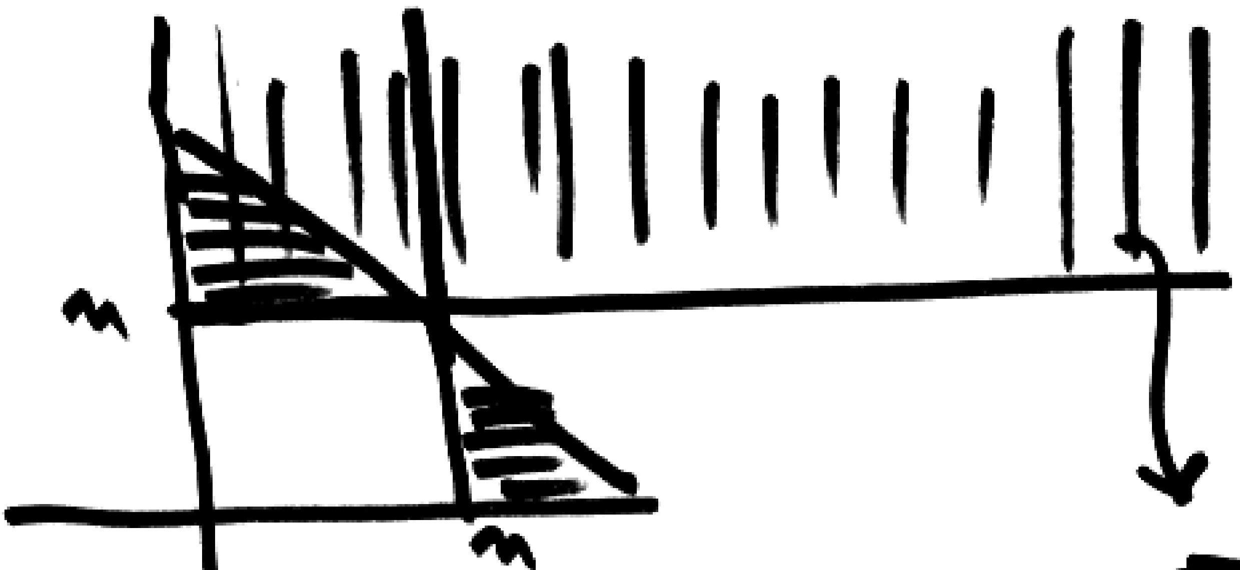
$$= \sum_{k\geq 0} |u_k| \sum_{j\geq 0} |v_j|$$

W_n is non-dereasing bounded sequence of real numbers so converges.

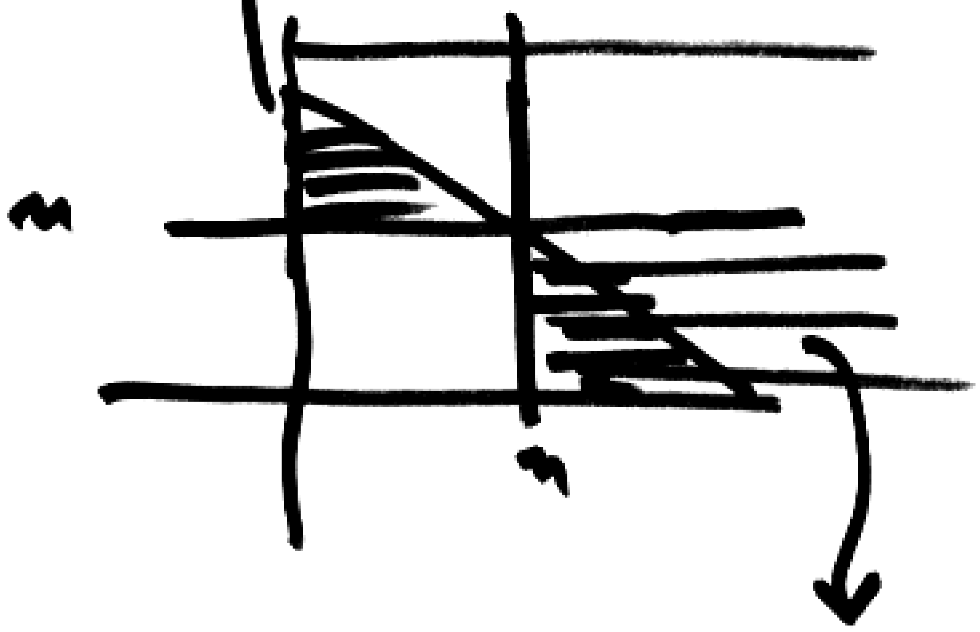


$$\sum_{r=0}^{2n} w_r = \sum_{0\leq k+j\leq 2n} u_k v_j$$

$$\sum_{k=0}^{n} u_k \cdot \sum_{j=0}^{n} v_j$$



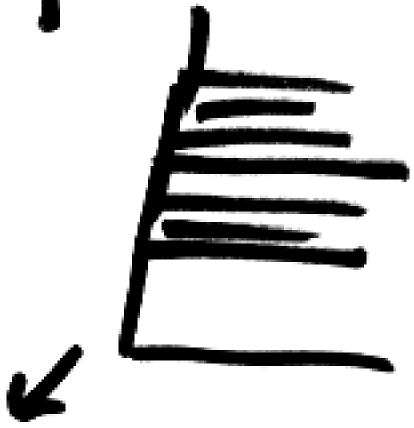
$$\sum_{k\geq 0}|u_k|\cdot\sum_{j\geq n}|v_j|$$



$$\sum_{k\geq n}|u_k|\cdot\sum_{j\geq 0}|v_j|$$

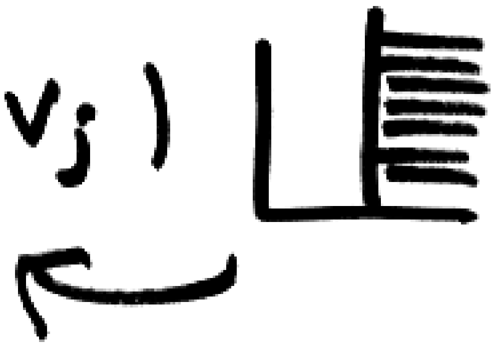
$$\left|\sum_{r=0}^{2n} w_r - \sum_{k=0}^{n} u_k \sum_{j=0}^{n} v_j\right|$$

↑ big △ ↑ □



$$\leq \sum_{k\geq 0}|u_k|\sum_{j\geq n}|v_j|$$

$$+\sum_{k\geq n}|u_k|\sum_{j\geq 0}|v_j|$$



⇝ □

$f = \sum_{n \geq 0} a_n z^n$, $g = \sum_{n \geq 0} b_n z^n$

radius R_1 , R_2

$f \cdot g$ has radius at least
$\min \{R_1, R_2\}$

$$\frac{(z-1)}{(z-2)} \cdot \frac{(z-2)}{(z-1)} = 1$$

geometric series $a \neq 0$

$$\frac{1}{1-az} = \sum_{n \geq 0} a^n z^n$$

has radius of convergence

$$R = \frac{1}{|a|}$$

$R = 2$ $R = 1$

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}$$

entire function.

$$e^z \cdot e^w = e^{z+w}$$

$$\sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$

$$= \left(\sum_{k=0}^{n} \frac{n!\, z^k w^{n-k}}{k!\,(n-k)!} \right) \frac{\text{[redacted]}}{n!}$$

$$= \frac{(z+w)^n}{n!}$$

$$\boxed{e^z \cdot e^w = e^{z+w}.}$$

$$1 = e^z \cdot e^{-z}$$

→ e^z is <u>never</u> 0.

$$e^{-z} = \frac{1}{e^z}$$

$$\lim_{z \to \infty} e^z = \text{does not exist}$$

$$\overline{e^z} = e^{\bar{z}}$$

$$\Rightarrow \quad |e^z|^2 = e^{z + \bar{z}}$$

$$\Rightarrow \quad |e^z| = e^{Re(z)}$$

$$\Rightarrow \quad z = \theta \in \mathbb{R}$$

$$Re(i\theta) = 0$$

$$|e^{i\theta}| = 1$$

Define

$$\cos z := \frac{1}{2}\left(e^{iz}+e^{-iz}\right)$$

$$\sin z := \frac{1}{2i}\left(e^{iz}-e^{-iz}\right)$$

$$e^{iz} = \cos z + i\sin z$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$\cos\theta$, $\sin\theta$ are the usual functions.

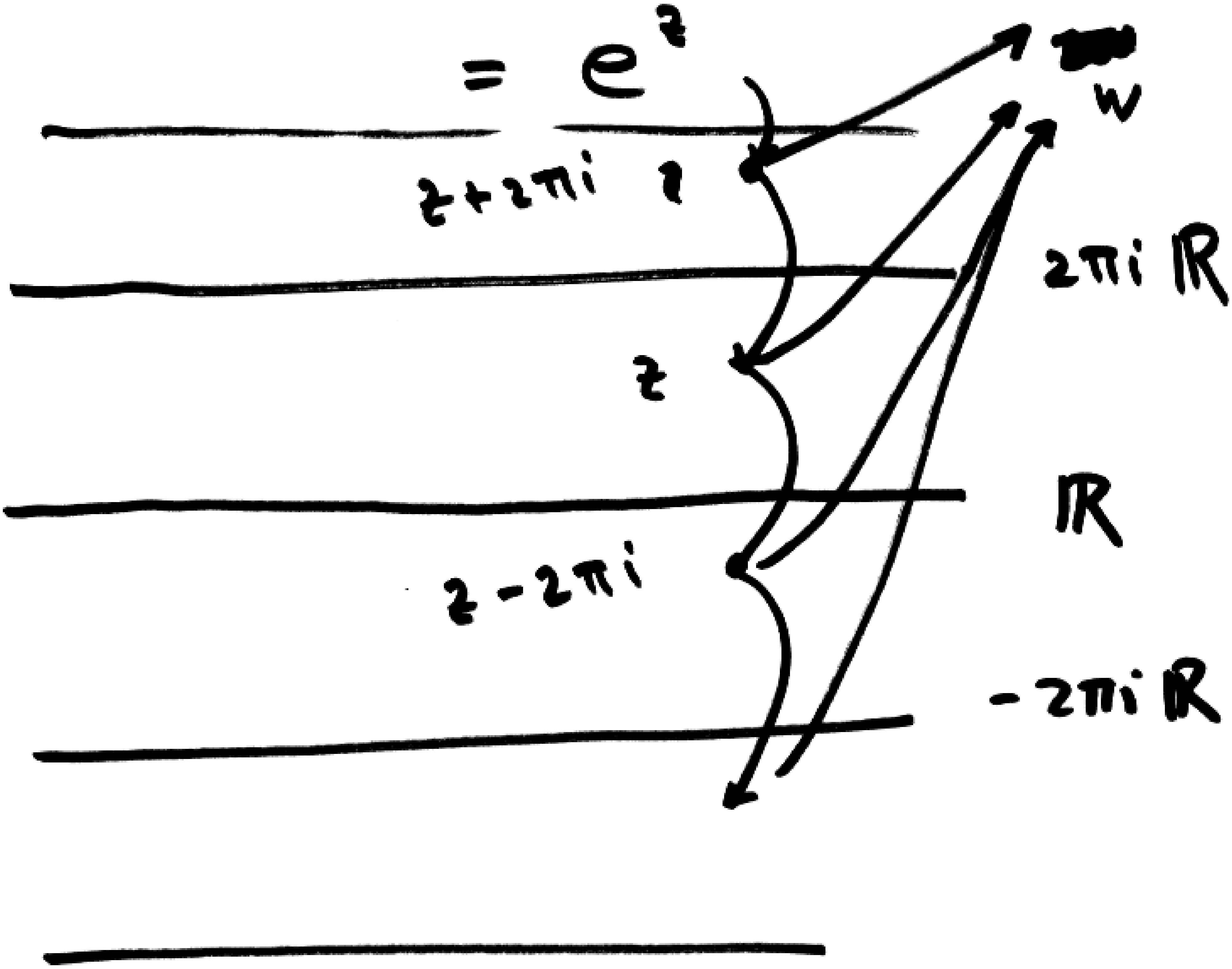
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\boxed{e^{2\pi i} = 1}$$

$$\boxed{e^{\pi i} + 1 = 0}$$

$\Rightarrow$ e^{z} is periodic

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i}$$
$$= e^z$$



Logarithm

$\boxed{w \neq 0}$

say $z = \log w$

iff $e^z = w$

"multivalued function."

$$z = x + iy$$

$$w = e^z = e^x \cdot e^{iy}$$

$$|w| = |e^z| = e^x$$

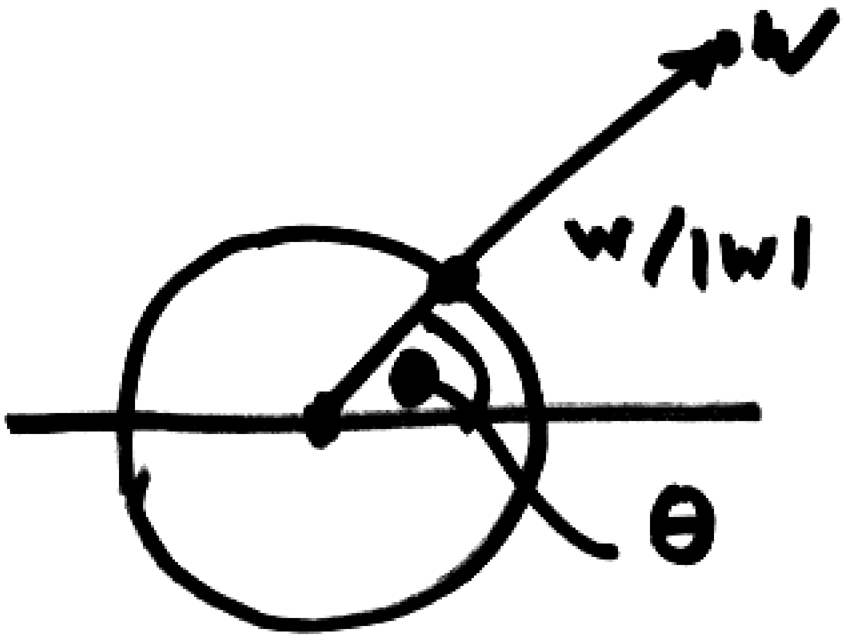
$$x = \log|w|$$

perfectly well defined

$$w = re^{i\theta} \qquad 0 \le \theta < 2\pi$$

$$y = \theta + 2\pi n$$

$$n \in \mathbb{Z}$$



$$\theta = \arg w$$

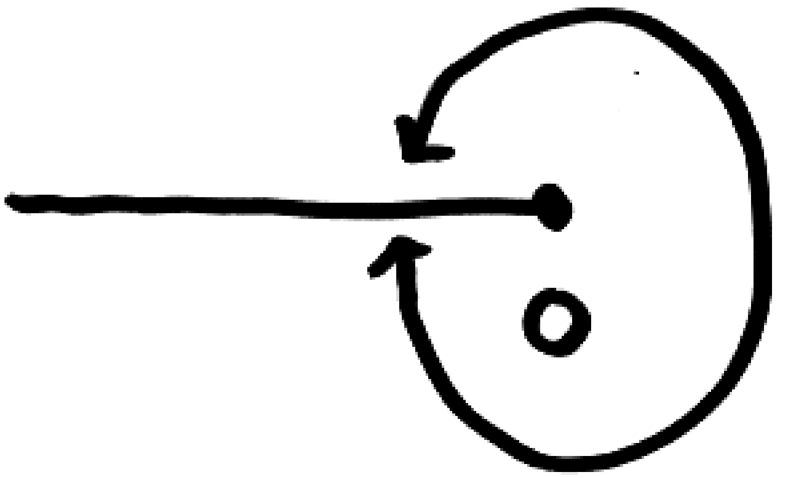
well defined modulo $2\pi\mathbb{Z}$

$$\log w = \log|w| + i \arg w$$

$$\log : \mathbb{C}^{\times} \longrightarrow \mathbb{C}/2\pi i\mathbb{Z}$$

$$\arg : \mathbb{C}^{\times} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

To fix multivaluedness



$$\mathbb{C} \setminus [0,-\infty) \longrightarrow \mathbb{C}$$

$$w \longmapsto \log w$$

$$= \log|w| + i\theta$$

$$-\pi < \theta < \pi$$

$$a^b = \exp(b \log a)$$

$$a \neq 0.$$

$k = b \in \mathbb{Z}$

$$a^k = \underbrace{a \cdot \ldots \cdot a}_{k} \qquad k > 0$$

$$a^k = (\underbrace{a^{-1} \cdots a^{-1}}_{|k|}) \qquad k < 0.$$

any $a \in \mathbb{C}$.

Feb 6, 2006

$a \neq 0$

$a^b := \exp(b . \log a)$

$\log a + 2\pi i n, \quad n \in \mathbb{Z}$

$$\exp(b \log a + b 2\pi i n)$$

$$= a^b \cdot e^{2\pi i n b}$$

- $b \in \mathbb{Z} \rightsquigarrow a^b$ unique value.
- $b = p/q \quad q > 0, \ \gcd(p,q) = 1$

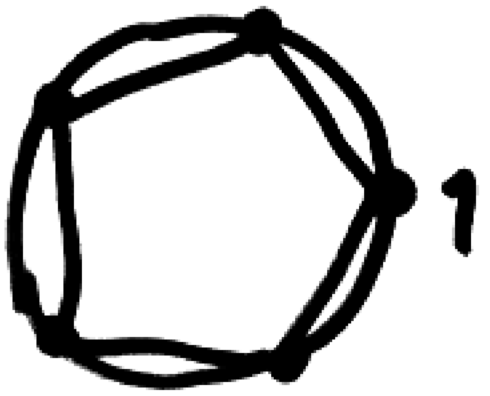
q different values for

$$e^{2\pi i \frac{np}{q}} \qquad n_1 = n \bmod q$$

$$\Rightarrow \quad '' \; e^{2\pi i \frac{n_1 p}{q}}$$

$e^{\frac{2\pi i n}{q}}$ $n = 0, 1, \ldots, q-1$ ②

$q = 5$



q^{th}-roots of unity.

$$a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$$

- otherw. will have infinitely many choices for a^b

If $a \in \mathbb{R}_{>0}$ then $\log a$ we naturally <u>pick</u> to be real.

$$a^b = \exp(b \log a)$$

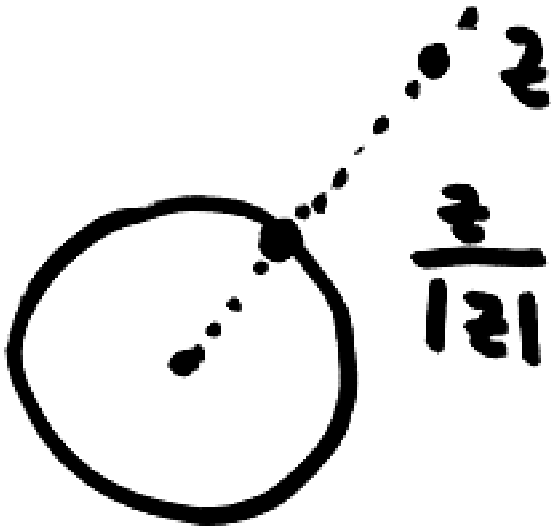
- $z \mapsto |z|$ is continuous

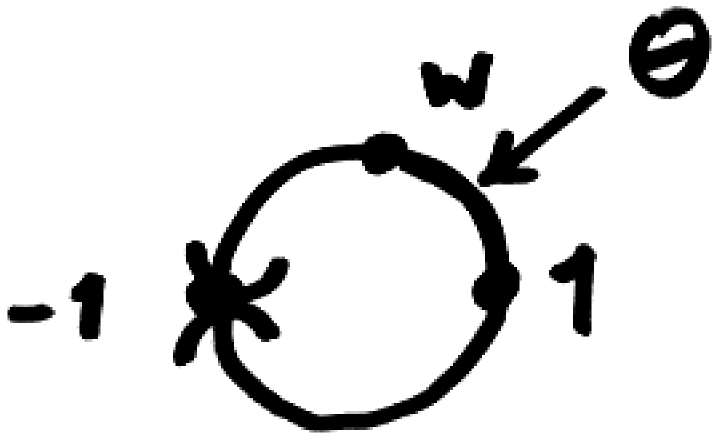
$$\big|\,|w|-|z|\,\big| \le |w-z|$$

- $z \in \mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$

$$z \longmapsto \frac{z}{|z|}$$

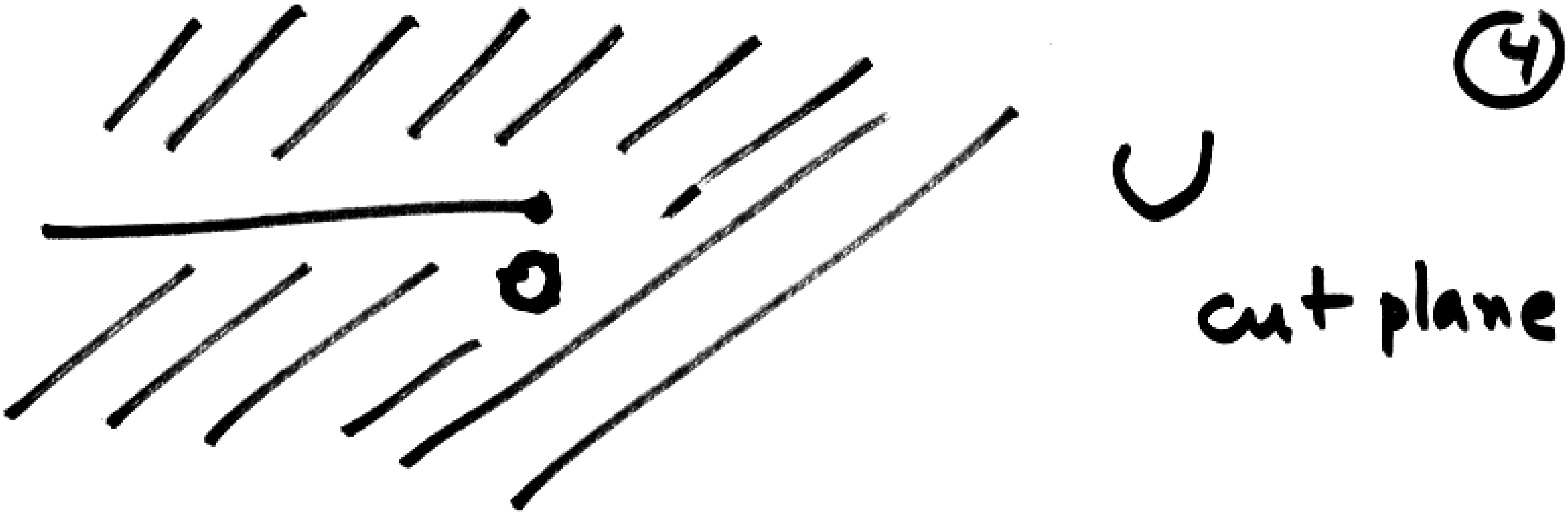
is continuous



- 

θ is a continuous function of w.

- $z \longmapsto \theta$ is continuous



$$\begin{aligned} U &\longrightarrow (-\pi, \pi) \\ z &\longmapsto \theta \end{aligned}$$

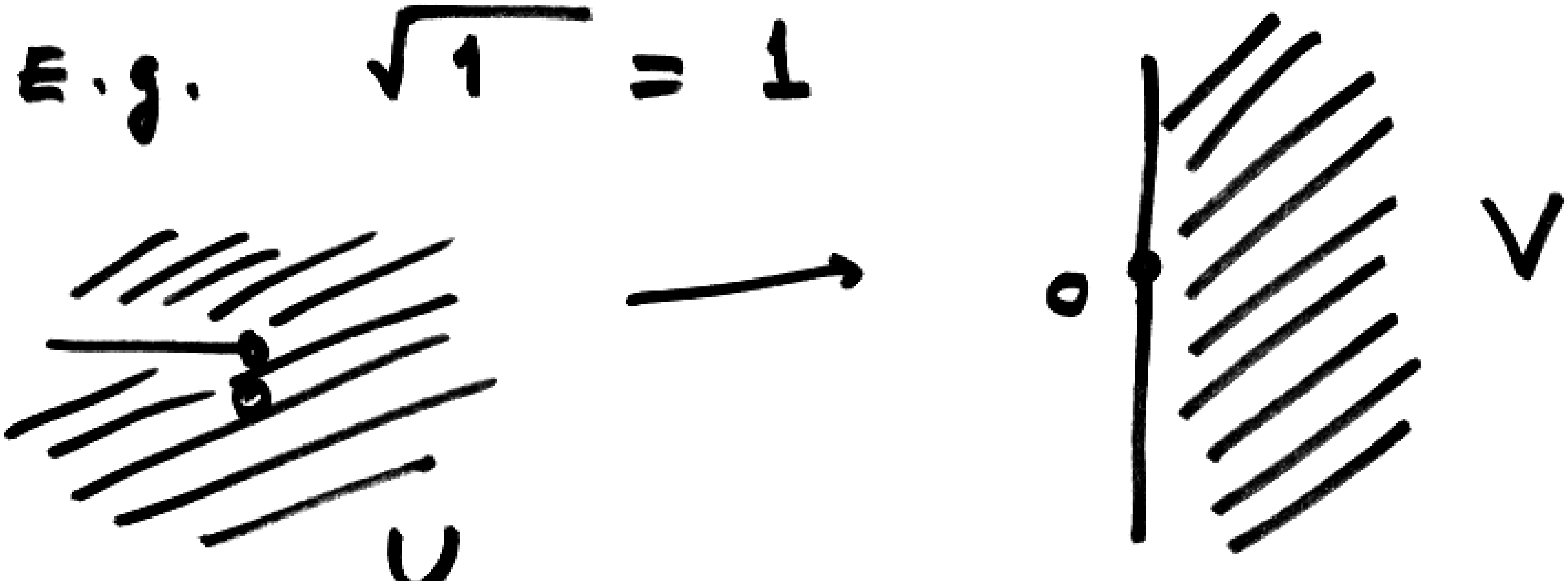
is continuous.

Define on U

$$\sqrt{z} := |z|^{1/2} e^{i\frac{\theta}{2}}$$

Is continuous.

E.g. $\sqrt{1} = 1$



$Re(z) > 0$

THM $f : \overset{\mathbb{C}}{\overset{\cup\!|}{U}} \longrightarrow \overset{\mathbb{C}}{\overset{\cup\!|}{V}}$

$g : V \longrightarrow U$

- $g \circ f(z) = z \, , \quad z \in U$
- f continuous
- g analytic, $g'(z) \neq 0$ on V

$\Rightarrow$ f is analytic &

$$f'(z) = \frac{1}{g'(f(z))}$$

Apply this $\quad g(w) = w^2$

$\qquad f(z) = \sqrt{z}$

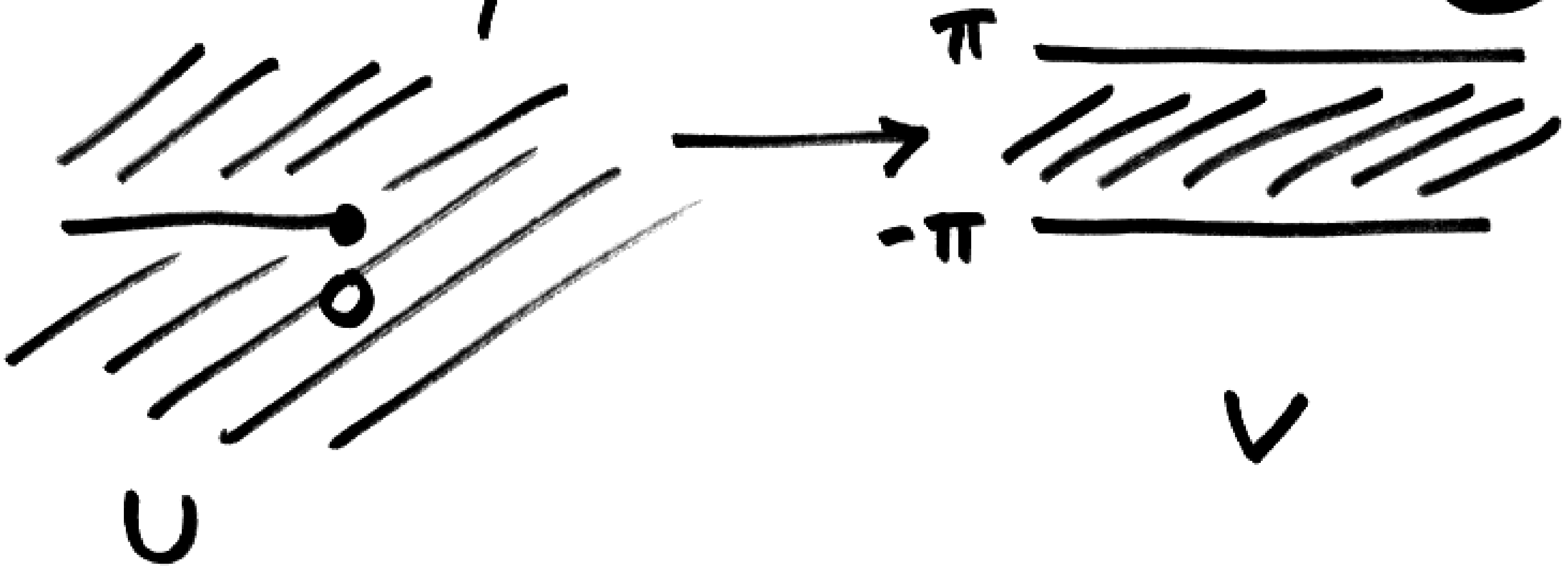
$g \circ f(z) = z, \quad z \in U$

$g'(w) = 2w$ not zero on V

$\Rightarrow$ f is analytic &

$$f'(z) = \frac{1}{2\sqrt{z}}$$

Similarly



$$f(z) = \log z := \log|z| + [illegible]\, i\theta$$

continuous

$$g(w) = e^{w}$$

$$g \circ f(z) = z \qquad z \in U$$

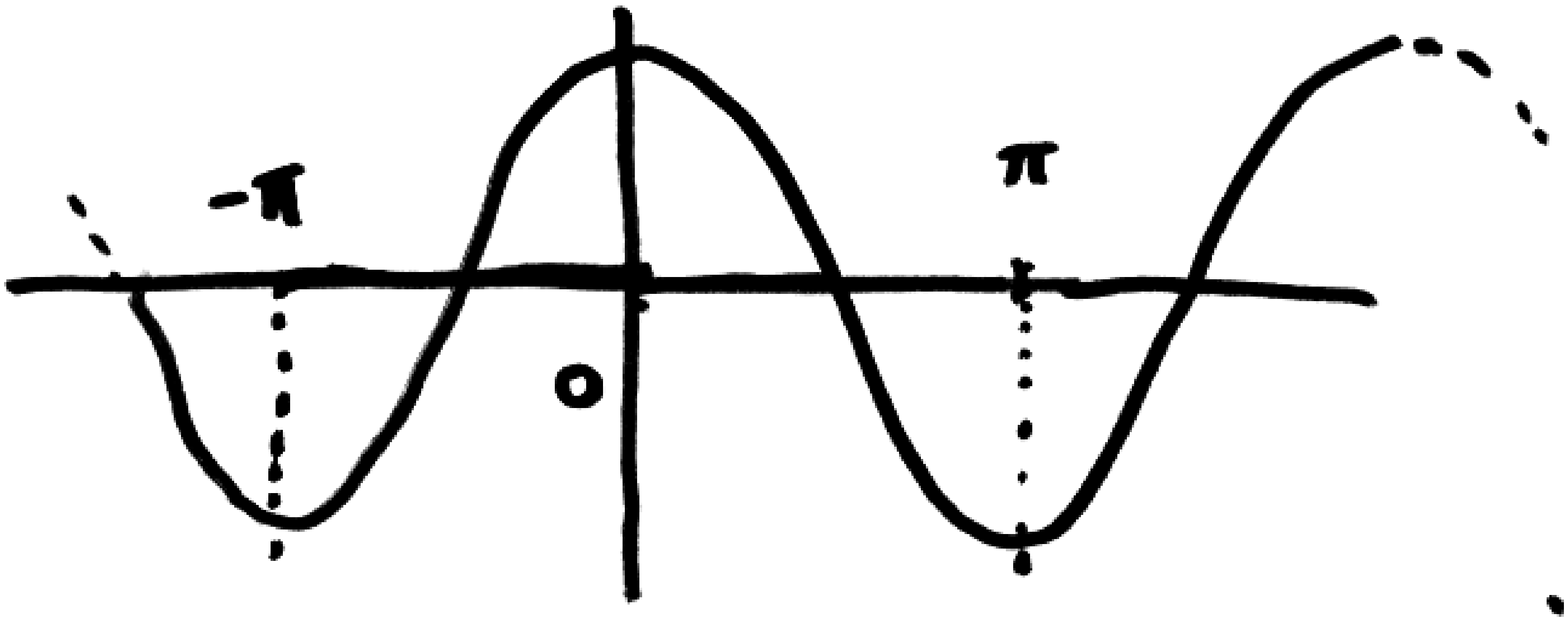
$\Rightarrow$ f is analytic in U &

$$f'(z) = \frac{1}{z}$$

$\log 1 = 0 \leftarrow$ pins down branch

<u>principal branch of log</u>

Define arccos z



$$w = \cos z := \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$u = e^{iz} \qquad = \frac{1}{2}\left(u + \frac{1}{u}\right)$$

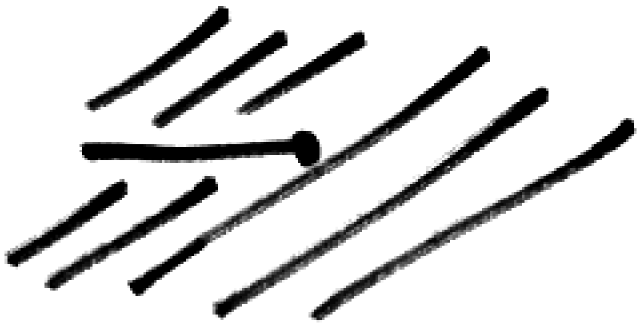
$$u^2 - 2wu + 1 = 0$$

$$e^{iz} = u = w \pm \sqrt{w^2 - 1}$$

$$iz = \log\left(w \pm \sqrt{w^2 - 1}\right)$$

$$z = \pm i \log\left(w + \sqrt{w^2 - 1}\right)$$

First define: $\sqrt{1-z^2}$

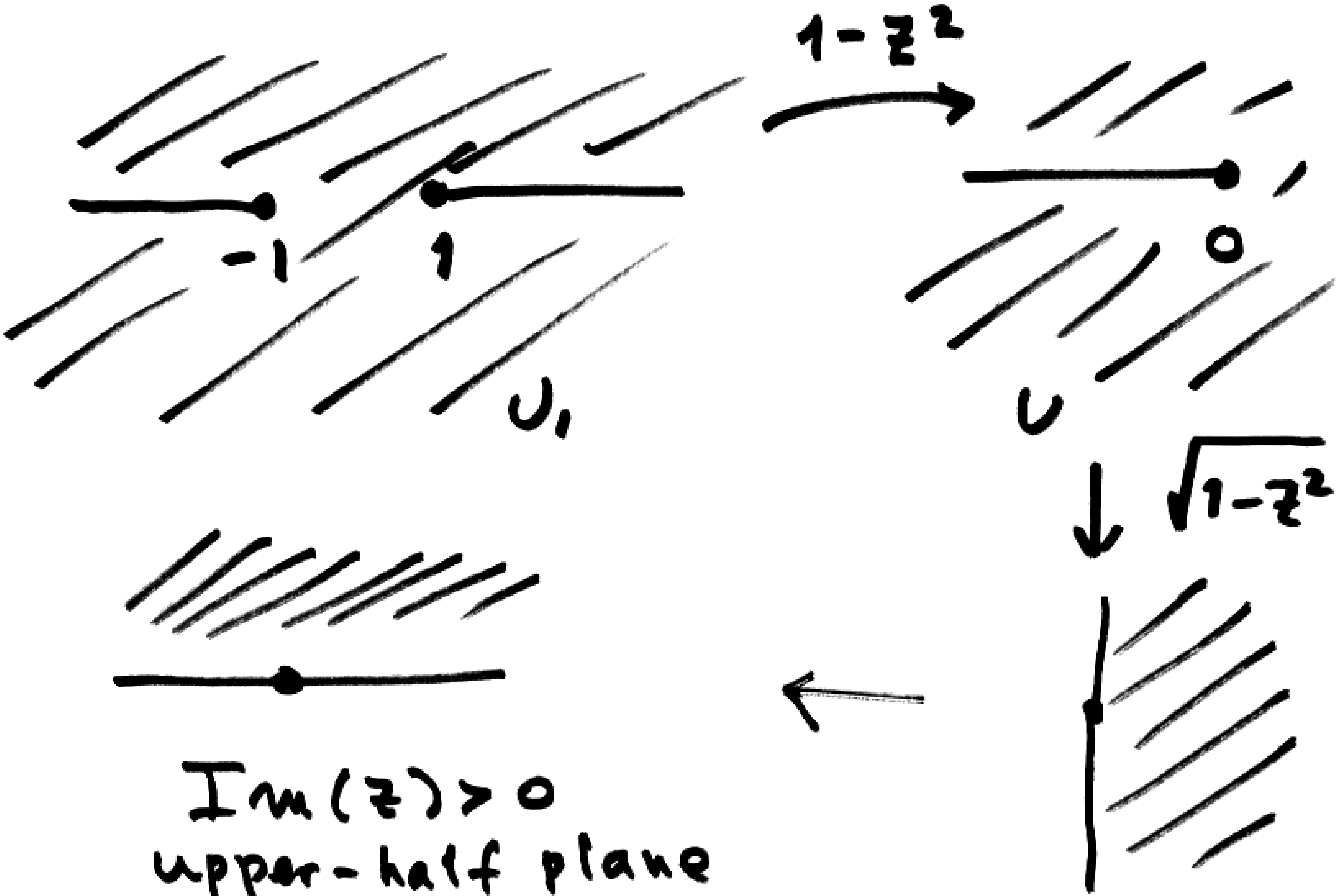
$1-z^2 \in$  $\cup$

$$1-z^2 = -t \quad , \quad t \geq 0$$

$$z^2 = 1+t$$

$$\to \quad z^2 \in [1,\infty)$$

$$\to \quad z \in (-\infty,-1] \cup [1,\infty)$$



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$$f : \overset{z_0}{\overset{\in}{U}} \longrightarrow V$$

Think of it as $\overset{\mathbb{R}^2}{\overset{\cup\!|}{U}} \longrightarrow \overset{\mathbb{R}^2}{\overset{\cup\!|}{V}}$

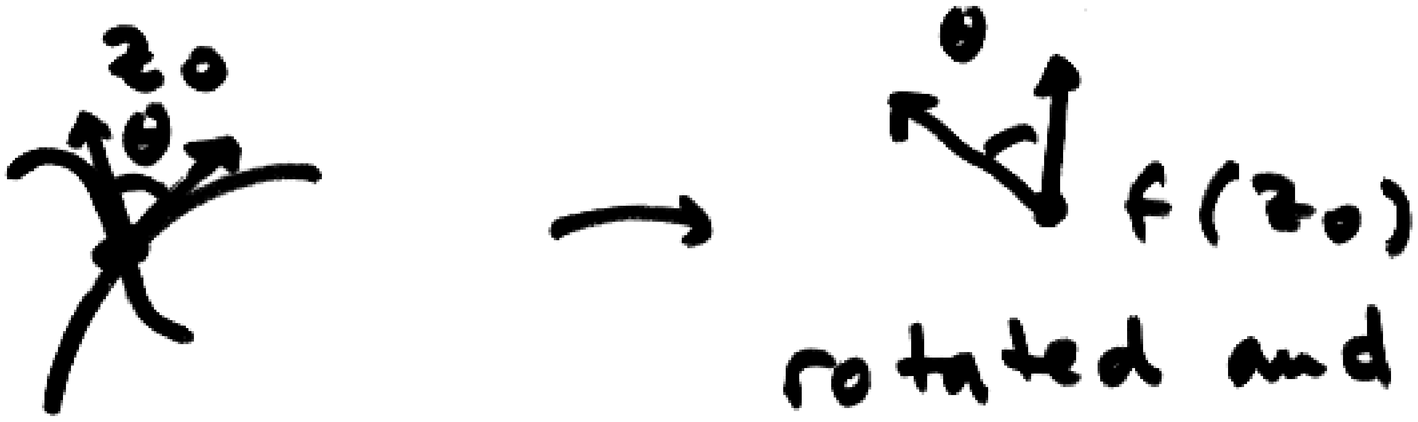
THM f analytic at z_0 $\Rightarrow$ f is differentiable at z_0 and its differential is multiplication by $f'(z_0)$

Differential is a linear map

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

sequence of a rotation followed by scaling by a positive real number.

If $f'(z_0) \neq 0$

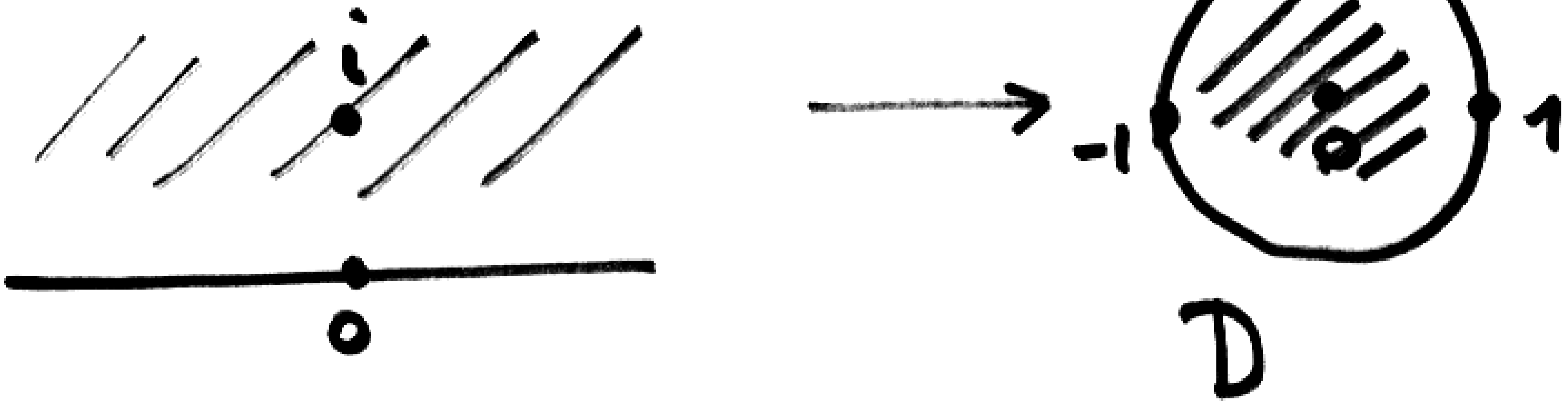


rotated and scaled
(multiplication by $f'(z_0)$)

Angle is locally preserved

conformal map.

$z \longmapsto \bar{z}$ preserve angles
but changes orientation
not conformal



$\mathfrak{h}$ = $\mathrm{Im}(z) > 0$
upper half plane

$\mathbb{D}$
$|z| < 1$

Möbius transformation

$$w = \frac{z - i}{z + i}$$

$$z \in \mathbb{R} \longrightarrow w \in S^1$$

$$\bar{w} = \frac{z + i}{z - i} = \frac{1}{w}$$

$$z = 0 \rightarrow w = -1$$

$$z = \infty \rightarrow w = 1$$

Möbius transformations take circles to circles.

("circles" includes lines)

$$S(z) = \frac{az + b}{cz + d}$$

$S(z) \in \mathbb{R}$?

$$\frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$

$$(a\bar{c} - \bar{a}c)|z|^2 + (a\bar{d} - \bar{b}c)z$$
$$+ (b\bar{c} - d\bar{a})\bar{z}$$
$$+ (\underbrace{b\bar{d} - \bar{b}d}_{it}) = 0 \qquad t \in \mathbb{R}$$

- $a\bar{c} - \bar{a}c = 0$

then equation is linear in x, y $(z = x+iy)$

z are in a line.

$u := a\bar{d} - \bar{b}c = r + si$

$u z - \bar{u}\bar{z} = it$

$ry - sx = t$

Need to check $u \neq 0$

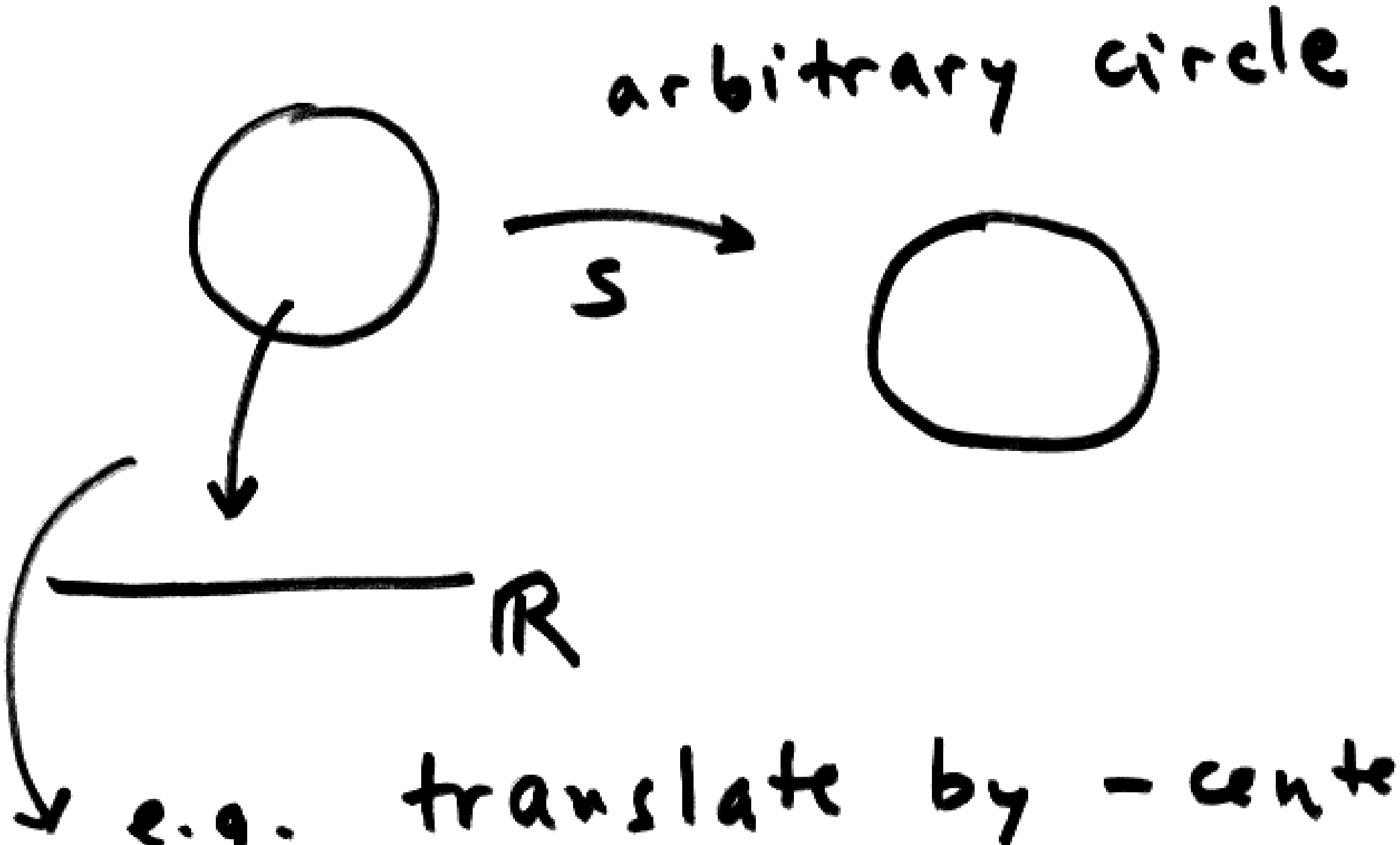
$ad - bc \neq 0 \Rightarrow u \neq 0$

- $a\bar{c} - \bar{a}c \neq 0$ divide out

$$|z|^2 - \gamma z - \bar{\gamma}\bar{z} + \delta = 0$$

$$|z - \gamma|^2 = \gamma\bar{\gamma} - \delta$$

$\Rightarrow$ <u>circle</u>. $\square$



e.g. translate by − center

scale

$$\begin{tikzcd} \text{Transf.} \arrow[r, squiggly] & S^1 \arrow[d, "\frac{z-i}{z+i}"] \\ & \mathbb{R} \end{tikzcd}$$

- $$Gl_2(\mathbb{C}) \longrightarrow Aut(\mathbb{P}^1)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$$

homomorphism

$$\begin{tikzcd}(z_0 : z_1) \arrow[d, mapsto] \\ (az_0+bz_1 : cz_0+dz_1)\end{tikzcd}$$

scalars in $Gl_2(\mathbb{C})$ act trivially

$$PGL_2(\mathbb{C}) := Gl_2(\mathbb{C})/_{\text{scalars}}$$

(this completely algebraic)

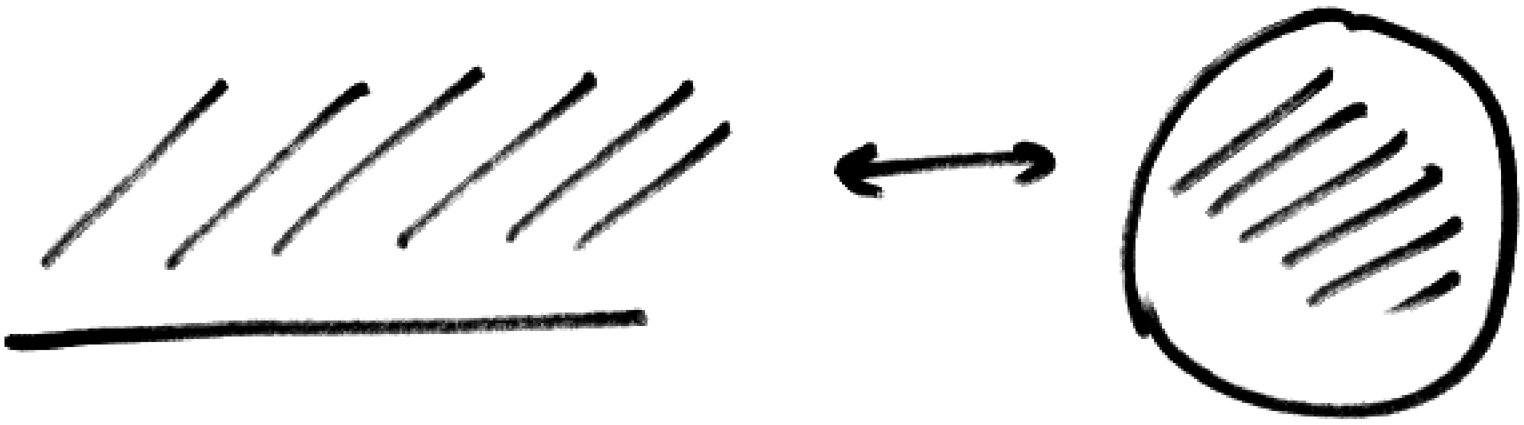
- Möbius transformation preserve cross-ratio
- cross-ratio real

$$\Updownarrow$$

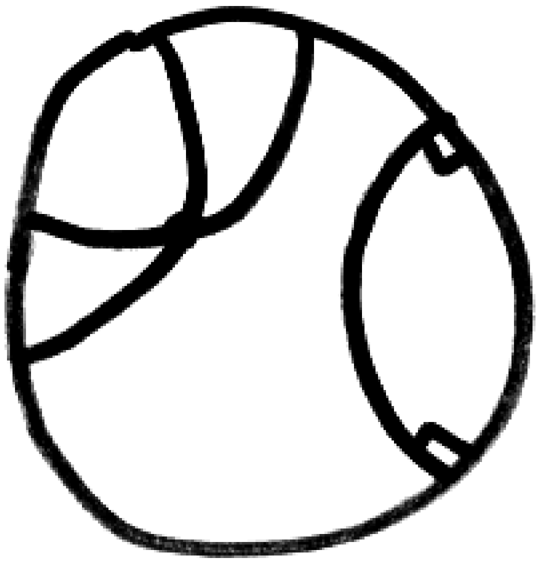
points on a circle

- $PGl_2(\mathbb{C})$

we can send any three distinct points to any other distinct three points in a unique way.



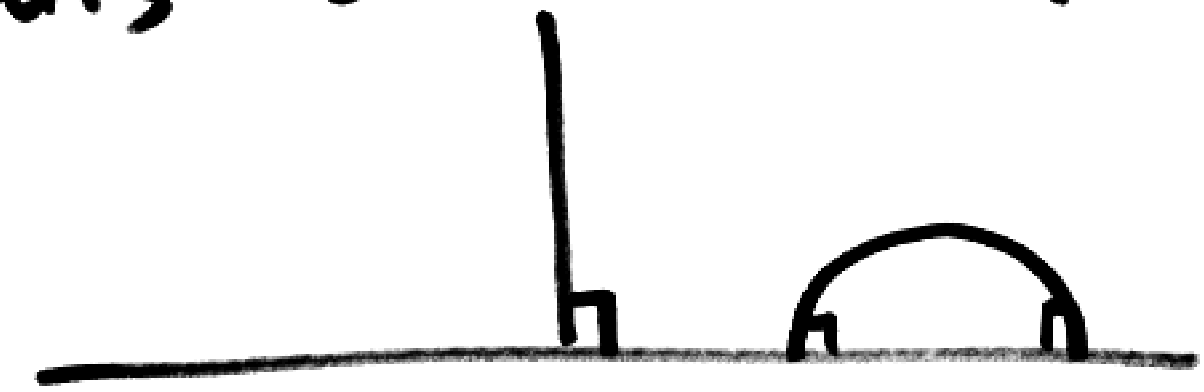
Hyperbolic geometry



- points in D
- lines are circles meet $\partial \bar{D} = S^1$ at right angle.

(Poincaré model)

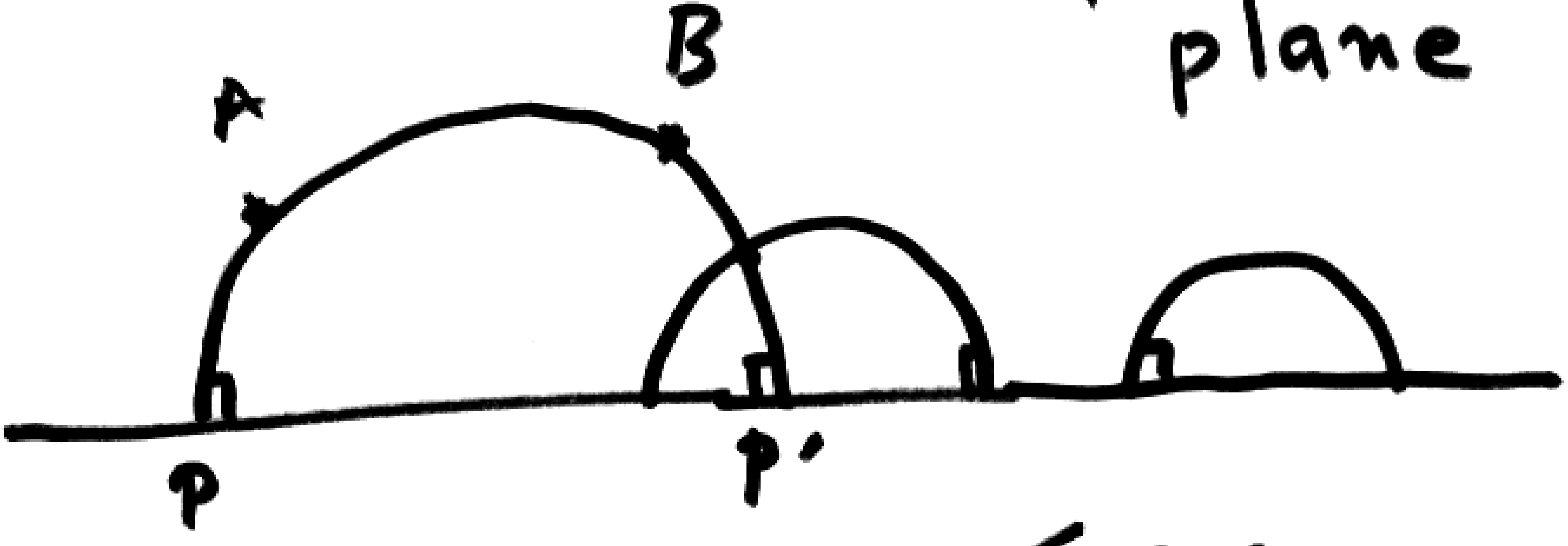
fails 5th Euclid postulate.

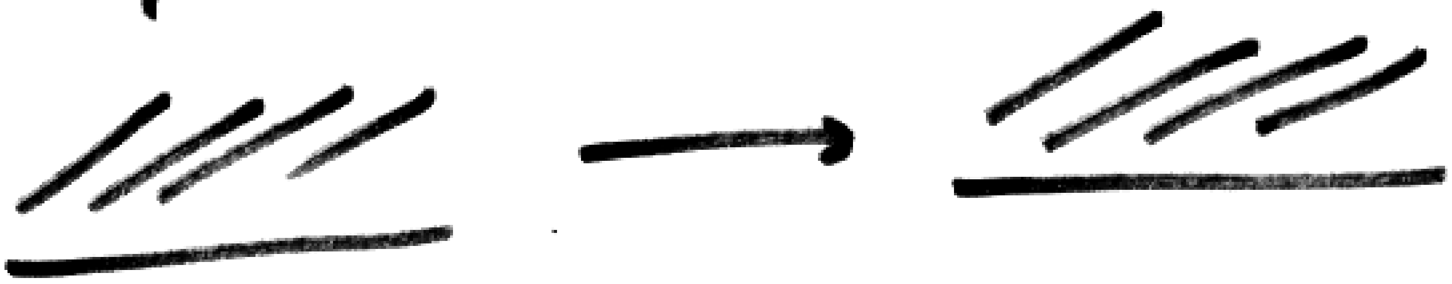


Feb 10, 2006

Poincaré

upper half plane





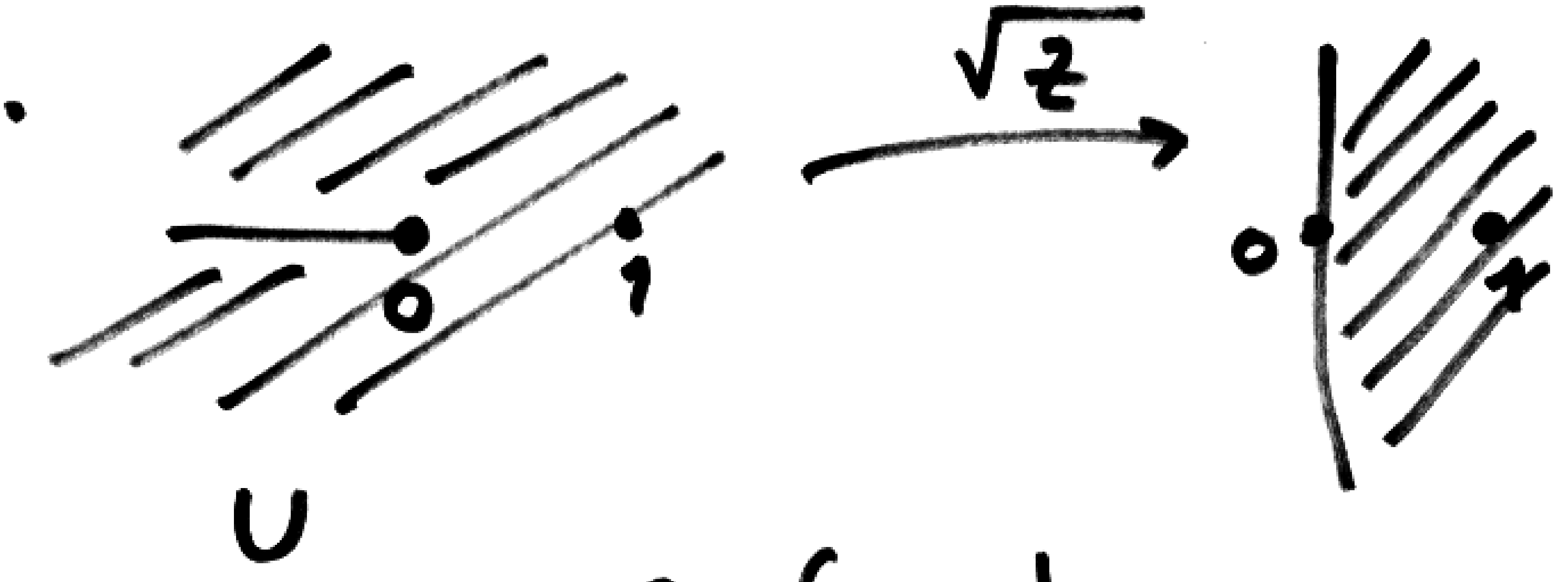
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad a, b, c, d \in \mathbb{R}$

$\det = ad - bc > 0$

$PSL_2(\mathbb{R})$

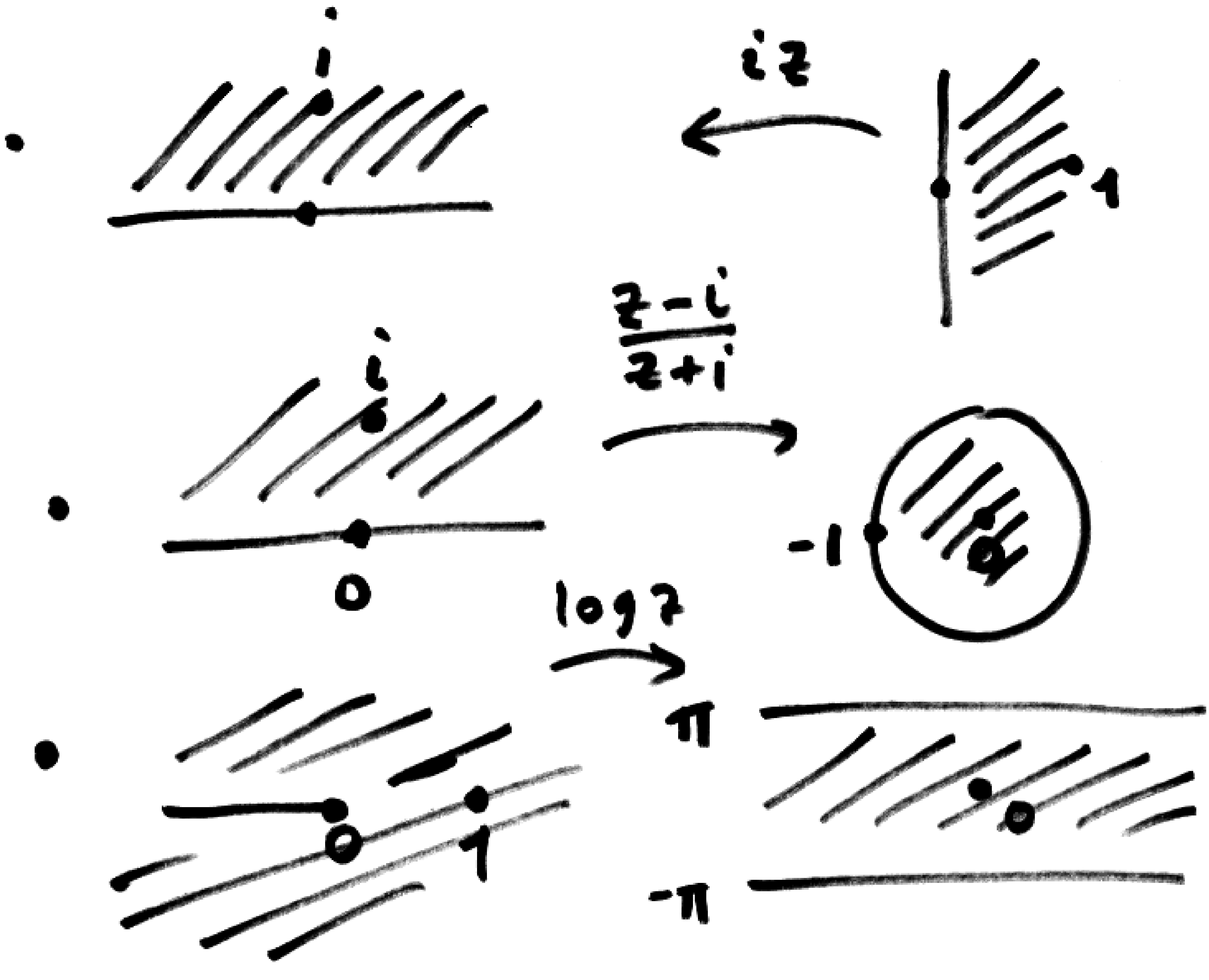
$d(A,B) := \log |(P, P', A, B)|$

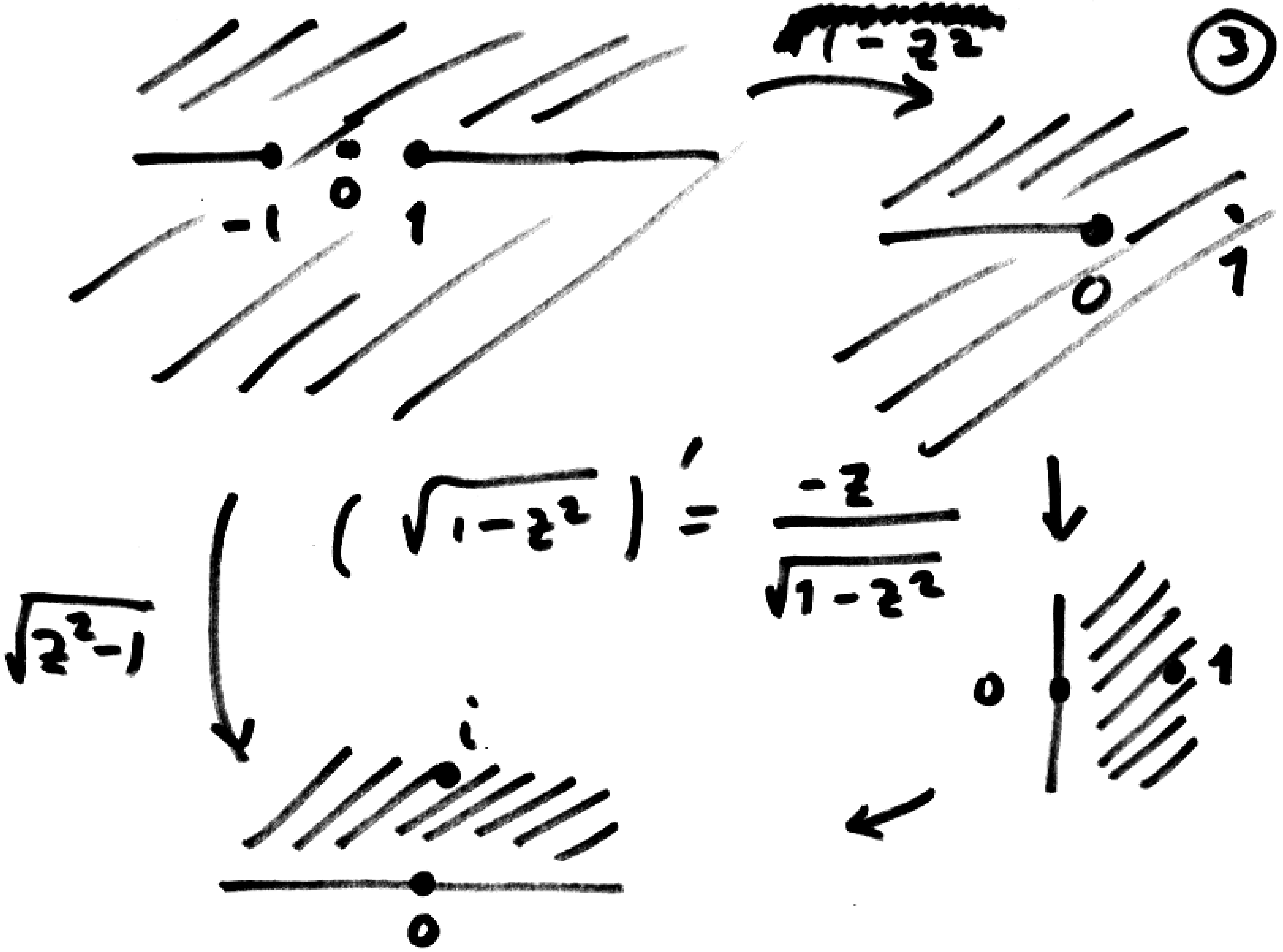
metric preserved by the group.



conformal

$$(\sqrt{z})' = \frac{1}{2\sqrt{z}}$$





$$\left(\sqrt{1-z^2}\right)' = \frac{-z}{\sqrt{1-z^2}}$$

$u = z + \sqrt{z^2-1}$, $u = e^{iw}$

$$u^2 - 2zu + 1 = 0$$

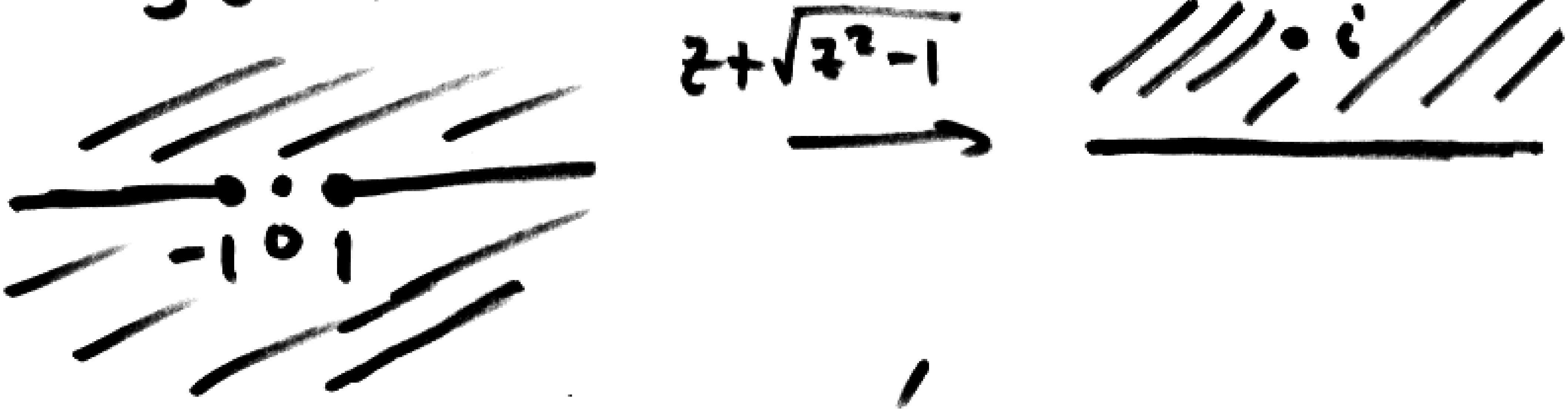
$$z = \frac{1}{2}\left(u + \frac{1}{u}\right)$$

what z's mapp to [redacted] $u \in \mathbb{R}$?

$$u \in \mathbb{R} \Rightarrow \frac{1}{2}\left(u + \frac{1}{u}\right) = z$$

$$\Leftrightarrow \quad |z| \geq 1$$

So this branch $z + \sqrt{z^2-1}$

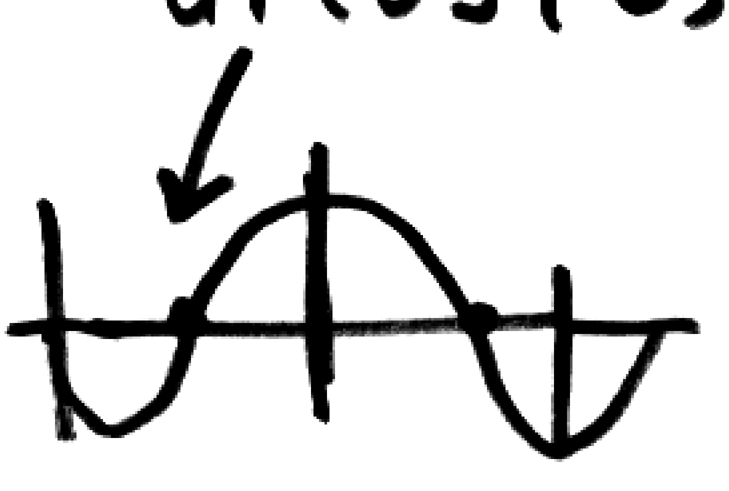


$$\left(z + \sqrt{z^2-1}\right)' = 1 + \frac{z}{\sqrt{z^2-1}}$$

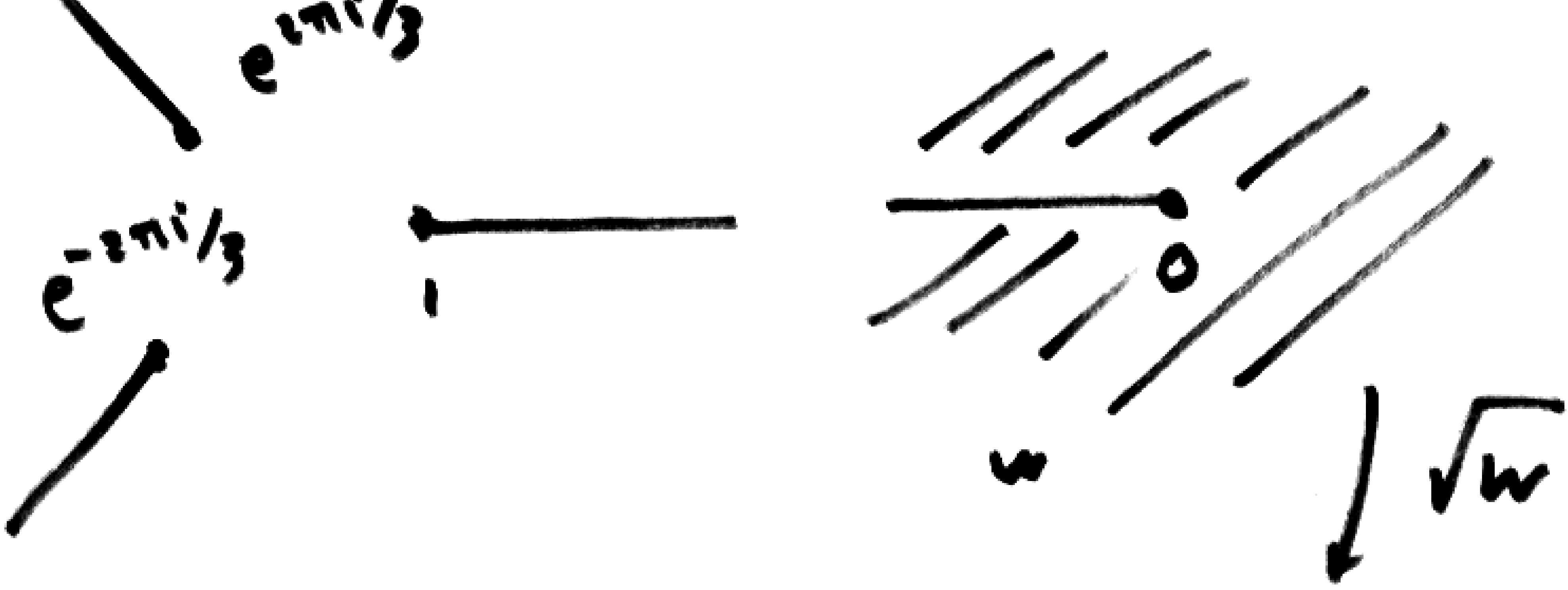
$$\operatorname{arcos} z = i \underbrace{\log\left(z + \sqrt{z^2-1}\right)}_{0 \leq \mathrm{Im} \leq \pi}$$

$$\operatorname{arcos}(0) = i \log i$$

$$= i \cdot i \frac{\pi}{2} = -\frac{\pi}{2}$$



$\sqrt{1 - z^3}$?



$1 - z^3 = w$

z's for which w is excluded?

$1 - z^3 \in (-\infty, 0]$

$1 - z^3 = -t \qquad t \geqslant 0$

$z^3 = 1 + t \geqslant 1$

—|||—

Path Integral

open set

$\gamma : [a,b] \longrightarrow U \subseteq \mathbb{C}$

piecewise smooth
continuos

$f : U \to \mathbb{C}$ continuous

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

__Claim__ This is independent of the parametrization.

$t : [\alpha, \beta] \longrightarrow [a, b]$

increasing piecewise smooth continuous

$\gamma \circ t$

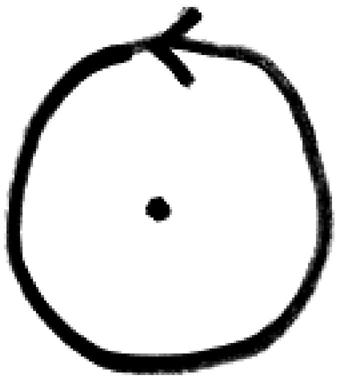
$$\int_{\gamma\circ t} f(z)\,dz = \int_{\gamma} f(z)\,dz$$

Examples

1) $f(z) = \dfrac{1}{Z}$

$\gamma(t) = Re^{it}$ $\quad R>0$

$\gamma : [0, 2\pi] \to \mathbb{C}$



$\gamma'(t) = R i e^{it}$

$$\int_{\gamma} f(z)\,dz = \int_0^{2\pi} \frac{1}{Re^{it}}\, iRe^{it}\,dt$$

$$= i\int_0^{2\pi} dt = 2\pi i$$

2) If $F : U \to \mathbb{C}$ analytic

$\underline{f = F'}$ on U

$$\int_\gamma f(z)\,dz = \int_a^b f\circ\gamma(t)\gamma'(t)\,dt$$

$$= \int_a^b (F\circ\gamma)'(t)\,dt$$

$$= F(\gamma(b)) - F(\gamma(a))$$

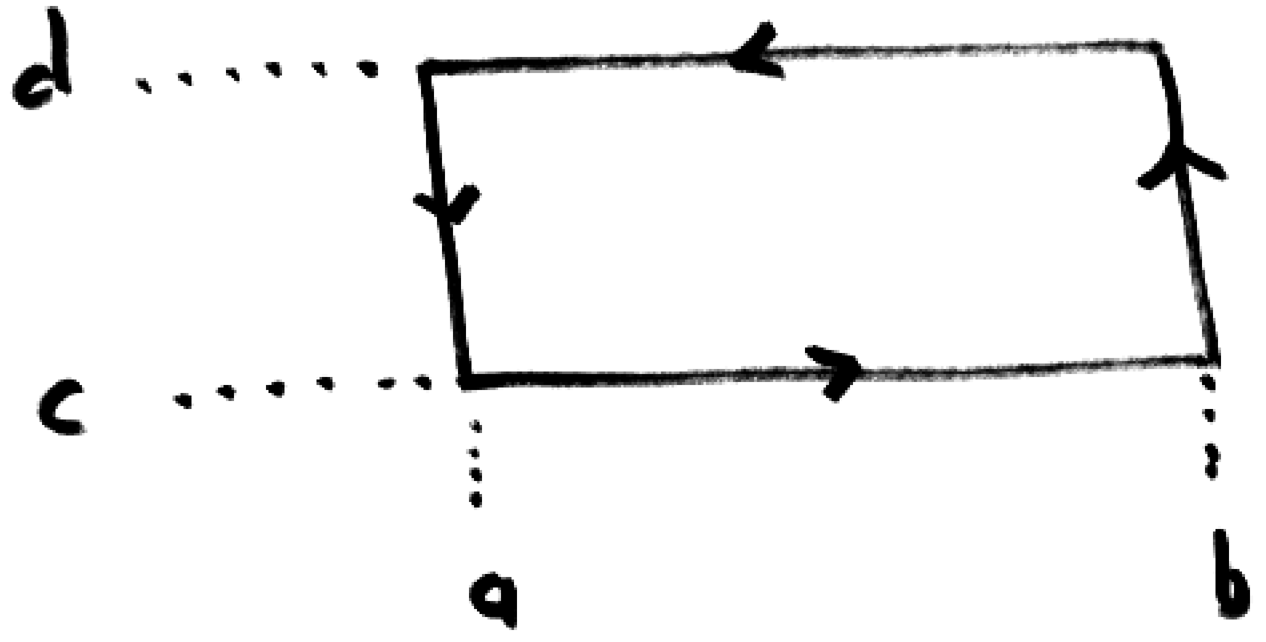
In particular $\int_\gamma f(z)\,dz = 0$

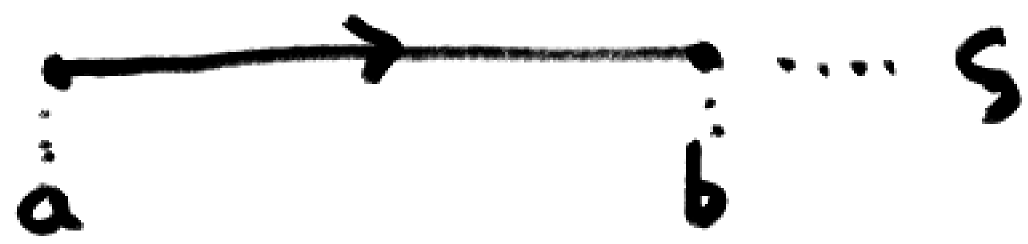
if $\gamma(a) = \gamma(b)$ i.e.

if γ is $\underline{\text{closed}}$.

3) $f(z) = \frac{1}{z}$

$a, b, c, d \neq 0$





$$\gamma : [a, b] \longrightarrow \mathbb{C}$$
$$t \longmapsto t + is$$

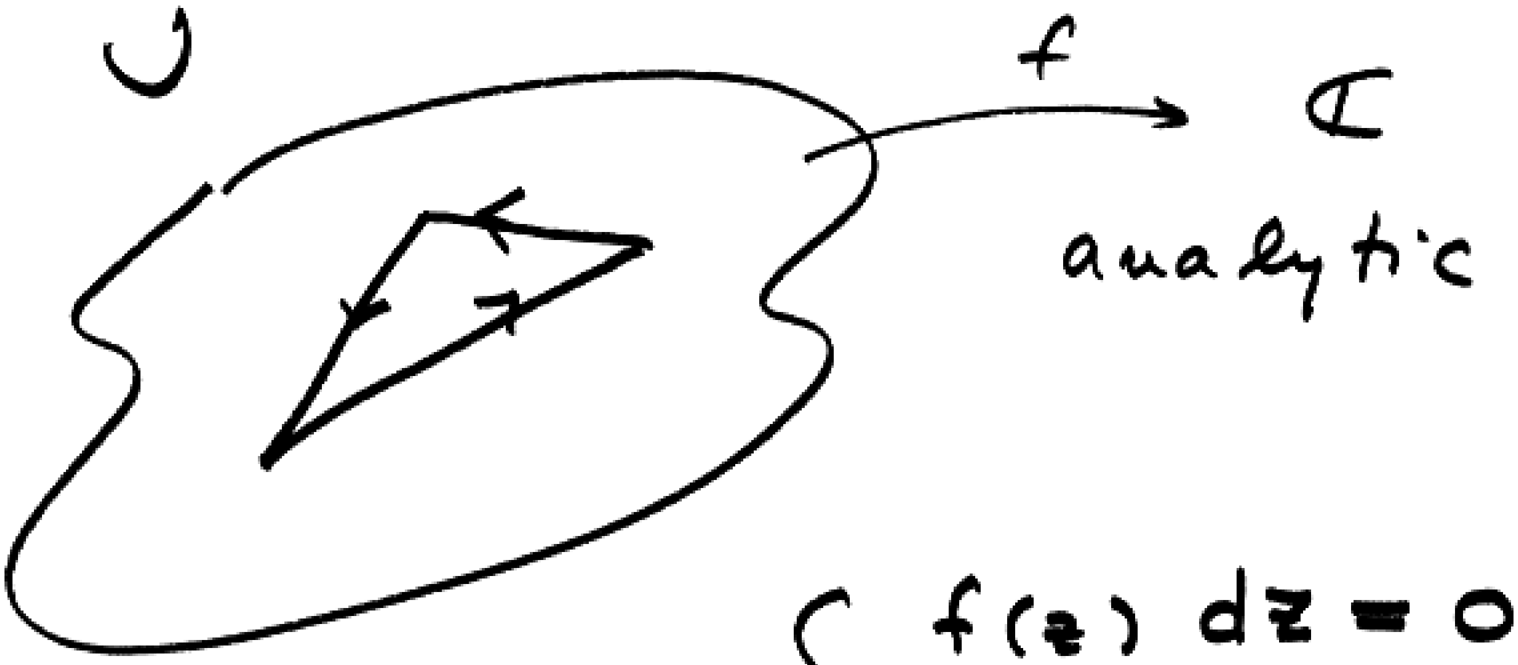
$$\gamma'(t) = dt$$

$$\int_\gamma f(z)\, dz = \int_a^b \frac{t - is}{t^2 + s^2}\, dt$$

$$\int \frac{t}{t^2 + s^2}\, dt = \frac{1}{2} \log(t^2 + s^2) + C$$

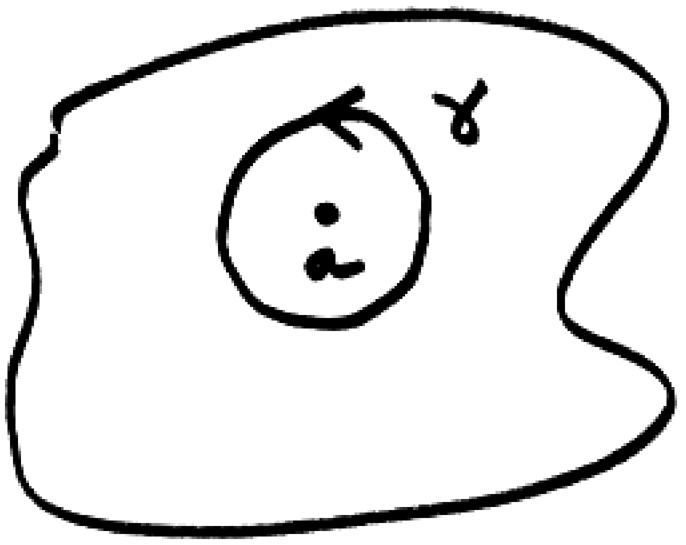
$$\int \frac{s}{t^2 + s^2}\, dt = \arctan(t/s) + ($$

Wed 15, 2006



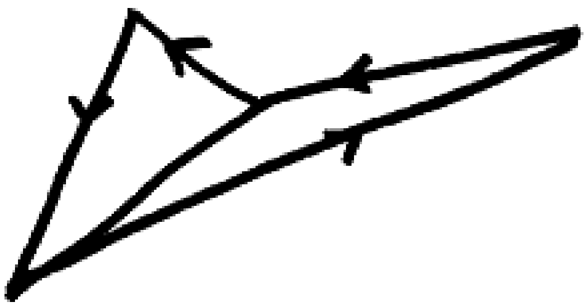
$$\int_{\triangle} f(z)\,dz = 0$$

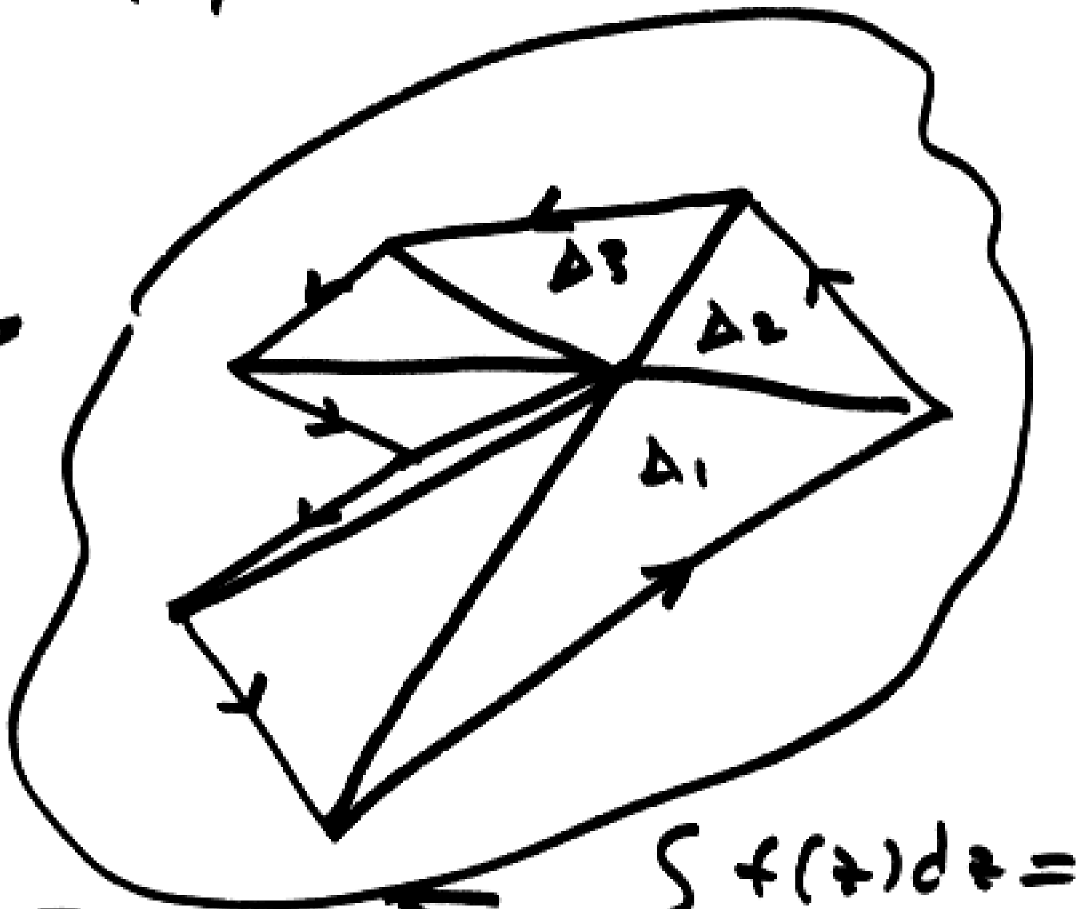
If $f(z) = \dfrac{1}{z-a}$ then



$$\int_{\gamma} f(z)\,dz = 2\pi i$$

$a \notin U \quad U$





$$\int_{\text{polygon}} f(z)\,dz = \sum_i \int_{\Delta_i} f(z)\,dz = 0$$

$\gamma : [\alpha, \beta] \longrightarrow \mathbb{C}$

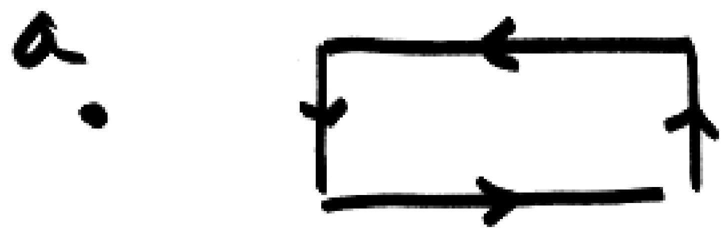
a not on γ

$$n(\gamma, a) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

Claim $n(\gamma, a) \in \mathbb{Z}$ **winding number**

E.g.

$n(\gamma, a) = 1$



$n(\gamma, a) = 0$

$n(\gamma, a) = 1$

$$h(t) = \int_{\alpha}^{t} \frac{\gamma'(u)}{\gamma(u) - a} \, du$$

$$h(\beta) = n(\gamma, a)$$

$$\frac{dh}{dt}(t) = \frac{\gamma'(t)}{\gamma(t) - a}$$

$$\left[e^{-h(t)}(\gamma(t)-a)\right]'$$

$$= -h'(t)\,e^{-h(t)}(\gamma(t)-a)$$

$$+\; e^{-h(t)}\cdot\gamma'(t)$$

$$= 0$$

Hence constant since it is continuous.

$$e^{-h(t)}(\gamma(t)-a) = c$$

$$h(\alpha) = 0$$

$$c = \gamma(\alpha) - a$$

$$e^{h(t)} = \frac{\gamma(t)-a}{\gamma(\alpha)-a}$$

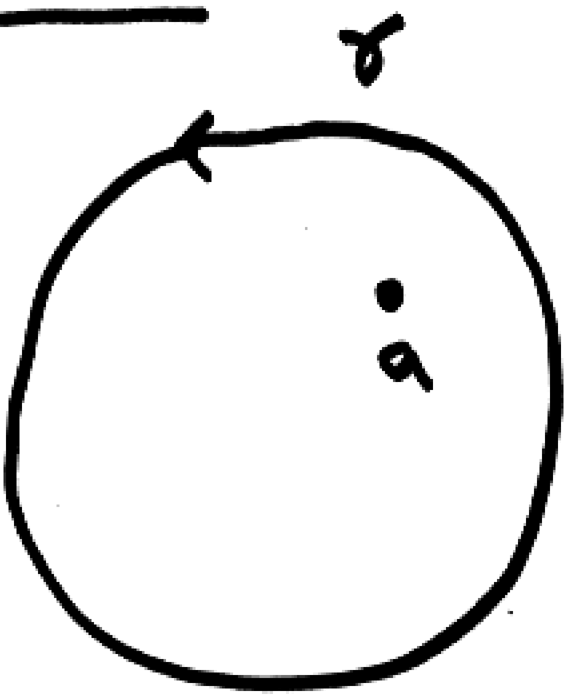
$$e^{h(\beta)} = \frac{\gamma(\beta)-a}{\gamma(\alpha)-a} = 1$$

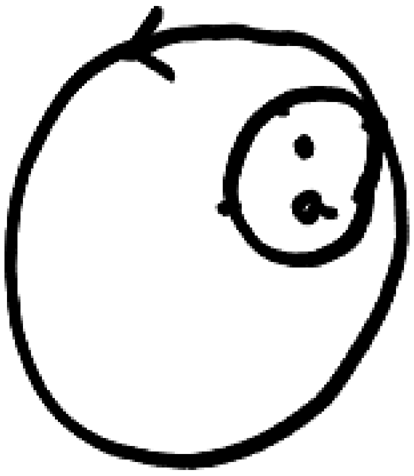
$$\Rightarrow \quad h(\beta) = 2\pi i m$$

for some integer m.

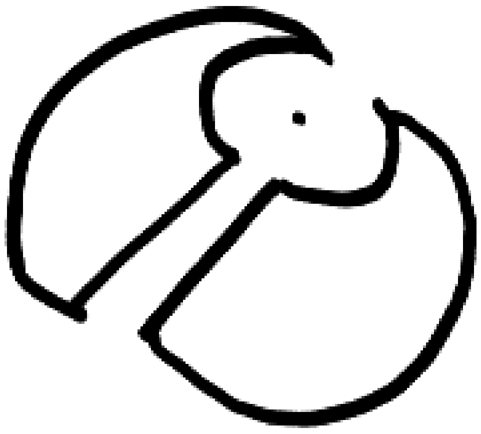
Claim

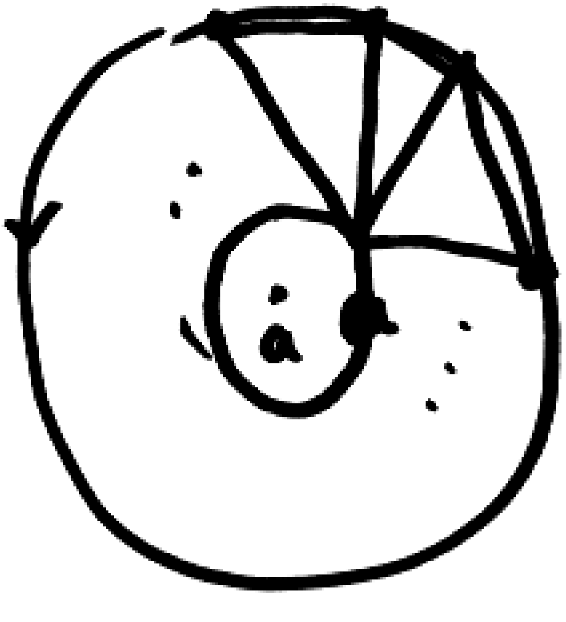
$n(\gamma, a) = 1$



- 

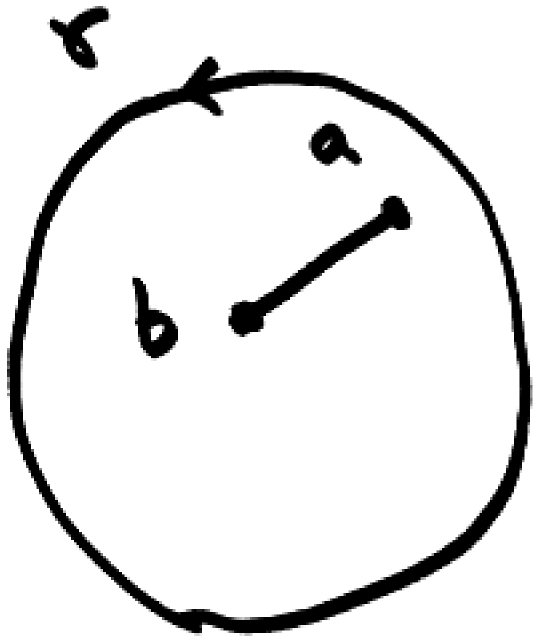
  Like before

- 

  Approx. w/triangles

•



show

$n(\gamma, a) = n(\gamma, b)$

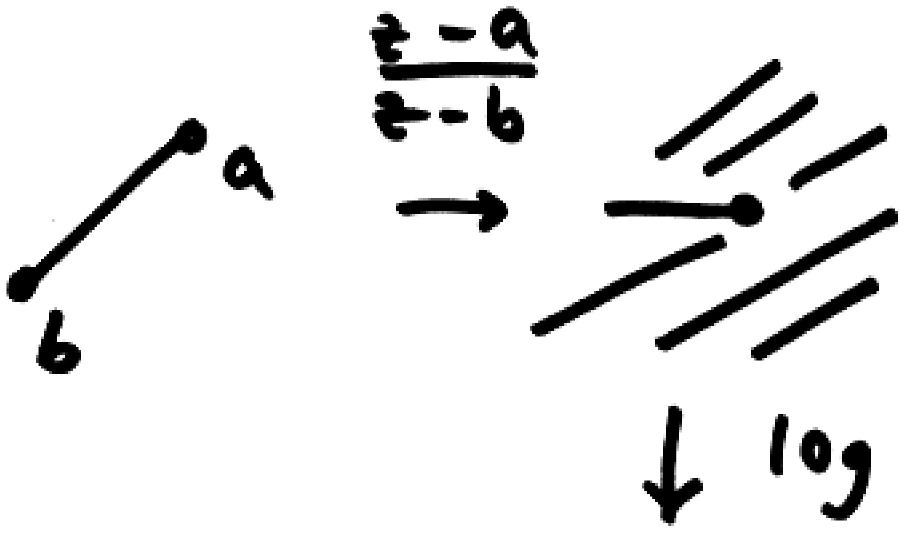
$$\frac{z-a}{z-b}$$

↙

on segment is in $(-\infty, 0]$

$a \longmapsto 0$

$b \longmapsto \infty$



$$\log\left(\frac{z-a}{z-b}\right)$$

is analytic

$$\log\left(\frac{z-a}{z-b}\right)'$$

$$= \frac{1}{z-a} - \frac{1}{z-b}$$

$$\Rightarrow \int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{dz}{z-b}$$

$\Rightarrow$ $n(\gamma, a)$ is constant on the connected components of $\mathbb{C} \setminus \{\gamma\}$

Do this in $\hat{\mathbb{C}}$: unique unbounded connected component (that that contains ∞)

and $n(\gamma, a) = 0$ for points in this component

$\{\gamma\} \subset D(0, R)$

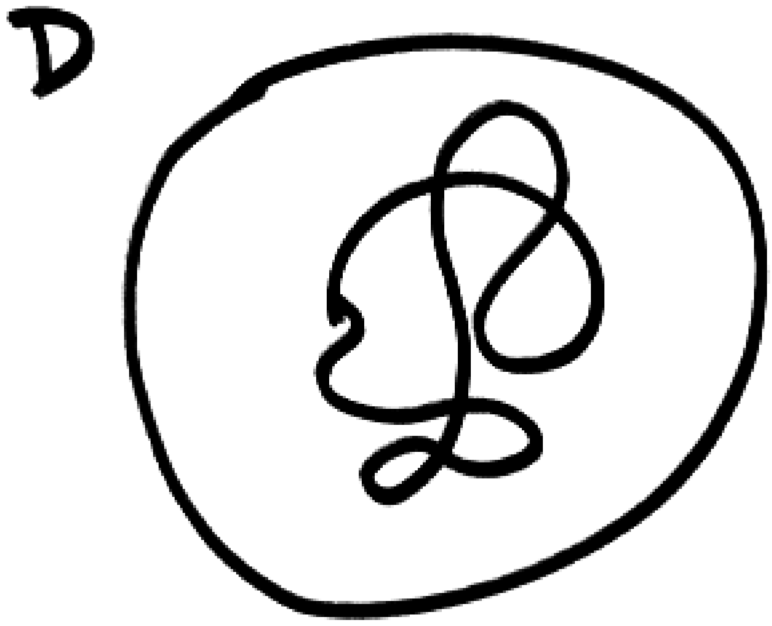
for some large enough R

$\cdot a$

$\Rightarrow n(\gamma, a) = 0$

Feb 17, 2006

Cauchy Thm for a disk



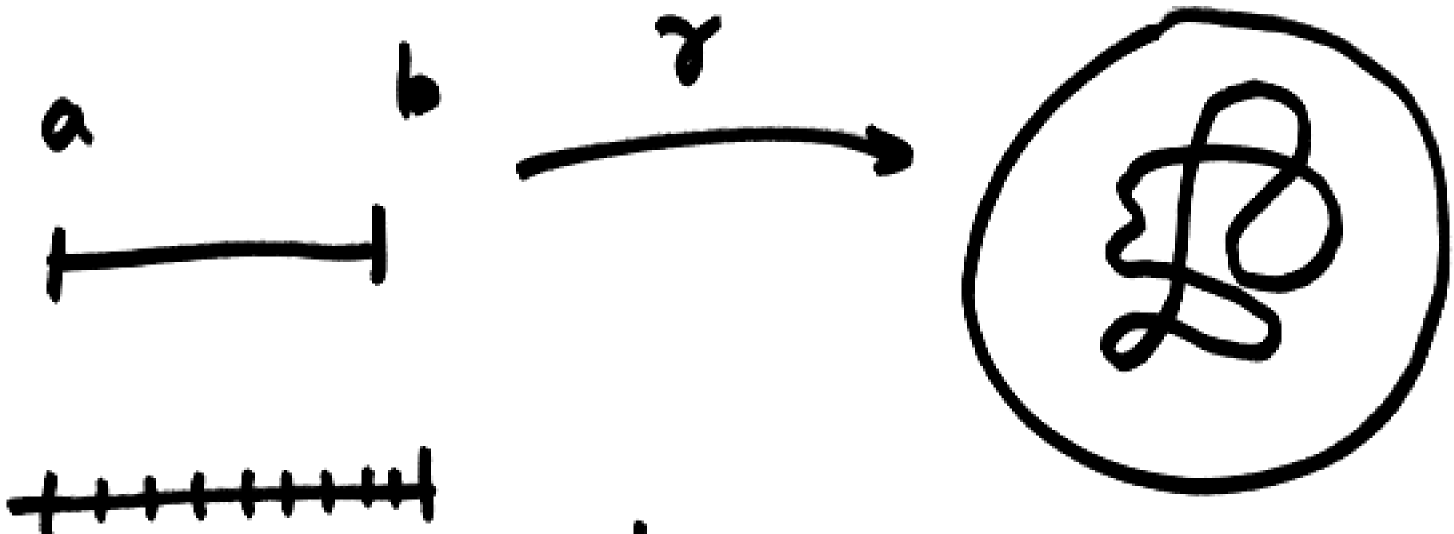
γ closed

f analytic on D

then $\int_\gamma f(z)\,dz = 0$

We proved $\int_\Delta f(z)\,dz = 0$

$\Delta \subset D$ triangle in D.

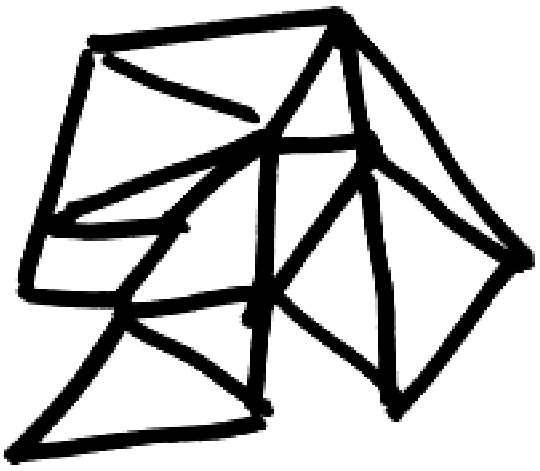


$a = \alpha_0 \ldots\ldots \alpha_N = b$

Approx. γ by polygon P

- $\int_P f(z)\,dz = 0$
- $\int_P f(z)\,dz \longrightarrow \int_\gamma f(z)\,dz$

by refining the partition



We showed

$a, b \in \{\gamma\}$

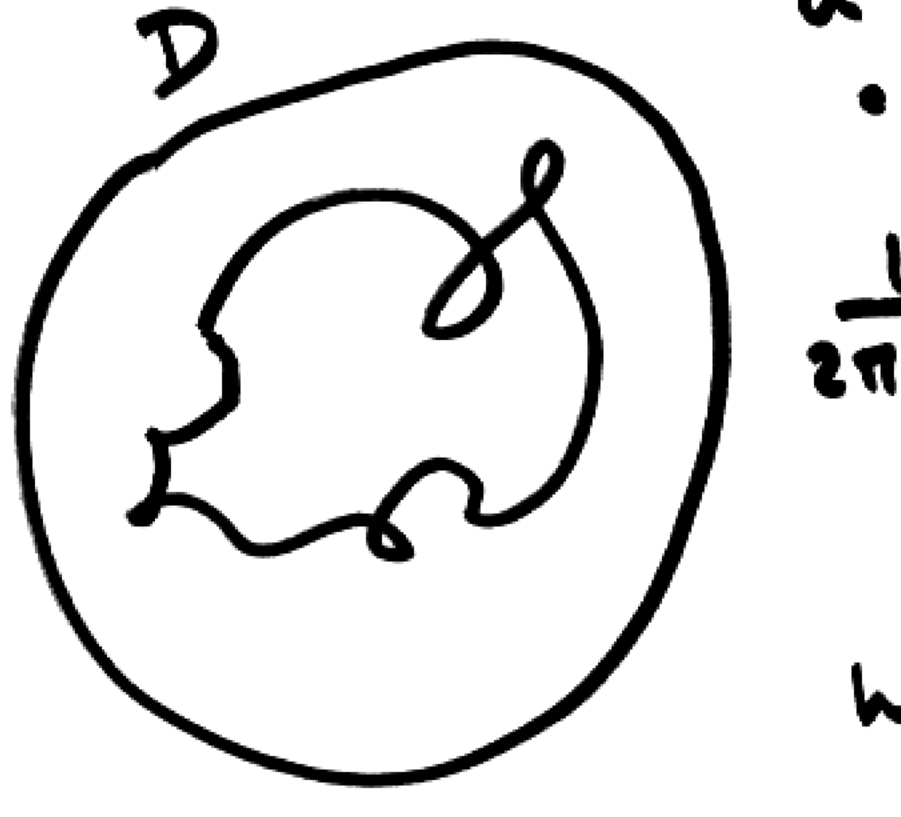
$$n(\gamma, a) = n(\gamma, b)$$

$\Rightarrow$ $n(\gamma, \cdot)$ is constant on connected components

- $n(\gamma, a) = 0$ if a is in the unbounded component (component containing ∞ in $\mathbb{P}^1$)

- $\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$

  let $|a| \to \infty$ integral → 0

- 

  $\frac{1}{2\pi i} \int_\gamma \frac{dz}{\underset{\uparrow}{z-a}}$

  analytic on D
  hence integral is 0

Defn closed path γ

in $U \subseteq \mathbb{C}$ open set

is **homologous to 0**

$\gamma \sim 0$

iff $n(\gamma, a) = 0$

$a \notin U$

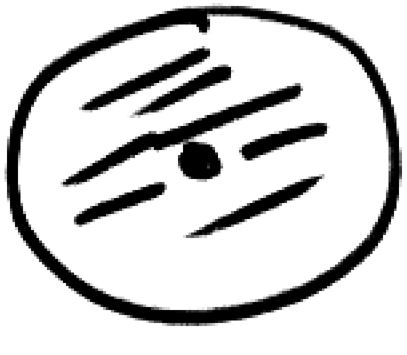
Defn U domain open
region connected

is **simply connected** iff
every closed path γ in U
is homologous to 0: $\gamma \sim 0$

E.g. D disk is simply connected

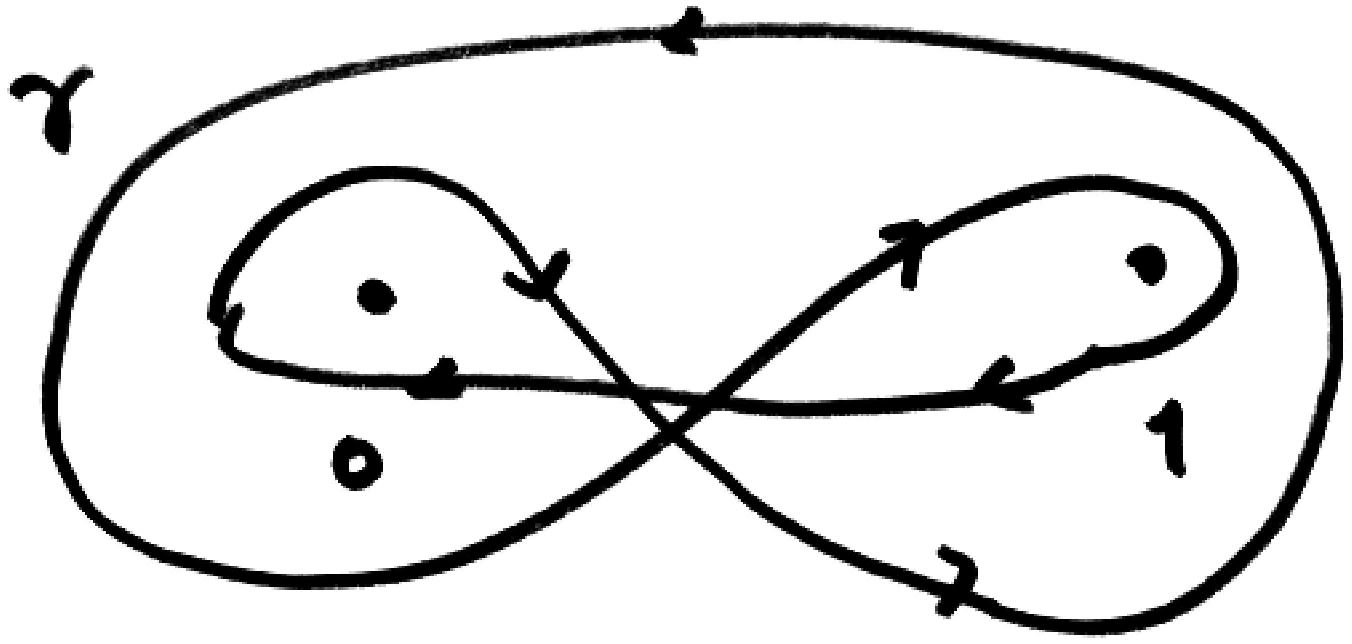

is

s.c.


is not
s.c.

$U \quad \mathbb{C} \setminus \{0, 1\}$

Pochamer



$$\oint \qquad n(\gamma, 0) = 0$$
$$n(\gamma, 1) = 0$$

not homotopic

$$\pi_1 \to H_1 = \pi_1^{ab}.$$

<u>General Cauchy thm</u>

If γ is homologous to 0 (in U) then

$$\int_\gamma f(z)\, dz = 0$$

Cauchy Integral Fmla

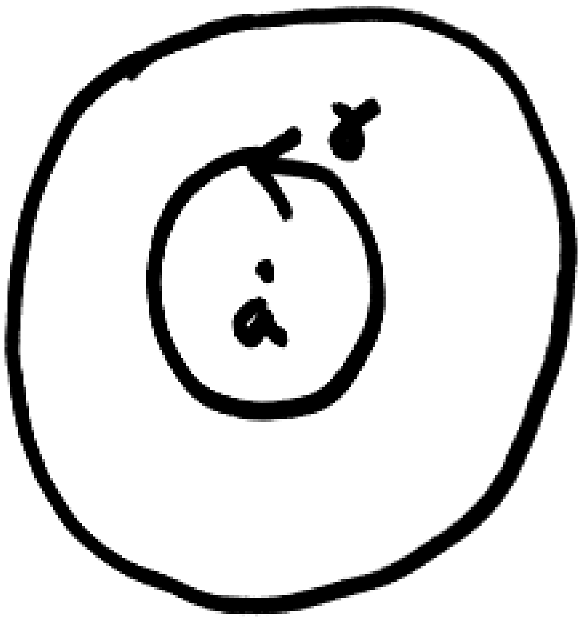
D disk, f analytic on D

γ closed path in D

$a \notin \{\gamma\}$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = n(\gamma,a)\,f(a)$$

E.g.



$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = f(a)}$$

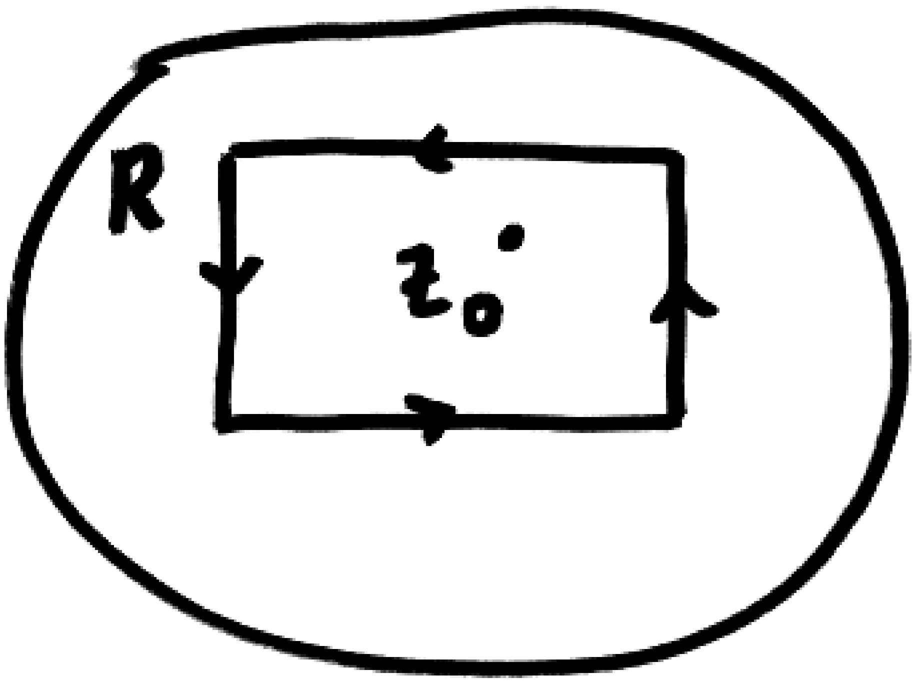
This case first

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 1$$

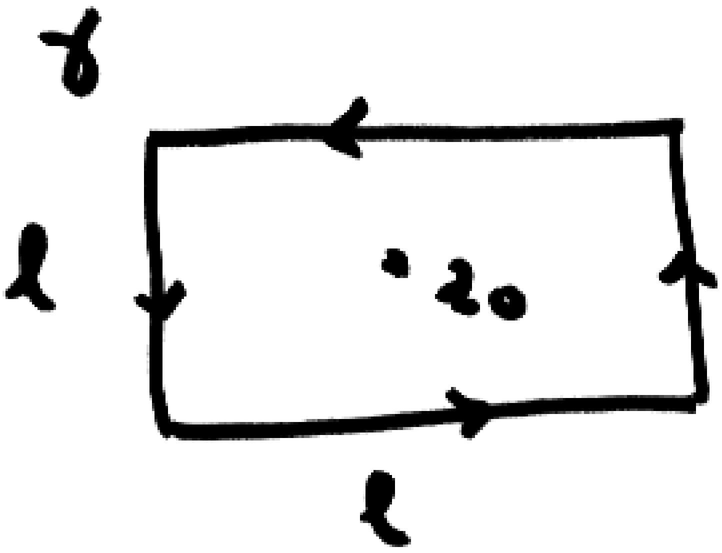
$$\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)-f(a)}{z-a}\,dz = 0$$

- Suppose f is analytic except at a point z_0

but $\lim\limits_{z\to z_0}(z-z_0)\,f(z)=0$



E.g. f is analytic $U\setminus\{z_0\}$ but is bounded



$$|z-z_0|\geqslant\frac{\ell}{2}$$

$$\left|\int_{\gamma} f(z)\,dz\right|\leq\varepsilon\int_{\gamma}\frac{|dz|}{|z-z_0|}$$

$$\leq\varepsilon\,\frac{2}{\ell}\,4\ell=8\varepsilon$$

Shrink the square enough so that

$$|f(z)| \leq \frac{\varepsilon}{|z-z_0|}$$

$$0<|z-z_0|<\delta$$

Above $F(z) := \dfrac{f(z)-f(a)}{z-a}$

$$(z-a)\,F(z) = f(z)-f(a)$$

$$\to 0$$

$$\text{as } z \to a$$

because f is continuous

$$\Rightarrow \quad \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$$

Lemma

$$\int_\gamma \frac{\varphi(w)\,dw}{(w-z)^n} =: F_n(z)$$

φ is continuous on $\{\gamma\}$

Then F_n is analytic on $\mathbb{C} \setminus \{\gamma\}$ and

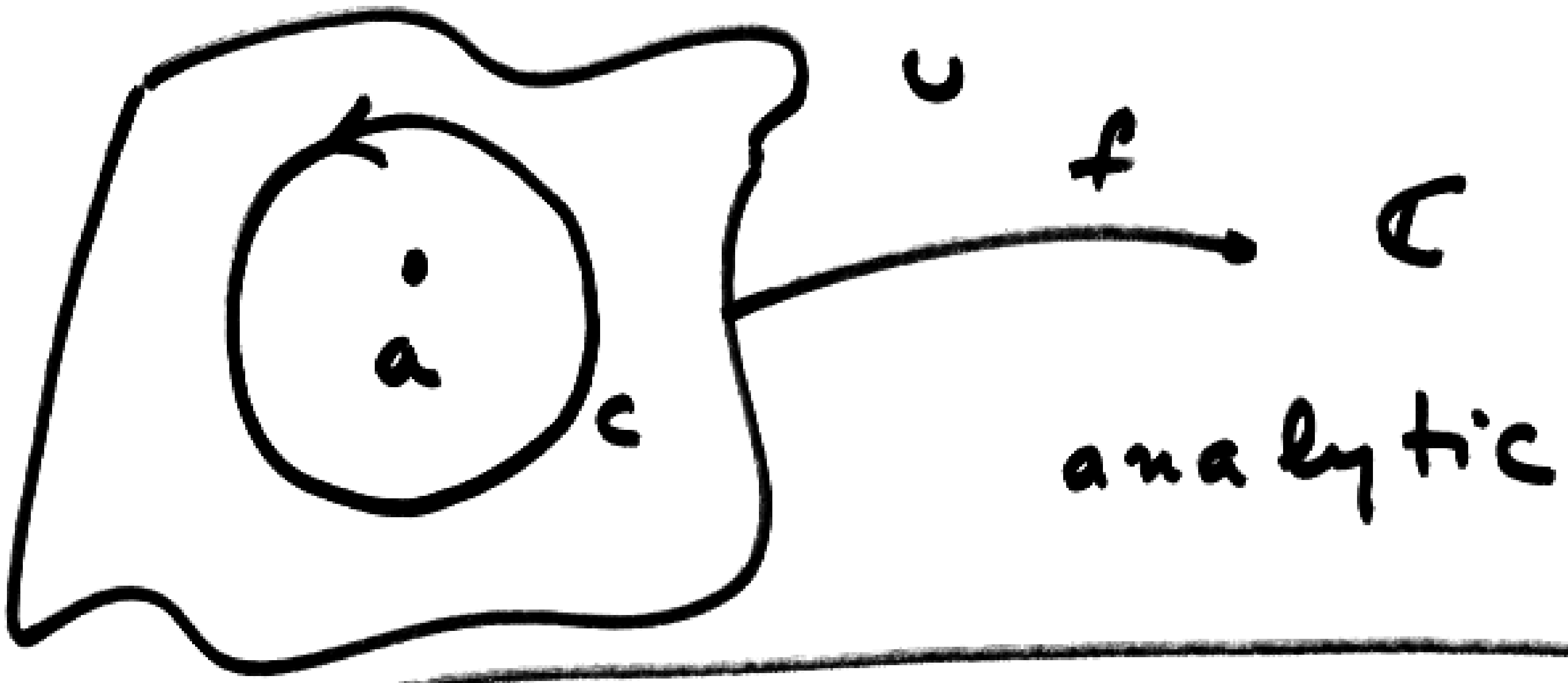
$$F_n'(z) = n \dot{F}_{n+1}(z)$$

i.e. diff. inside the integral.

Applied to the Cauchy fmla we get

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-z)^{n+1}}$$

Feb 20, 2006



$$\boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dZ}$$

Lemma

$$F_n(z) = \int_\gamma \frac{\varphi(w)}{(w-z)^n}\,dw$$

φ continuous $\Rightarrow$ F_n is analytic and in $\mathbb{C} \setminus \{\gamma\}$. $F_n'(z) = n\,F_{n+1}(z)$

$\Rightarrow$ f is infinitely differentiable at a

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{(2\pi i)}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$C: \gamma(\theta) = a + Re^{i\theta}$

$0 \le \theta \le 2\pi$

$\gamma'(\theta) = iRe^{i\theta}$

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+Re^{i\theta})\,d\theta$$

i.e. $f(a)$ is the average of f on the circle.

Cauchy estimate

$|f^{(n)}(a)| \le n!\, M_R R^{-n}$

$$M_R := \max_{|z-a|\le R} |f(z)|$$

Liouville's theorem

If f bounded, entire (analytic in all of Z) then f is constant.

$$|f'(a)| \leq \frac{M}{R}$$

($n=1$ of Cauchy estimate)

$|f(z)| \leq M$ all z

$R \to \infty$

$\Rightarrow f'(a) = 0$

$\Rightarrow f$ is constant.

Almost trivial proof of FTA

$0 \neq P$ polynomial w/ $\mathbb{C}$ coefficients

Consider $\dfrac{1}{P(z)} =: f(z)$

P w/ no zeroes in $\mathbb{C}$

$\Rightarrow f$ is analytic in $\mathbb{C}$

If $f(z)$ was bounded then P has deg 0.

since $\deg P > 0$

$$P(\infty) = \infty$$

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots$$

$a_n \neq 0$

$$= a_n z^n \underbrace{\left(1 + a_{n-1} \frac{1}{z} + \dots \right)}_{\substack{\downarrow \\ 1 \\ \text{as } z \to \infty}}$$

$n \geq 1 \Rightarrow P(z) \to \infty$ as $z \to \infty$ $\square$

<u>THM</u> (Morera)

$f : U \longrightarrow \mathbb{C}$ U domain

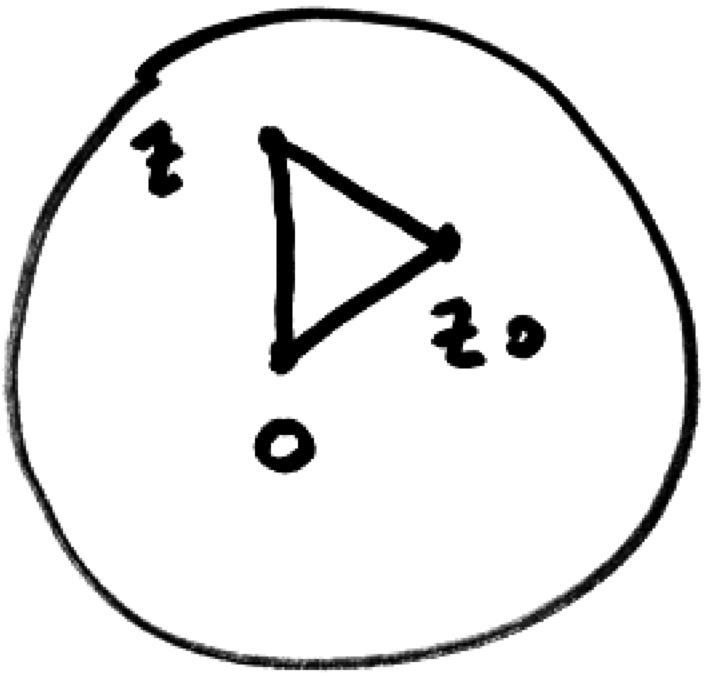
continuous and $\int_\gamma f(z)\,dz = 0$

for closed paths in U.

Then f is analytic

Pf In fact: true for triangles

w.l.o.g



$$F(z) := \int_{[0,z]} f(w)\,dw$$

$$F(z) - F(z_0) = \int_{[z_0,z]} f(w)\,dw$$

because $\int_{\Delta} f(w)\,dw = 0$

$\underline{z \neq z_0}$

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)$$

$$= \frac{1}{z - z_0} \int_{[z_0,z]} \left[f(w) - f(z_0) \right] dw$$

f continuous at z_0

$|w - z_0| < \delta \Rightarrow |f(w) - f(z_0)| < \varepsilon$

$$\left| \frac{F(z)-F(z_0)}{z-z_0} - f(z_0) \right| \le$$

$$\le \frac{\varepsilon}{|z-z_0|} \int_{[z_0,z]} |dw|$$

$$= \varepsilon$$

$\Rightarrow$ F is differentiable at z_0

and $F'(z_0) = f(z_0)$

$\Rightarrow$ F is analytic in disk

$F' = f$

$\Rightarrow$ f is analytic in disk □

Removable singularity

THM $f : U \setminus \{a\} \longrightarrow \mathbb{C}$ analytic

$a \in U$ ~~[illegible]~~ **and**

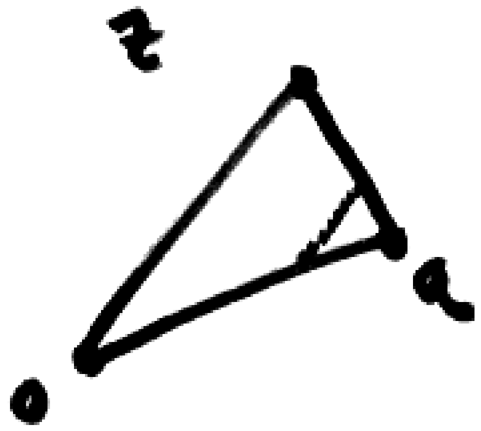
$$(z-a)f(z) \longrightarrow 0$$

then f can be extended to an analytic function on U (uniquely) and conversely.

Pf

Lemma $f : U \to \mathbb{C}$ analytic on $U \setminus \{a\}$ continuous at a then f is analytic at a

Pf



$$\int_{\partial\Delta} f(w)\,dw = 0$$

$$\int_{\text{big}} f(w)\,dw = 0$$

$$\int_{\text{small}} f(w)\,dw \leq \text{small} \ldots$$

□

Apply Lemma

$$g(z) := \begin{cases} (z-a)f(z) & z \neq a \\ 0 & z = a \end{cases}$$

g is analytic on $U \setminus \{a\}$

and continuous on U.

$\Rightarrow$ g is analytic on U

and $g(a) = 0$

$$f_1(z) = \begin{cases} \dfrac{g(z)}{z-a} & z \neq a \\ g'(a) & z = a \end{cases}$$

■ next time ...

Feb 22, 2006

Lemma $f : U \to \mathbb{C}$

U domain

- f analytic on $U \setminus \{a\}$
- f is continuous at a

$\Rightarrow$ f is analytic on U

Cor $g : U \to \mathbb{C}$ analytic

$g(a) = 0$

$$f(z) := \begin{cases} \dfrac{g(z)}{z-a} & z \neq a \\ g'(a) & z = a \end{cases}$$

$\Rightarrow$ f is analytic on U

Pf
$$\frac{g(z)}{z-a} = \frac{g(z)-g(a)}{z-a} \to g'(a)$$

THM $f : U \setminus \{a\} \longrightarrow \mathbb{C}$

analytic

and $(z-a)\, f(z) \longrightarrow 0$
$z \to a$

then f can be extended to an analytic function on U.

Pf Consider

$$g(z) := \begin{cases} (z-a)\, f(z) & z \neq a \\ 0 & z = a \end{cases}$$

by the ~~Cor.~~ Lemma g is analytic on U

By the corollary

$$f_1(z) := \begin{cases} f(z) & z \neq a \\ g'(a) & z = a \end{cases}$$

$f_1 = f$ on $U \setminus \{a\}$
and is analytic on U. $\square$

Such a point a is called a <u>removable singularity</u>

$$\frac{\sin x}{x}$$

$$\frac{(x^2-1)}{x-1}$$

$f : U \longrightarrow \mathbb{C}$ analytic on domain U

$$f_1(z) := \begin{cases} \dfrac{f(z) - f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

$$f(z) = f(a) + (z-a) f_1(Z)$$

with f_1 analytic and $f_1(a) = f'(a)$

By induction

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) \cdots$$

$$+ \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}$$

$$+ (z-a)^n f_n(z)$$

f_n is analytic and

$$f_n(a) = f^{(n)}(a).$$

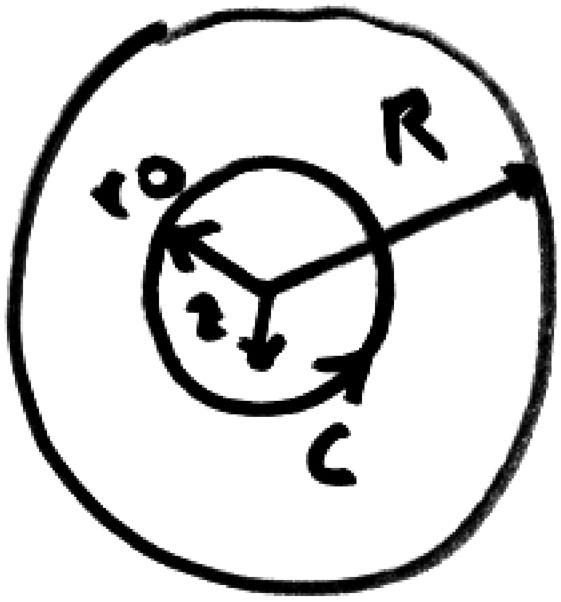
Power series expansion

THM f analytic on $D(0,R)$ $R>0$ then

$$f = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!} z^n.$$

Pf

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$$



$C = \{ |w| = r_0 \}$

$|z| = r$

$0 < r < r_0 < R$

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1 - z/w} = \frac{1}{w}\left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots\right)$$

$|z| < |w|$

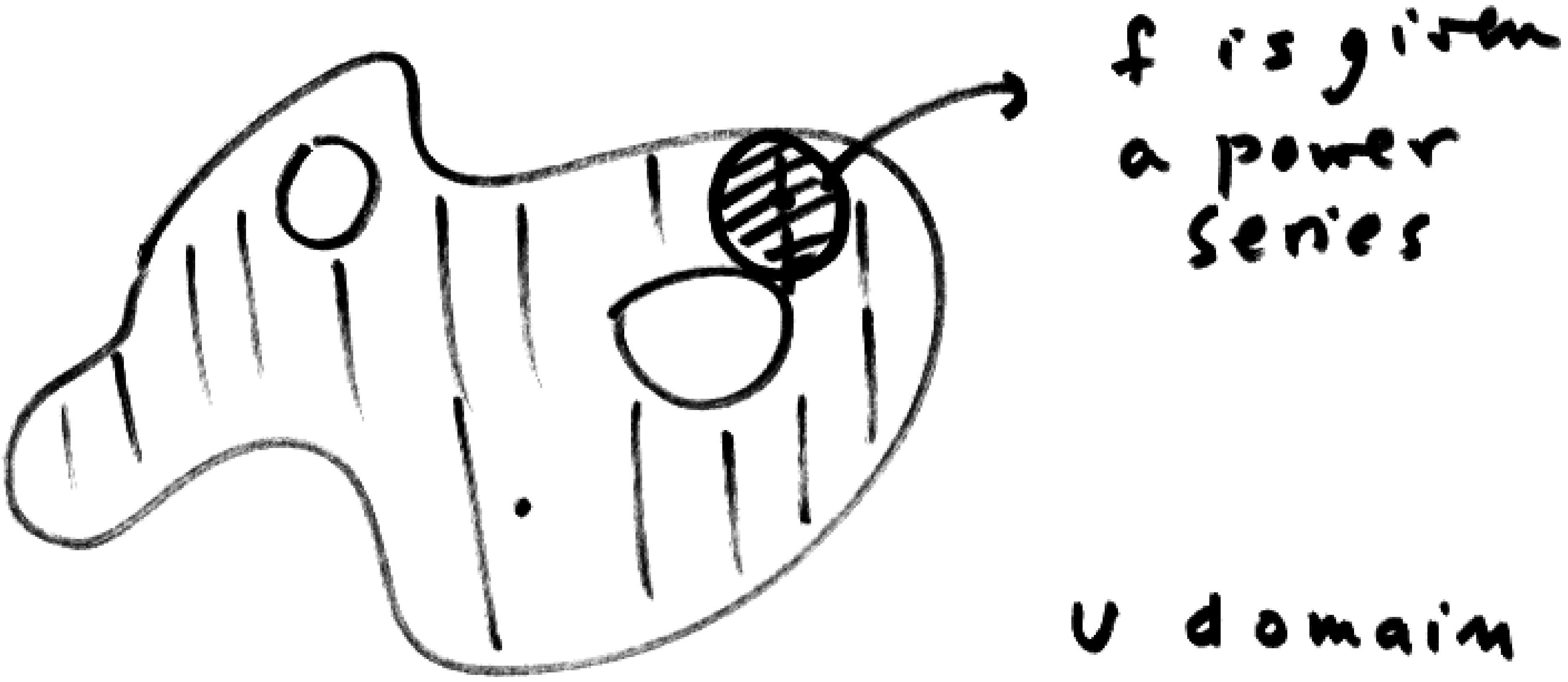
$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w\,(1 - z/w)}\, dw$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{w}\left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots\right) dw$$

Interchange sum and integral

$$= \sum_{n \geq 0} z^n \underbrace{\frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}}\, dw}_{\displaystyle\frac{f^{(n)}(0)}{n!}}$$

$$= \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n \quad \square$$



U domain

THM $f : U \to \mathbb{C}$ analytic

TFAE

1) $f \equiv 0$ on U

2) $f^{(n)}(a) = 0$ all n for some $a \in U$

3) $f(z_k) = 0$ for $z_k \to a$, $z_k \neq a$

Pf 3) $\Rightarrow$ 2)

By continuity $f(z_k) \to \underset{0}{\overset{}{f(a)}}$

Assume $f^{(j)}(a)=0$ $j=0,1,..n$ ⑦

$$f(z) = (z-a)^n f_n(z)$$

$$f_n(a) = f^{(n)}(a)$$

$$0 = f(z_k) = (z_k - a)^n f_n(z_k)$$

$$z_k \neq a \Rightarrow f_n(z_k) = 0$$
$$\Rightarrow f_n(a) = 0$$

since f_n is analytic (hence continuous).

2) ⇒ 1)

$$A = \{ z \in U \mid \underset{\text{all } n \geq 0}{f^{(n)}(z) = 0} \}$$

$a \in A$, $A \neq \phi$

A is closed $A = \bigcap_{n \geq 0} (f^{(n)})^{-1}(\{0\})$

A is open by power series expansi
about z. Hence $A = U$ □

One way to view this:

the values of f at z_k

determine f

Feb24, 2006

Examples of power series

Binomial theorem

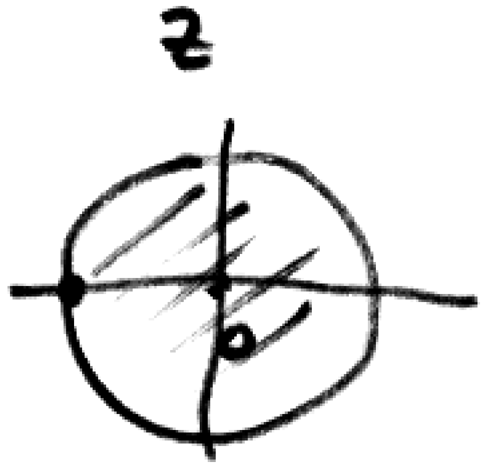
$$(1+z)^a := 1 + az + \binom{a}{2} z^2 + \cdots$$

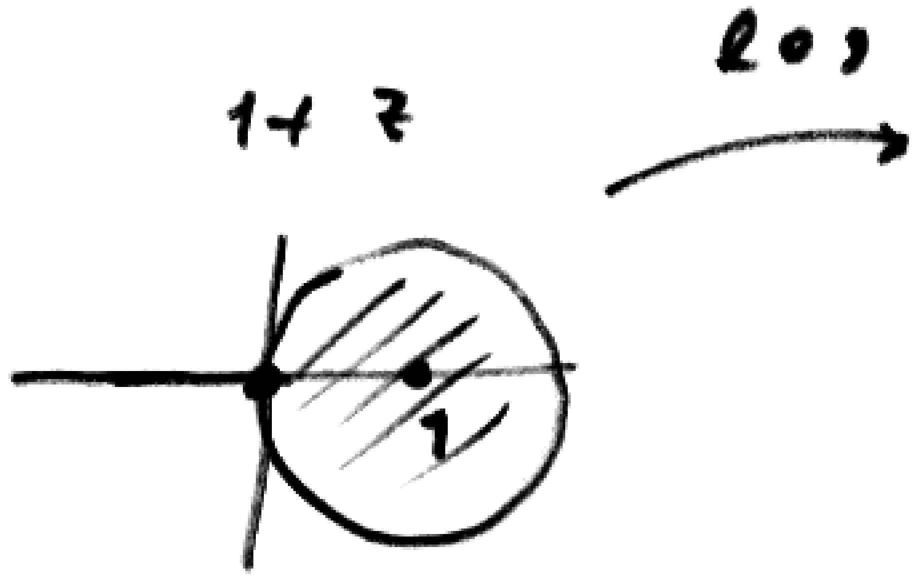
$$a_n = \binom{a}{n} := \frac{a(a-1)\cdots(a-n+1)}{n!}$$

$a \in \mathbb{C}$

$|z| < 1$ $\qquad \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n+1}{a-n}\right| \to 1$

$$(1+z)^a = \exp(a \log(1+z))$$





$\underline{a=-1}$

$$(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$$

$\underline{a=1/2}$

$$(1+z)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} z^n$$

$$= 1 + \frac{1}{2} z + \frac{1/2\,(1/2-1)}{2!} z^2 + \dots$$

$$\binom{1/2}{n} = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n\, n!}$$

$\underline{a=-1/2}$

$$\frac{1}{\sqrt{1-4z}} = \sum_{n\geq 0} \binom{2n}{n} z^n$$

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geq 0} C_n z^n = 1 + z + 2z^2 + 5z^3 + \dots$$

C_n Catalan numbers

$$= \frac{1}{n+1}\binom{2n}{n} \in \mathbb{Z}$$

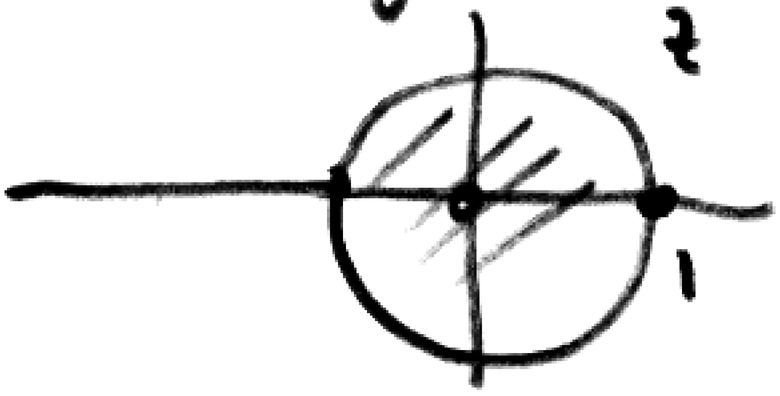
probably the most common combinatorial numbers...

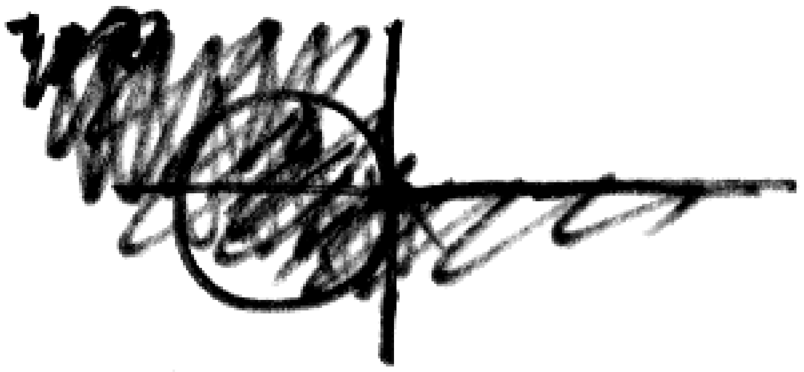
$$\left.\begin{array}{l} ((a\,b)\,c)\,d \\ (a\,(b\,c))\,d \\ (a\,b)\,(c\,d) \\ a\,((b\,c)\,d) \\ a\,(b\,(c\,d)) \end{array}\right\} \quad 5 = C_3$$

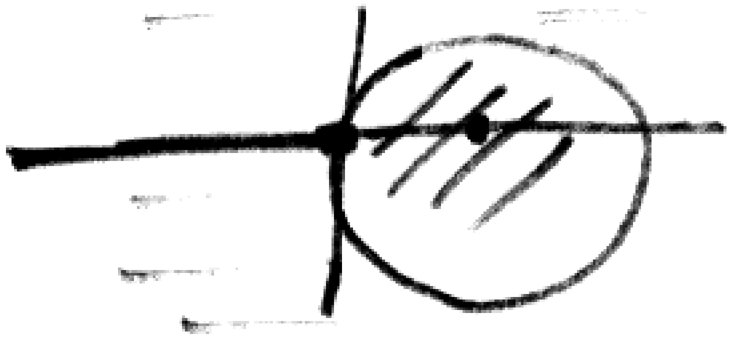
$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad |z|<1$$

Integrate term by term

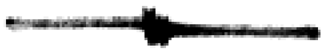
$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad |z|<1$$







$$\frac{\sin z}{z}$$



zeros of analytic functions are isolated.

If we had $z_1, z_2 \dots$ converging to a z_K distinct

$f(z_K) = 0$ $z_1, z_2, \dots, a \in U$

$\Longrightarrow f \equiv 0.$

entire $\sin(\pi z) = 0$ $z \in \mathbb{Z}$.

$\sin\left(\frac{\pi}{z}\right)$ analytic in $\mathbb{C} \setminus \{0\}$

zeros at $z = \frac{1}{n}$. $n \in \mathbb{Z}$.

a is <u>pole</u> $\lim_{z \to a} f(z) = \infty$

$\frac{1}{f(z)}$ has a zero at a

poles are isolated as well.

f is analytic on $D(a,R) \setminus \{a\}$

$R > 0$

f has an isolated singularity

- removable singularity

$$(z-a)f(z) \xrightarrow[z \to a]{} 0$$

- pole $\quad f(z) \xrightarrow[z \to a]{} \infty$

- essential singularity

$z \longmapsto \sqrt{z}$ cannot be defined on $D(0,R) \setminus \{0\}$

0 is a branch point

Casorati-Weierstrass

f has e.s. at a

$D(a, R)$ $R > 0$

$\setminus \{a\} =: U_R$

$f(U_R)$ is dense in $\mathbb{C}$

for all $R > 0$.

<u>pf</u> suppose not. $c \in \mathbb{C}$

$$|f(z) - c| > \varepsilon \quad \text{all } z \in U_R$$

$$|z - a|^{-1} |f(z) - c| \to \infty$$

as $z \to a$

$g(z) := (z-a)^{-1}(f(z) - c)$ has a pole at a

$$f(z) = (z-a) g(z) + c$$

$$g(z) = \frac{h(z)}{(z-a)^K} \qquad K \geq 1$$

h analytic $h(a) \neq 0$ in some disk

$$f(z) = \frac{h(z)}{(z-a)^{k-1}} + c$$

$\Rightarrow$ f has a pole or removable singularity $\square$

g pole at a

$\frac{1}{g}$ zero at a

$$\frac{1}{g} = (z-a)^k g_k(z)$$

$k \geqslant 0$

g_k is analytic

$g_k(a) \neq 0$

$$g = \frac{1/g_k(z)}{(z-a)^k}$$

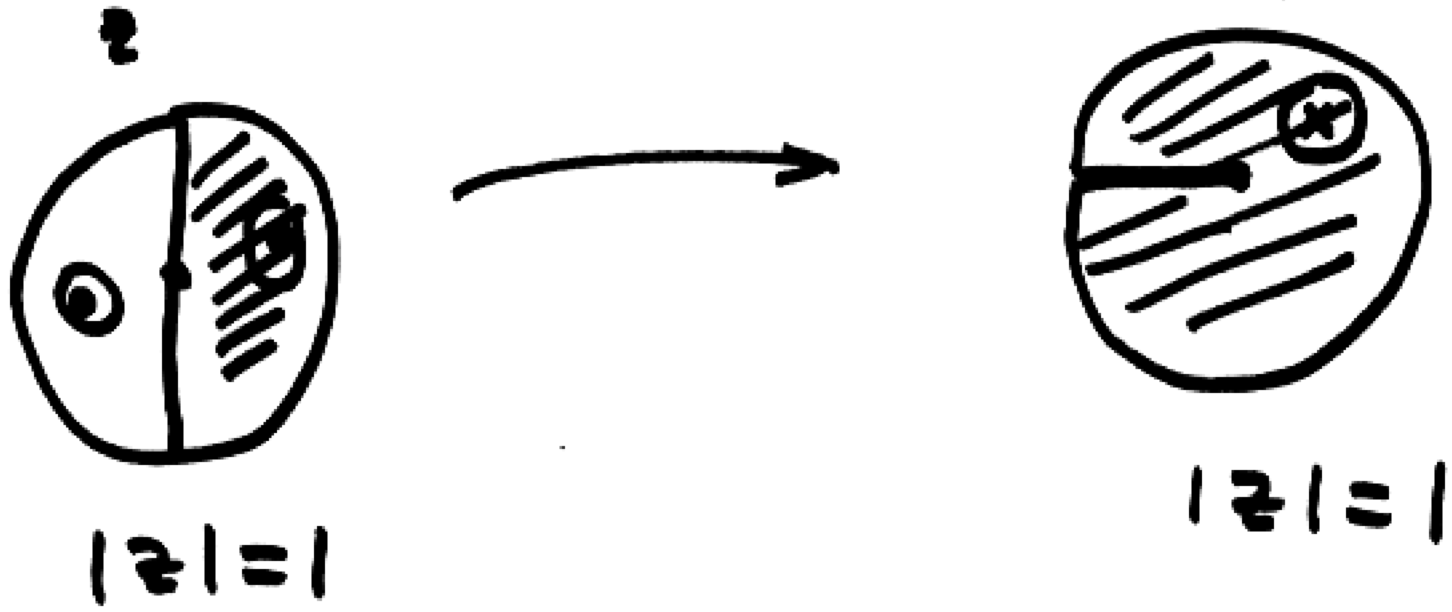
$1/g_k$ is analytic in some disk around a. $\square$

Feb 27, 2006

Locally ?

n=2

$z \longmapsto z^2$ around 0



$0 \longmapsto 0$

$w \neq 0$ two preimages

Locally it's given by a power series

$$f(0) = 0$$

$$w = f(z) = z^n \cdot f_n(z)$$

for some $n = 0, 1, 2, ..$

$$f_n(0) \neq 0$$

n order of 0 of f at $z=0$.

D

<u>Claim</u> There is a disk about $w=0$ such that $w \neq 0$ $w \in D$ there are exactly n pre-images of w in some disk about $z=0$

case <u>$n=1$</u>

$$f(z) = z f_1(z)$$

$f'(0) = f_1(0) \neq 0$

$$f(z) = a_1 z + a_2 z^2 + \dots$$

$a_1 \neq 0$.

f has a power series inverse

i.e. there is a power series

$g(z) = b_1 z + b_2 z^2 + \dots$

s.t. $f \circ g(z) = z$

$$g = b_1 z + O(z^2)$$

$$f \circ g(z) = f(b_1 z + O(z^2))$$

$$= a_1 b_1 z + O(z^2) = z$$

$$b_1 = a_1^{-1}$$

$$g = a_1^{-1} z + b_2 z^2 + O(z^3)$$

$$f \cdot g(z) = a_1(a_1^{-1} z + b_2 z^2)$$

$$+ a_2(a_1 z + a_2 z^2)^2$$

$$+ O(z^3)$$

$$= z + a_1 b_2 z^2$$

$$+ a_2 a_1^2 z^2 + O(z^3) = z$$

$$a_1 b_2 + a_2 a_1^2 = 0$$

$$b_2 = -a_2 a_1$$

$$\vdots$$

Do by induction

$b_1, b_2, \ldots, b_{n-1}$

Repeat process

__claim__ $(n+1)$-st coeff of the composition $f \circ g(z)$ set to zero will involve b_n like this

$$a_1 b_n + \cdots = 0$$

__Challenge__ $f(z) = z e^z$

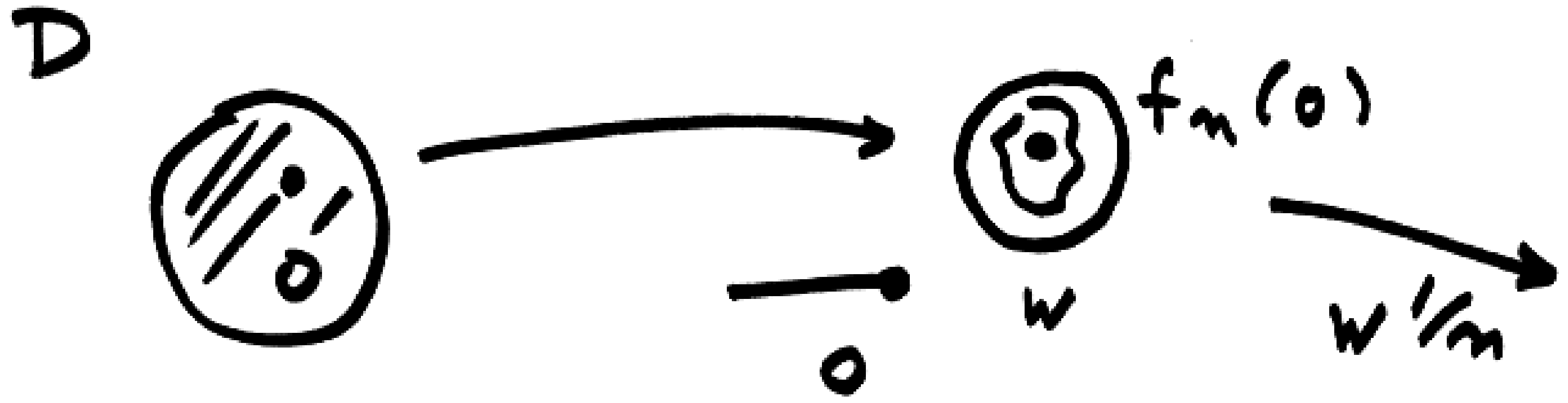
$$= z + \frac{z^2}{1!} + \frac{z^3}{2!} + \cdots$$

What is the inverse power series?

Back to general n

$$f(z) = z^n f_n(z)$$

$f_n(0) \neq 0$, f_n analytic



Pick disk D around $z=0$ so that image by f_n is contained in a disk.

i.e. on D there is an n^{th} root of f_n $\quad h(z)=z\, f_n^{1/n}(z)$

$$f(z) = h(z)^n,$$

for some h analytic on D_z

$$h(z) = a_1 z + a_2 z^2 + \dots$$

$a_1 \neq 0$. $\quad a_1 = f_n(0)^{1/n}$

Call g the inverse of h in some disk D_w about $w=0$

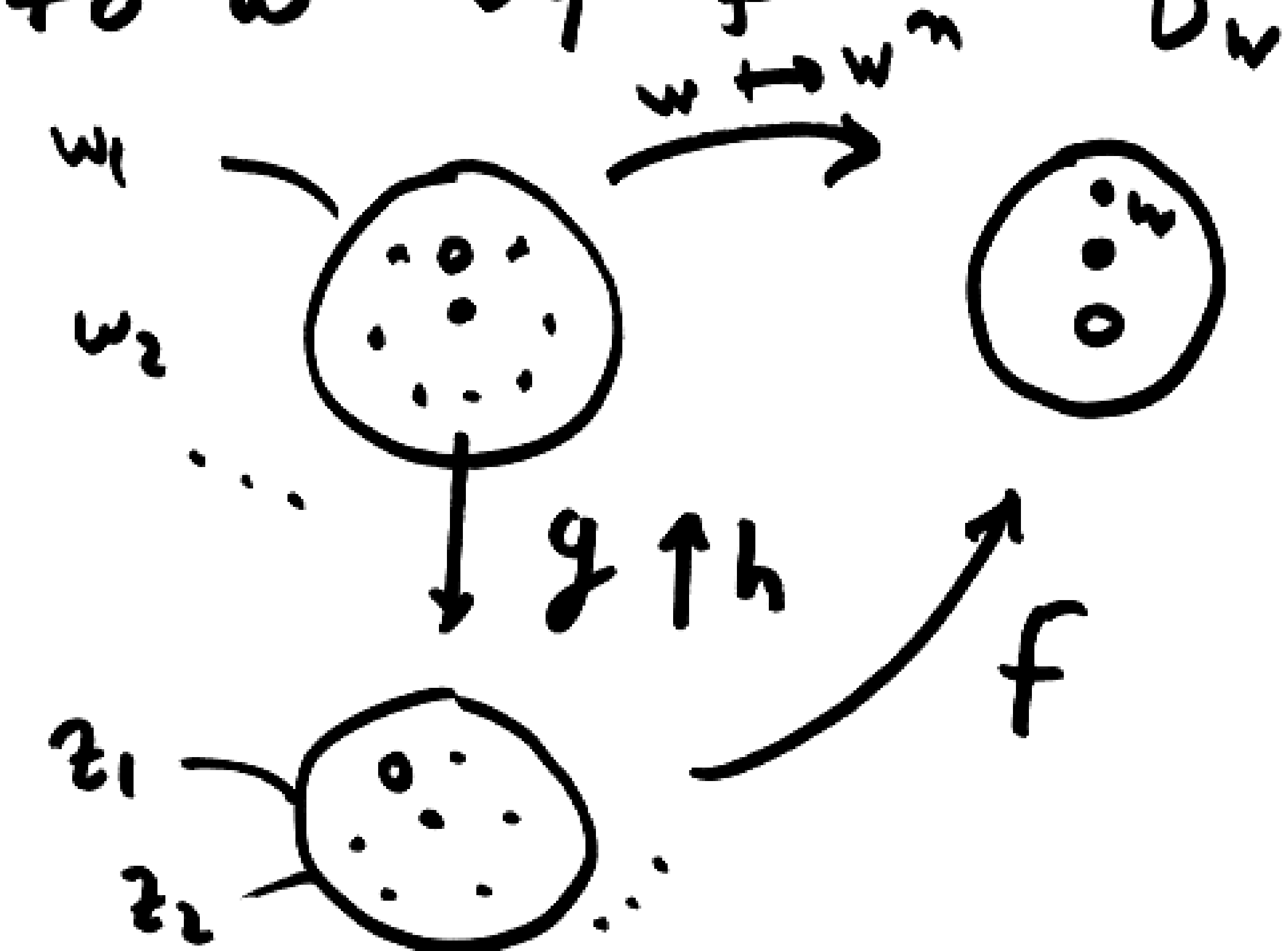
(Claim inverse power series converges in some disk if original power series does)

$w \in D_w \setminus \{0\}$

Take $w_1, w_2, \dots, w_n$
the n distinct n^{th}-roots of w
let $z_j = g(w_j)$

$$
\begin{aligned}
f(z_j) &= f \circ g(w_j) \\
&= (h \circ g(w_j))^n \\
&= w_j^n = w
\end{aligned}
$$

So $z_1, z_2, \dots, z_n$ are mapped
to w by f



If $f'(a) = 0$ then

f can't have a local inverse about a.

$f_0(z) := f(z) - f(a)$

$f_0(a) = 0$

$f_0'(z) = f'(z)$

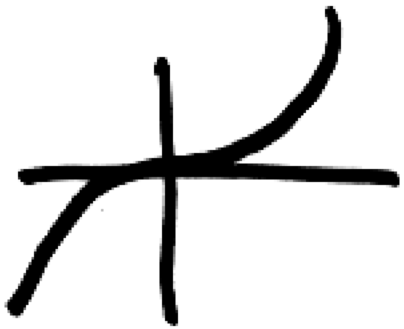
$f_0'(a) = 0$

$\Rightarrow$ f_0 has a zero at a

of $\underset{=n}{\text{order}} > 1$

f is a $n \mapsto 1$ map on any disk $D \setminus \{f(a)\}$ about $f(a)$

Constrast w/ real fctns



There is $x \mapsto x^{1/3}$ around 0

but no $z \mapsto z^{1/3}$ " (analytic)

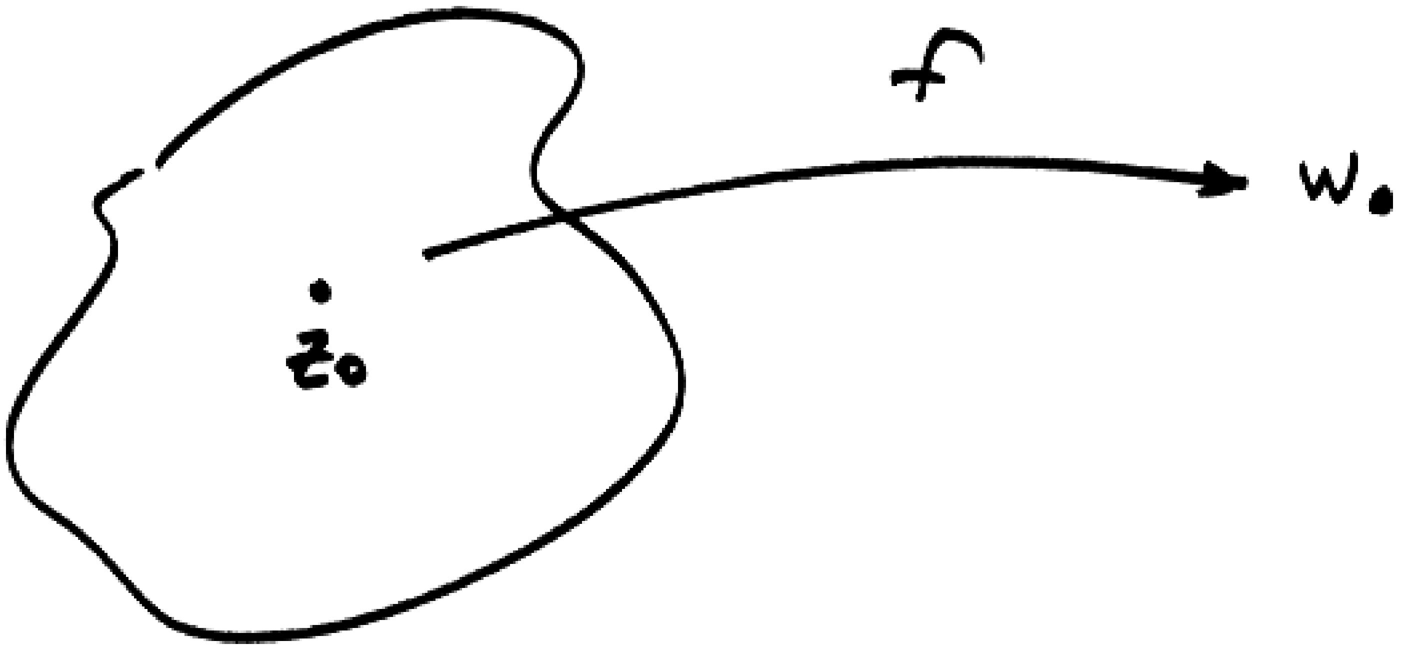
- locally 1-1 $\Leftrightarrow$ $f'(z) \neq 0$

$\not\Rightarrow$ globally 1-1

e.g. $z \mapsto e^z$

March 1, 2006

f analytic domain U



$$f(z) - f(z_0) = (z - z_0)^n f_m(z)$$

- f_m analytic
- $f_m(z_0) \neq 0$

$f(z) = w_0$ multiplicity n

Shrink disk about z_0 s.t. can take an n^{th} root of f_n

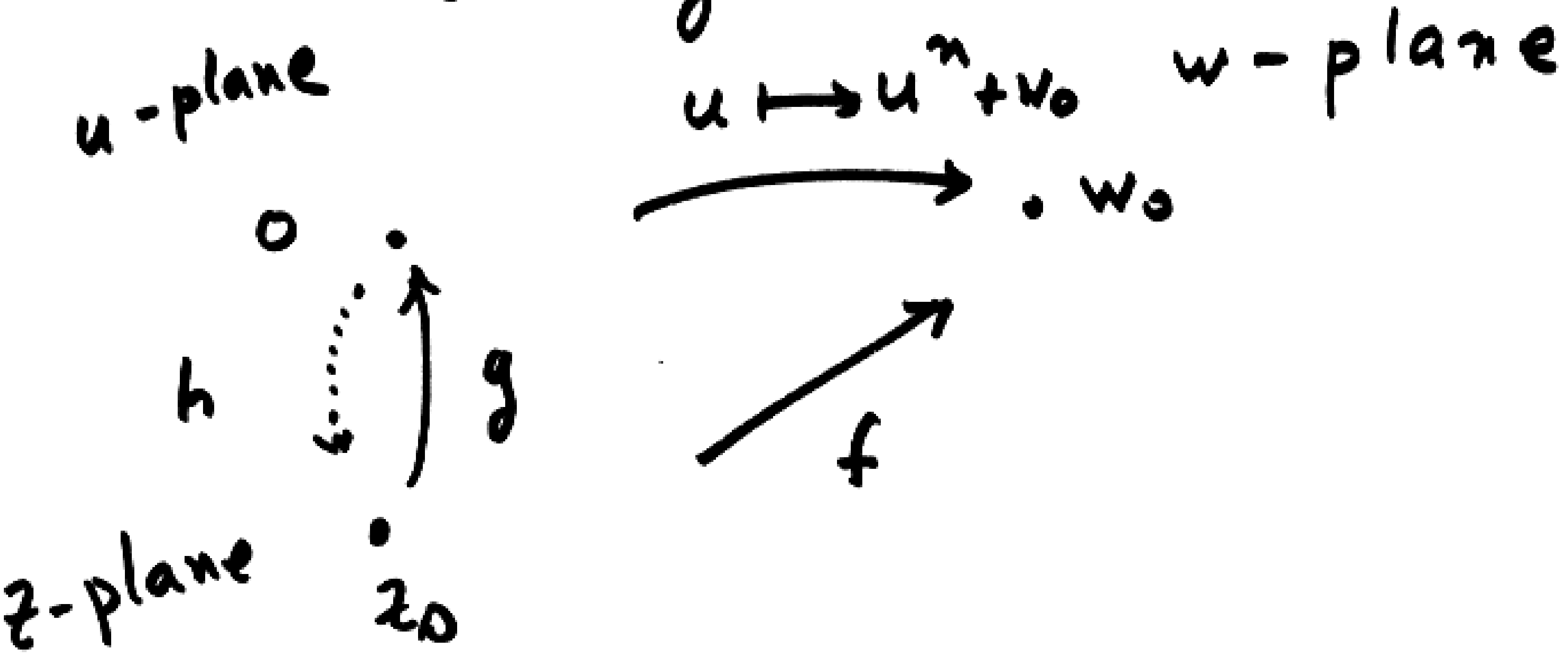
$$f(z) - f(z_0) = \Big(\underbrace{(z - z_0) \cdot f_n^{1/n}(z)}_{g}\Big)^n$$

$g(z_0) = 0$ multiplicity 1

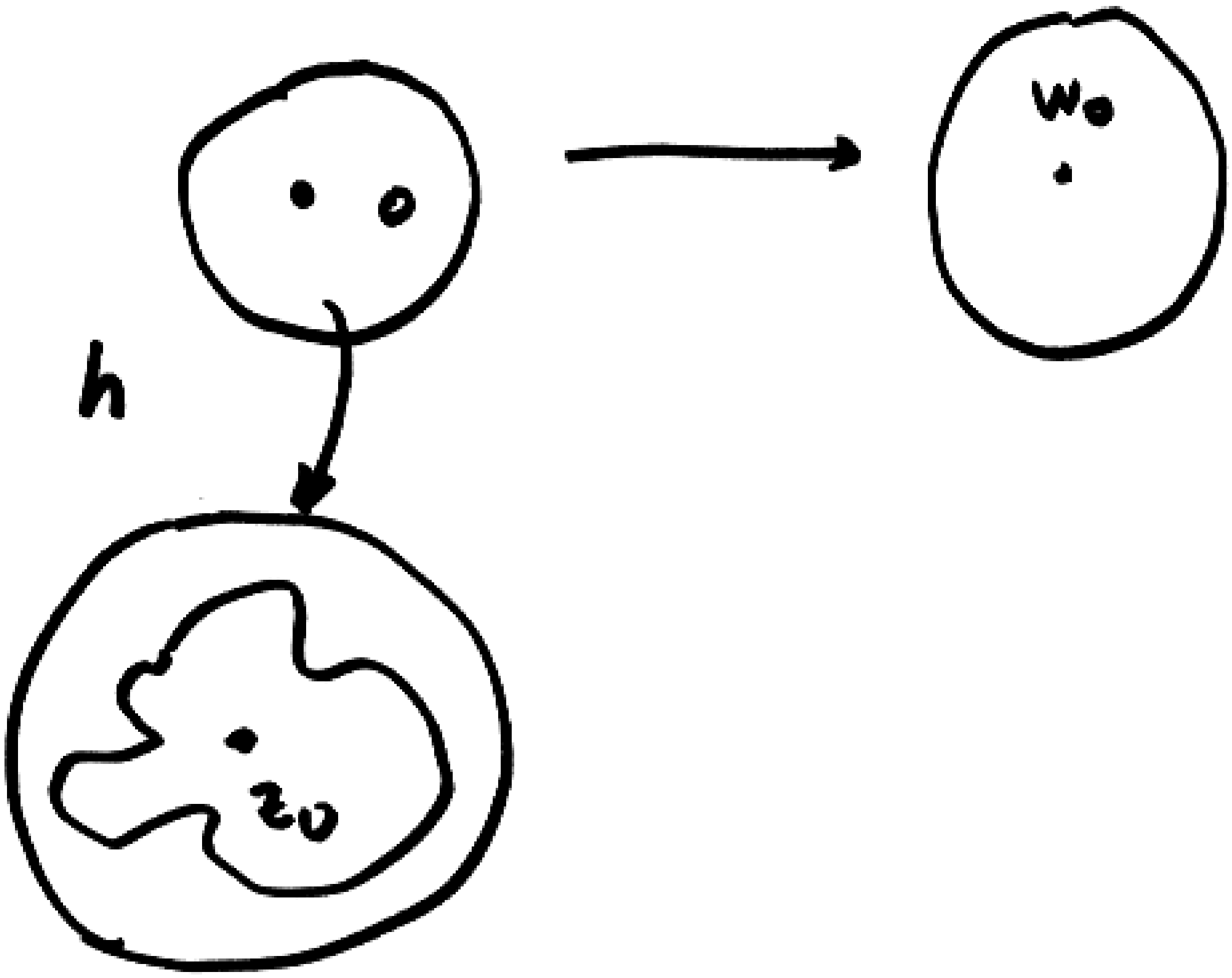
$$w = f(z)$$

$$w - w_0 = u^n$$

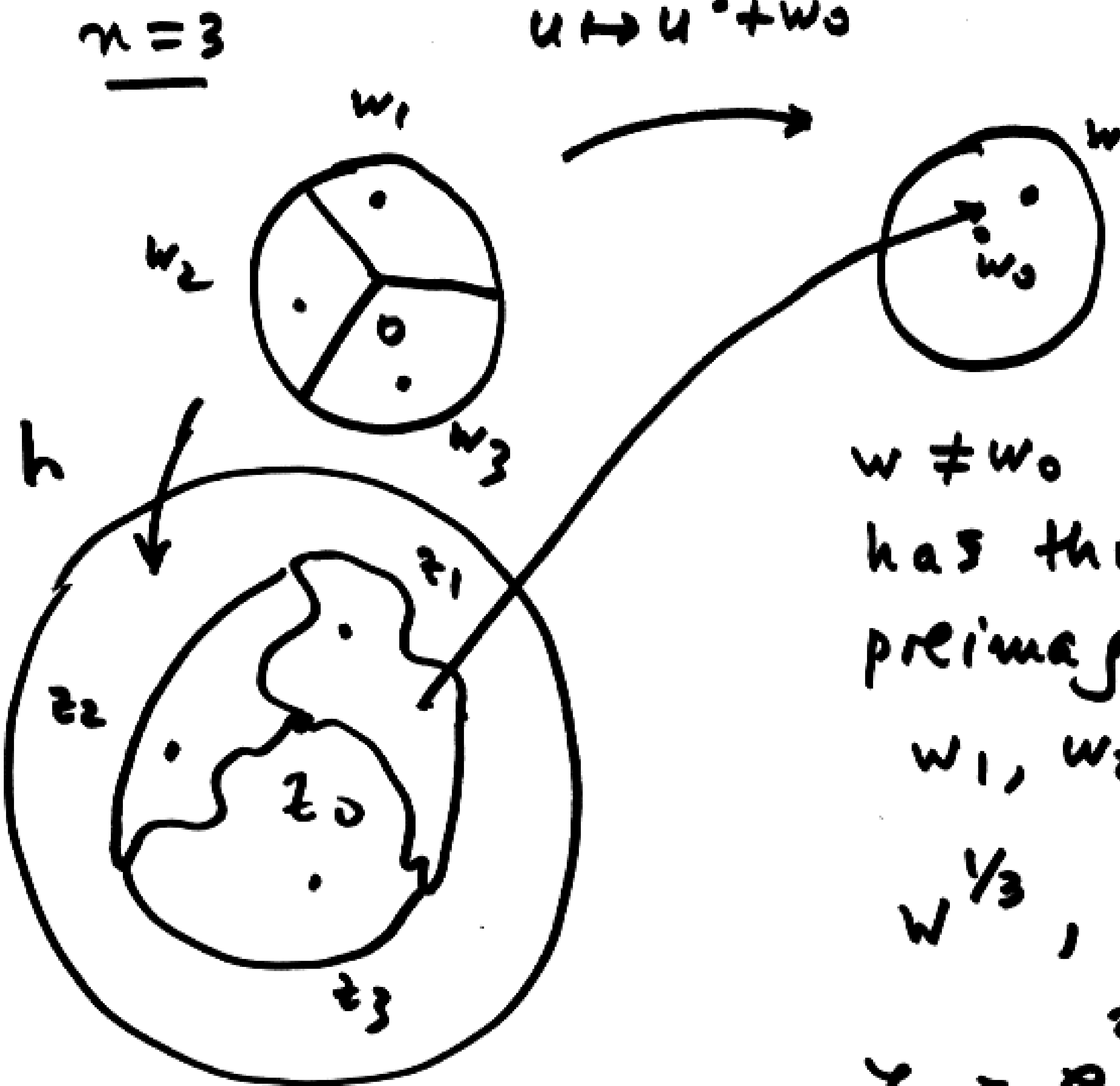
$$u = g(z)$$



g has locally an inverse h



$\underline{n = 3}$



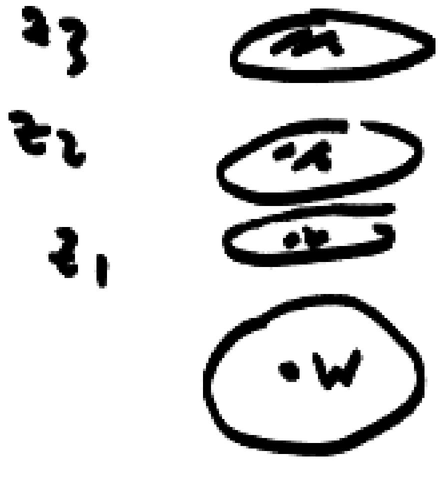
$w \neq w_0$ has three distinct preimages

w_1, w_2, w_3

$w^{1/3}, \zeta_3 w^{1/3}, \zeta_3^2 w^{1/3}$

$\zeta_3 = e^{2\pi i/3}$

$z_i = h(w_i)$



covering.

$w \neq w_0$

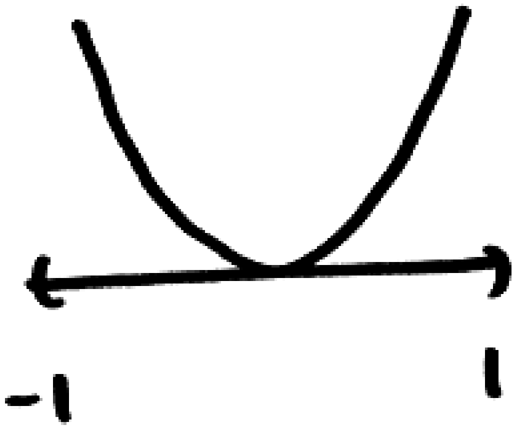
We can't have an inverse at a point with multiplicity > 1.

- $f : U \to \mathbb{C}$ analytic
  non-constant U domain

f takes open sets to open sets.

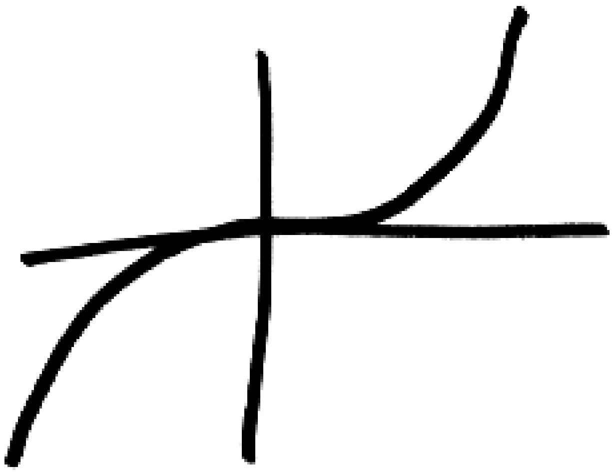
$\sin x$

x^2



takes $(-1, 1)$ to $[0,1)$

- For real functions
  e.g. $f(x) = x^3$ multiplicity 3 at 0
  which has inverse $x^{1/3}$



Let $V \subseteq U$ be an open set

$z_0 \in V$ $w_0 = f(z_0)$

We can find a nbhd of w_0 and one of z_0 s.t. $f(z) = w$

has n- solutions z, w in the corresponding nbhds.

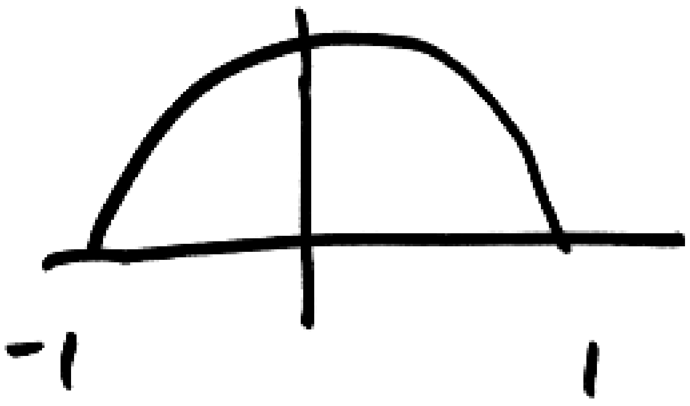
i.e. $w_0 \in V$ has a nbhd entirely contained in V.

- **Maximum principle**

$f: U \to \mathbb{C}$ f analytic
U domain

then if $|f(z)|$ attains its maximum on U then f is constant.

$1 - x^2$



By open mapping theorem

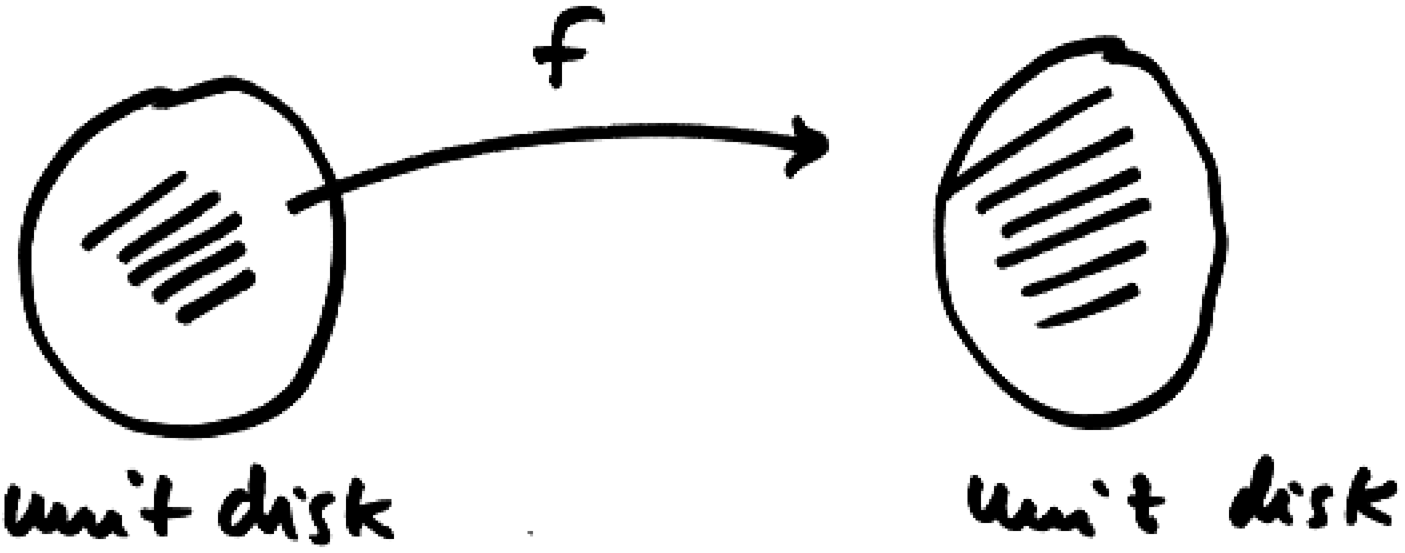
V = image of U by f

This disk contains points with larger absolute value.

(Another pf using Cauchy's fmla)

Schwarz Lemma

f analytic $|z| < 1$



$|f(z)| \leq 1$ for $|z| < 1$

$f(0) = 0$

Then

$$|f(z)| \leq |z|$$

and $|f'(0)| \leq 1$

Pf

$$f_1(z) := \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

is analytic in $|z| < 1$

want to prove

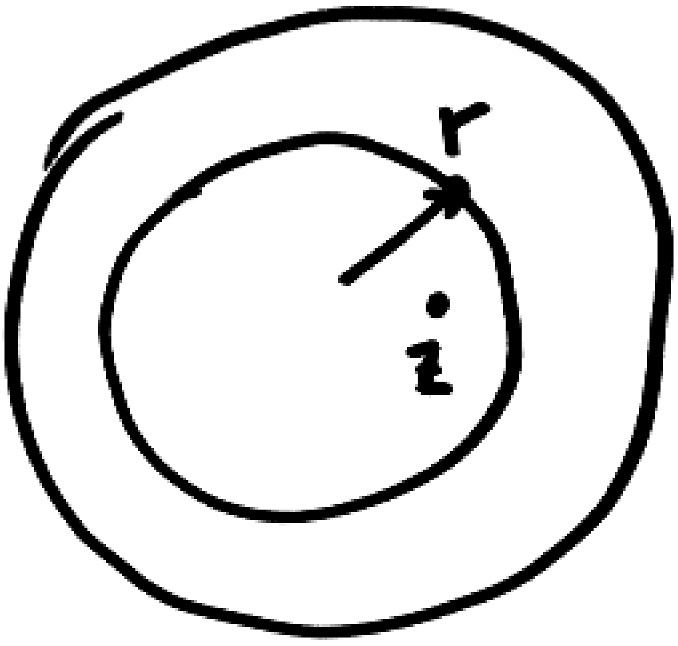
$$|f_1(z)| \le 1$$

take $|z| \le r < 1$

$|f_1(z)|$ takes its maximum on boundary

$$|f_1(z)| = \frac{|f(z)|}{|z|} \le \frac{1}{r}$$ [redacted] $|z| \le r$

$r \to 1$ $\quad |f_1(z)| \le 1.$



$$\Rightarrow \quad |f(z)| \le |z|$$

$$\& \quad |f'(0)| \le 1 \qquad \square$$

pblm

f entire

$$M(r) := \max_{|z|=r} |f(z)|$$

If $\dfrac{M(r)}{r^n} \to 0$ as $r \to \infty$

for some n then

f is a polynomial of deg < n

Cauchy fmla

$$\frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{m+1}}\, dz$$

$$m \geqslant n \qquad \leq \frac{M(r)}{r^m} = \frac{M(r)}{r^n} \frac{1}{r^{m-n}}$$

let $r \to \infty$ $\quad \dfrac{f^{(m)}(0)}{m!} = 0$

$m \geqslant n$

since f is entire it equals

its Taylor series which
such a polynomial.

March 3, 2006

f, g entire no common zeros

$\exists$ a, b entire

$$af + bg = 1$$

$$\frac{a}{g} + \frac{b}{f} = \frac{1}{f \cdot g}$$

Consider f, g polynomials

partial fraction expansion

$$\frac{1}{f \cdot g} = \sum_{\nu} h_{\nu}$$

h_ν has only one pole

i.e. $\dfrac{\alpha_k}{(z - a_k)^k}$ α_k polynomial of deg $< k$

or a polynomial.

Because f & g have no common zeros we can write the sum as

$$\sum_{\nu} f_{\nu} + \Big(\sum_{\mu} g_{\mu} + \text{polynomial} \Big)$$

poles of f_ν ⟷ zeros of f

poles of g_μ ⟷ zeros of g

$$\sum_\nu f_\nu = \frac{\tilde{a}}{f}$$

$\tilde{a}$, b polynomials

$$\text{polyn} + \sum_\mu h_\mu = \frac{\tilde{b}}{g}$$

$$\frac{\tilde{a}}{f} + \frac{\tilde{b}}{g} = \frac{1}{fg}$$

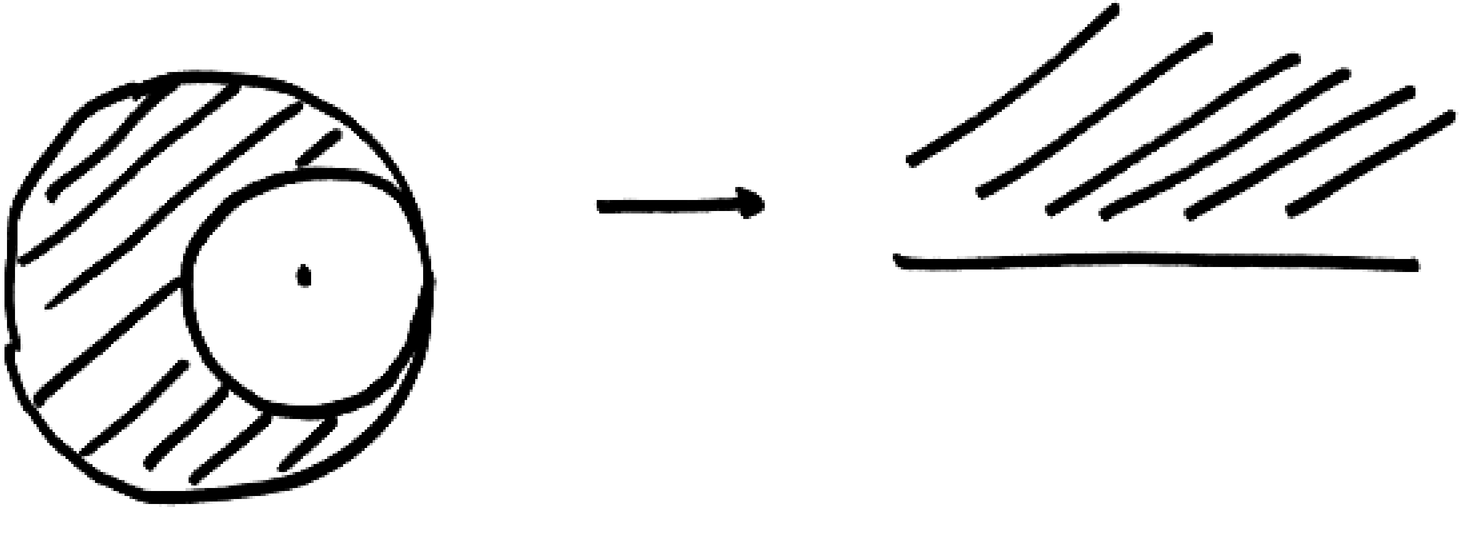
$$\Rightarrow \quad ag + bf = 1$$

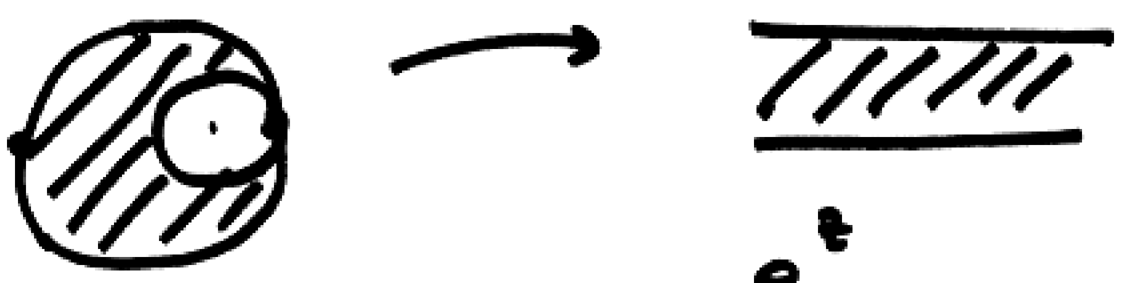
$$\{ z \mid \quad |z| < 2 \quad |z-1| > 1 \}$$

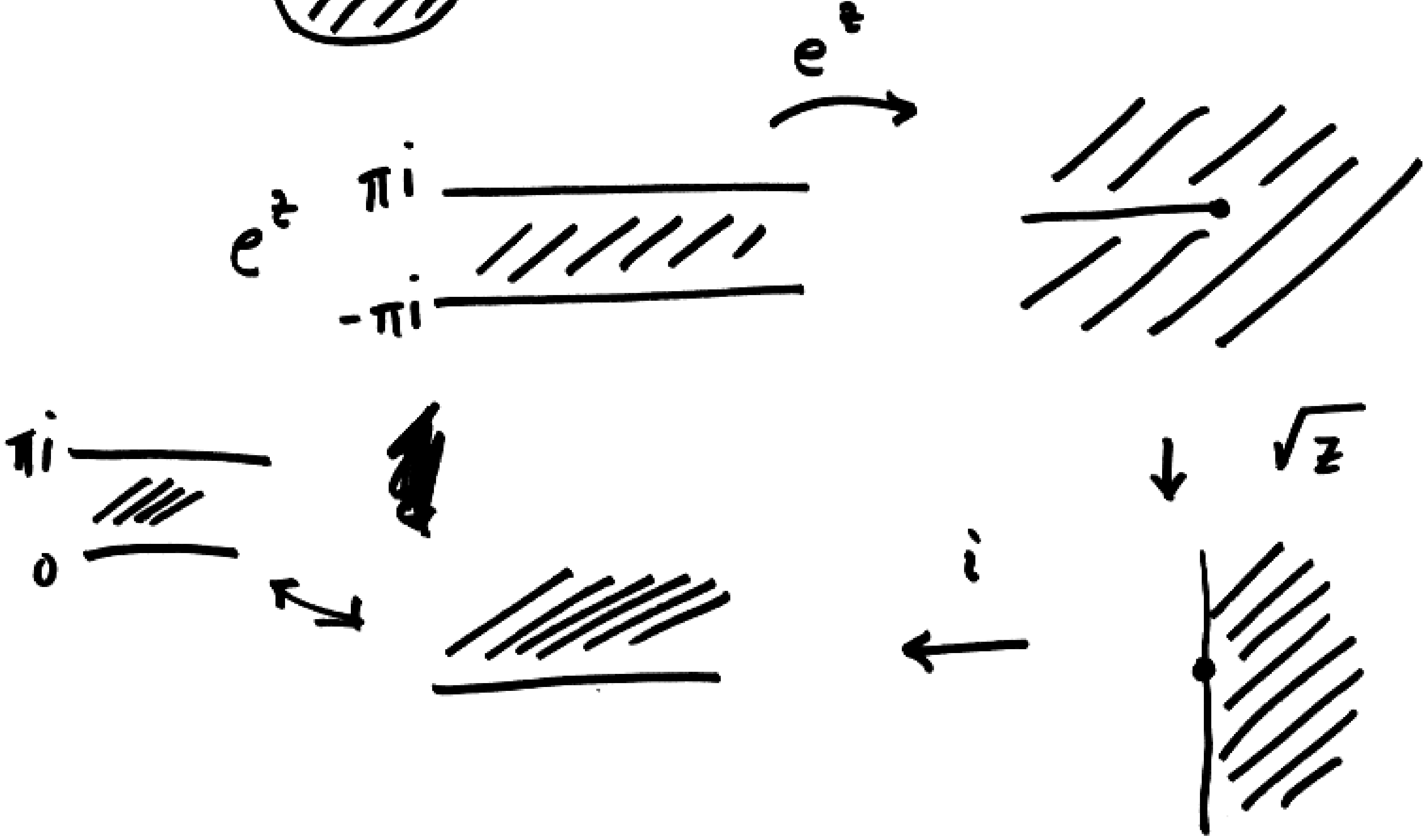
$$\downarrow$$

$$\{ z \mid \mathrm{Im}\, z > 0 \}$$

conformal homeo.







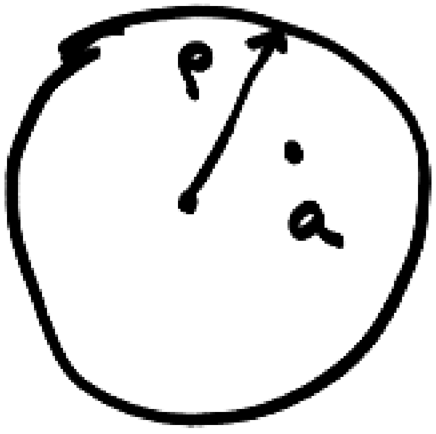
$$\frac{1}{i}\ \frac{z+2}{z-2}$$

$z = 2i$

$$\frac{2+i2}{2i-2} = \frac{1+i}{i-1} = -i$$

$$\frac{az+b}{z-2}$$

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^4}$$



$|z|=1$

$$|z-a|^4 = (z-a)^2(\bar{z}-\bar{a})^2$$
$$= (z-a)^2(z^{-1}-\bar{a})^2$$

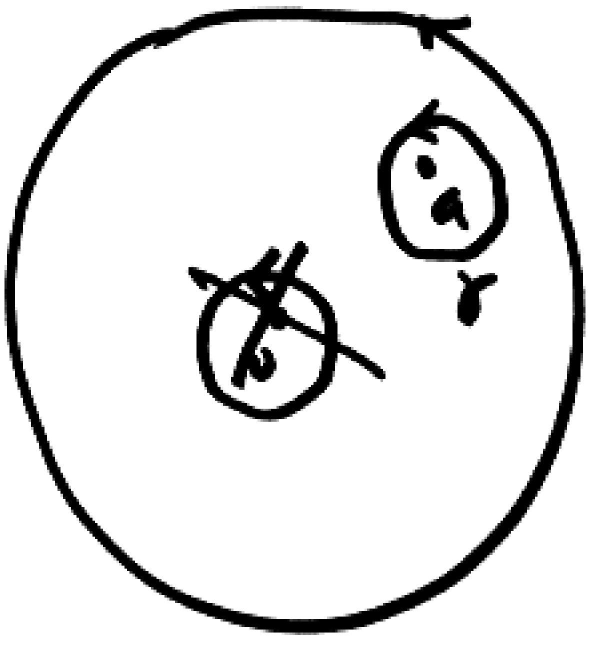
$$|dz| = -i\,\frac{dz}{z}$$

$$z = e^{i\theta}$$
$$dz = i e^{i\theta}\,d\theta$$

$$-\frac{dz}{iz} = d\theta$$

$$\frac{-1}{i}\int_{|z|=1} \frac{dz}{z(z-a)^2(z^{-1}-\bar{a})^2}$$

$$1-\bar{a}z \qquad \frac{1}{\bar{a}} = \frac{a}{|a|^2}$$



$$\frac{1}{2\pi i}\int_{\gamma} \frac{z\,dz}{(z-a)^2(1-\bar{a}z)^2}$$

$$f(z) = \frac{z}{(1-\bar{a}z)^2}$$

analytic at a

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)\, dz}{(z-a)^2}$$

$$= \frac{f'(a)}{1!}$$

similarly with a outside

$\rightarrow$ $1/\bar{a}$ is inside ...

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$z = e^{i\theta} \qquad = \frac{z + z^{-1}}{2}$$

$$= \frac{z - z^{-1}}{2i}$$

$$R(\cos\theta, \sin\theta) \rightsquigarrow R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)$$

* * *

f entire

For every $a \in \mathbb{C}$ the power series expansion of f at a has at least one zero coeff.

$\Rightarrow$ f is a polynomial.

$$\mathbb{C} = \bigcup_{n \geq 0} \{ z \mid f^{(n)}(z) = 0 \}$$

one of them is uncountable say $f^{(n)}$

hence $f^{(n)} \equiv 0$

Cplx Analysis SPRING '06

MIDTERM

Name: ____________________

Please give your answers in excruciating detail.

---

1. Let H be the ring of entire functions.

(a) Characterize the units of H and show there are infinitely many. non-constant units.

(b) Show that H is an integral domain (i.e., $f \cdot g = 0 \Rightarrow f = 0$ or $g = 0$)

(c) For $a \in \mathbb{C}$, prove $z - a \in H$ is irreducible (h is irreducible iff $h = f \cdot g \Rightarrow f$ or g are units)

(d) Show any irreducible of H is of the form $u(z-a)$ for some $a \in \mathbb{C}$ and some unit u in H.

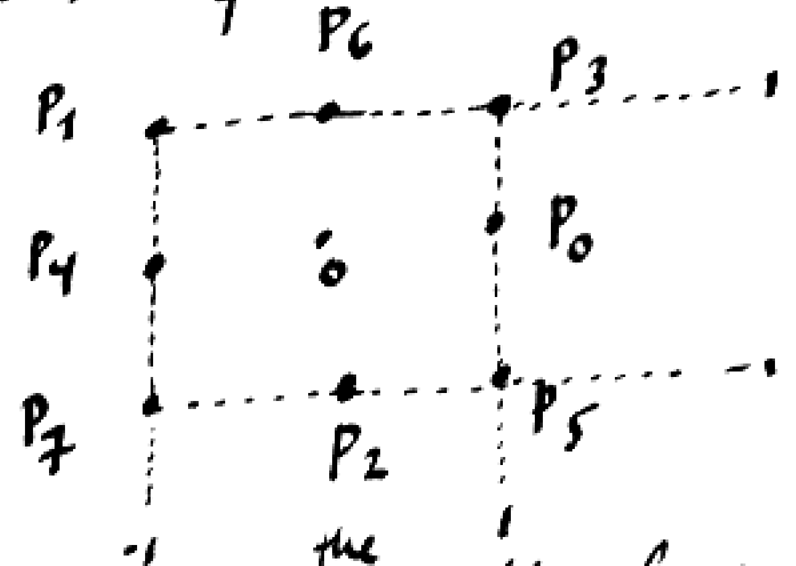
(e) Prove that not every $h \in H$ is a finite product $h = h_1 \cdots h_n$ of irreducibles h_i.

2. Let U be a domain and $f: U \to \mathbb{C}$ an analytic function. Prove that if f is injective then it has an analytic inverse

$$g : f(U) \to \mathbb{C}$$

$$g \circ f(z) = z$$

3. Consider the following points P_i on the unit square $i = 0, 1, \dots, 7$



Let γ be the path formed of segments

$$\overline{P_0 P_1}, \overline{P_1 P_2}, \dots, \overline{P_6 P_7}, \overline{P_7 P_0}$$

in this order.

Compute

$$\int_\gamma \frac{\cos z \cdot e^{z^2}}{\sin z \cdot (z-2)} \, dz$$

4. Let $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ be a power series with radius of convergence $R > 0$. Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$ be the power series with coefficients, given recursively by

$n \geq 1$, $$c_n = -(a_n + a_{n-1} c_1 + \dots + a_1 c_{n-1})$$

Let $\mu(r) := \sup_{|z|=r} |f(z)|$ for $r < R$.

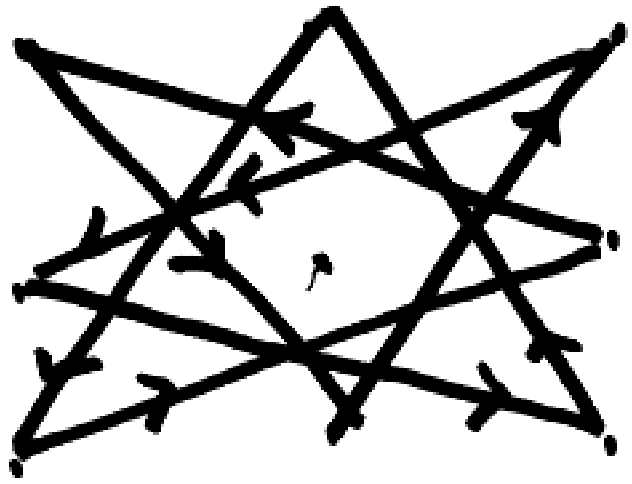
(a) Show that $f(z) \cdot h(z) = 1$ as formal power series.

(b) Show that h has radius of convergence at least $\frac{r}{1+\mu(r)}$ (HINT: Show $|c_k| \leq \frac{\mu(r)}{r} \left(\frac{1+\mu(r)}{r} \right)^{k-}$

$R > r > 0$

March 8, 2006

3.



$$\int_\gamma \frac{\cos z \cdot e^{z^2}}{\sin z\,(z-2)}\,dz$$

$\sin 0 = 0$

$\sin(\pm\pi) = 0$

$$f(z) = \frac{\cos z \cdot e^{z^2}}{\frac{\sin z}{z}(z-2)}$$

$$\int_\gamma \frac{f(z)}{z}\,dz = 3 \times 2\pi i \times f(0)$$

$$= 6\pi i \cdot \frac{1}{-2} = -3\pi i$$

4. $$f(z) = \overset{a_0=}{1} + a_1 z + a_2 z^2 + \dots$$

$$R > 0$$

$$\frac{1}{f(z)} = \underset{=c_0}{1} + c_1 z + c_2 z^2 + \dots$$

$$1 = (1 + a_1 z + a_2 z^2 + \dots)(1 + c_1 z + c_2 z^2 + \dots)$$

$$\sum_{k=0}^{n} a_{n-k} c_k = 0 \qquad \text{for } n \geq 1$$

$$c_n = -(a_n + a_{n-1} c_1 + \dots + a_1 c_{n-1})$$

$$\mu(r) := \sup_{|z|=r} |f(z)| \qquad 0 < r < R$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

$$|a_n| \leq \mu(r) \frac{1}{r^n}$$

Assume

$$|c_k| \le \frac{\mu(r)}{r}\left(1+\frac{\mu(r)}{r}\right)^{k-1}$$

$$|c_n| \le \frac{\mu(r)}{r^n} + \sum_{k=1}^{n-1} \frac{\mu(r)}{r^{n-k}} \cdot \frac{\mu(r)}{r}\left(1+\frac{\mu(r)}{r}\right)^{k-}$$

$$= \frac{\mu(r)}{r^n} + \sum_{k=1}^{n-1} \frac{\mu(r)^2}{r^n} \cdot (1+\mu(r))^{k-}$$

$$= \frac{\mu(r)}{r^n}\left[1 + \mu(r)\,\frac{(1+\mu(r))^{n-1}-1}{1+\mu(r)-1}\right]$$

$$= \frac{\mu(r)}{r^n}\,(1+\mu(r))^{n-1}$$

$$= \frac{\mu(r)}{r}\left(\frac{1+\mu(r)}{r}\right)^{n-1}$$

$\underline{K=1}$ $\qquad \underset{\overset{\parallel}{|a_1|}}{|c_1|} \le \dfrac{\mu(r)}{r}$

$$\frac{1}{1\mp x} = 1 \pm x + O(x^2)$$

Proved the hint

$$|C_k| \leq \frac{\mu(r)}{r}\left(\frac{1+\mu(r)}{r}\right)^{k-1}$$

$$\frac{1}{R} = \limsup_{k\to\infty} |C_k|^{1/k} \leq \frac{1+\mu(r)}{r}$$

$$R \geq \frac{r}{1+\mu(r)} \qquad 0 < r < R$$

$$\boxed{R \geq \sup_{0<r<R} \frac{r}{1+\mu(r)}} \qquad > 0$$

$$f(z) = \frac{1}{z-1}$$

$$\frac{1}{f(z)} = z-1 \qquad R = \infty$$

2. $f : U \to \mathbb{C}$

analytic injective

Prove it has analytic inverse

$g : f(U) \to \mathbb{C}$

[illegible] $g \circ f(z) = z$

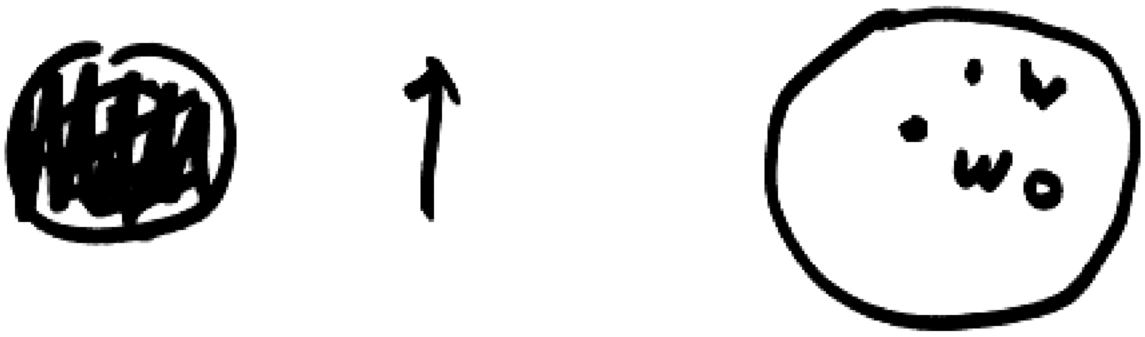
If $f'(z_0) = 0$ then f cannot be injective on a nbhd of z_0

$f'(z_0) = 0 \longrightarrow f(z) - \cancel{z_0} = f(z_0)$

has a zero of order at least 2

$= n$

$f(z) = w_0 \qquad w_0 \neq z_0$



n preimages

⑥

1. (a) u entire fctn

unit $uv = 1$

for some v entire.

$\Leftrightarrow$ u does not vanish in $\mathbb{C}$

$e^z, \quad e^{f(z)} \quad \ldots$

(b) $f \cdot g = 0 \Rightarrow f = 0$ OR $g = 0$

$f(z_0) \neq 0 \Rightarrow f(z) \neq 0$ on disk about z_0

$\Rightarrow g = 0$ on disk

$\Rightarrow g = 0$ in $\mathbb{C}$.

(c) $z - a = f \cdot g$

either f or g are units.

(d)

(e). $h_1 \cdots h_n = h \qquad e^z - 1$

March 10, 2006

①

$$f(z) = 1 + a_1 z + a_2 z^2 + \dots$$

Radius of convergence ≥ 1

$$|a_n| \leq 1$$

$$f(z) \neq 0 \quad \text{for} \quad |z| \leq 0.1715\dots$$

If $f(z) \neq 0$ on $|z| \leq r$

then $\dfrac{1}{f(z)}$ is analytic there

hence ~~its~~ ~~ra~~ the radius of convergence of the series

$$g(z) = \frac{1}{f(z)} = 1 + c_1 z + c_2 z^2 + \dots$$

is at least r

R = Radius of convergence of g is

$$\geq \frac{r}{1+\mu(r)} \quad , \ \mu(r) = \max_{|z|=r} |f(z)|$$

$|z| = r$

$$|f(z)| \leq 1 + |a_1| r + |a_2| r^2 + \cdots$$

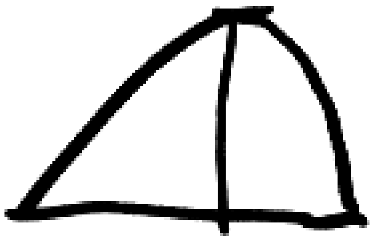
$$\leq \frac{1}{1-r} \qquad r < 1$$

$$\Rightarrow \quad \mu(r) \leq \frac{1}{1-r}$$

$$0 < r < 1$$

$$R \geqslant \frac{r}{1 + \frac{1}{1-r}}$$

$$= \frac{r(1-r)}{2-r} = \frac{(r-1)r}{(r-2)}$$



$$\text{deriv} = \frac{r^2 - 4r + 2}{(r-2)^2}$$

$$\frac{(2r-1)(r-2) - r(r-1)}{(r-2)^2}$$

max occurs at root of $r^2 - 4r + 2$ in $[0, 1]$ $r = 0.5857\ldots$

max $= 0.171528\ldots$

polynomials? $1 + a_1 z$

zero at $z = -\frac{1}{a_1}$.

—m—

$$f(z) = z e^z = z + z^2 + \frac{z^3}{2} + \cdots$$

$$g(z) = z + b_2 z^2 + \cdots$$

$$f \circ g(z) = z$$

Identify b_n?

$$b_n = \frac{(-n)^{n-1}}{n!}$$

$$f = a_1 z + a_2 z^2 + \cdots, \quad a_1 \neq 0$$

$$f_1(z) = a_1 + a_2 z + \cdots = \frac{f(z)}{z}$$

$$\frac{1}{f_1(z)} = c_0 + c_1 z + \cdots$$ radius of convergence $R>0$

Cauchy estimates

$$|c_n| \leq \frac{M_1(r)}{r^n} \qquad n = 0, 1, 2 \ldots$$

$0 < r < R$

$$\mu_1(r) := \max_{|z|=r} \left| \frac{1}{f_1(z)} \right|$$

$k = 1, 2, \ldots$

$$\frac{1}{f_1(z)^k} = c_0^{(k)} + c_1^{(k)} z + \cdots$$

$$|c_m^{(k)}| \leq \frac{\mu_1(r)^k}{r^m}$$

$$g(z) = b_1 z + b_2 z^2 + \cdots$$

$$f \circ g(z) = z$$

$$\boxed{b_n = \frac{1}{n} c_{n-1}^{(n)}}$$

E.g. $f(z) = z e^z$

$f_1(z) = e^z$

$$\frac{1}{f_1(z)^k} = e^{-kz} = 1 - \frac{k}{1!} z + \frac{k^2}{2!} z^2 + \cdots$$

$$C_n^{(k)} = \frac{(-k)^n}{n!}$$

$$b_n = \frac{1}{n} C_{n-1}^{(n)} = \frac{1}{n} \frac{(-n)^{n-1}}{(n-1)!}$$

$$= \frac{(-n)^{n-1}}{n!}$$

Apply to our bound

$$|b_n| \leq \frac{1}{n} \frac{\mu_1(r)^n}{r^{n-1}}$$

$$\underset{n \to \infty}{\limsup} |b_n|^{1/n} \leq \frac{\mu_1(r)}{r}$$

$$\begin{array}{c} \| \\ \rho^{-1} \end{array}$$

$\rho :=$ radius of convergence of g

$$\Rightarrow \quad \rho \geq \frac{r}{\mu_1(r)} > 0$$

$$\mu_1(r) = \max_{|z|=r} \left| \frac{z}{f(z)} \right|$$

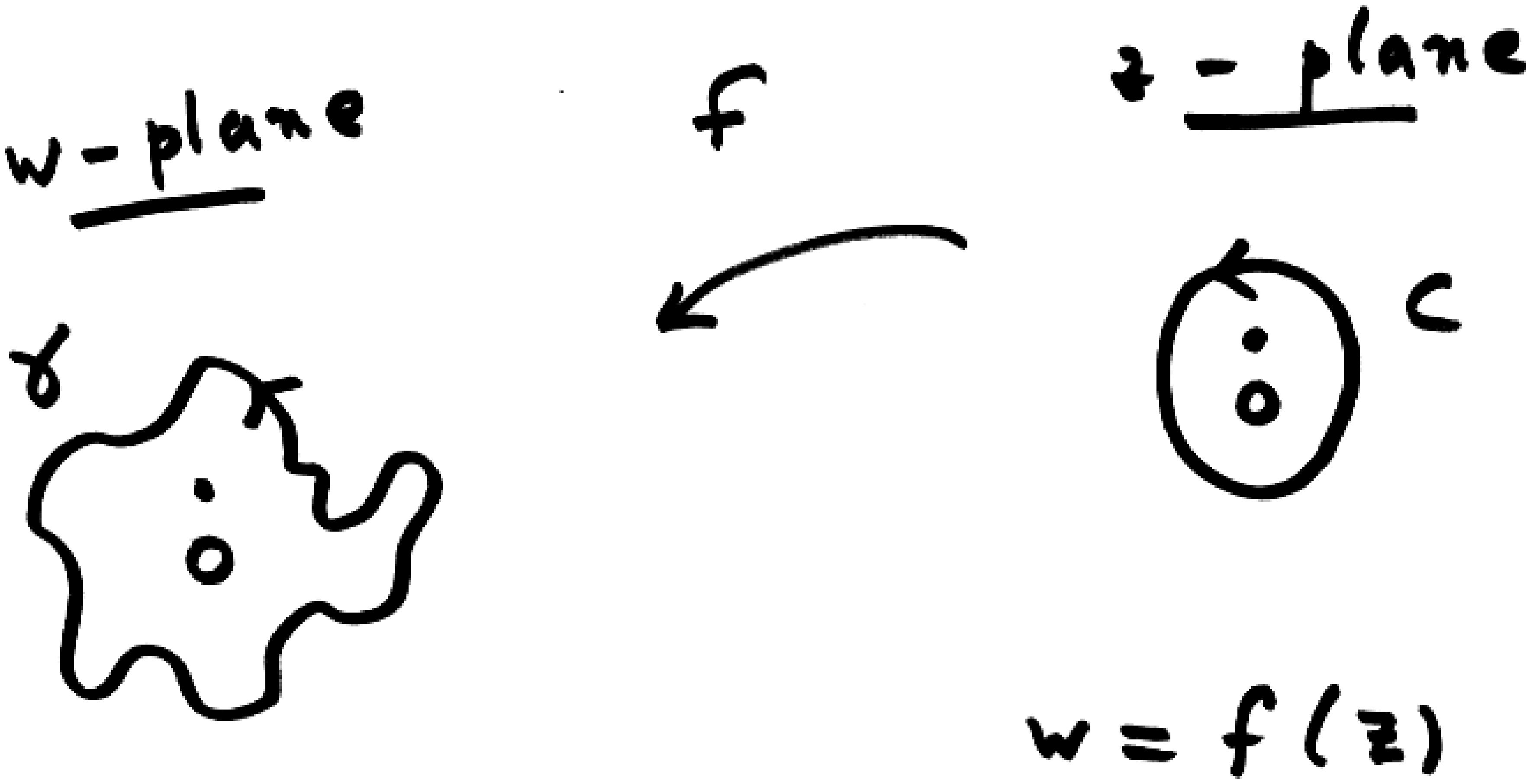
Proof of claim

Find coeff of $z \cdot g'(z) =: h(z)$

by Cauchy's fmla.

n^{th} coeff $= n b_n$

$$n b_n = \frac{1}{2\pi i} \int_\gamma \frac{h(w)\,dw}{w^{n+1}}$$



$w = f(z)$

$$\gamma(t) = f(C(t))$$
$$\gamma'(t) = f'(C(t)) \cdot C'(t)$$

$$n b_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{h(\gamma(t))}{\gamma(t)^{n+1}} \gamma'(t)\, dt$$

$$
\begin{aligned}
h(\gamma(t)) &= \gamma(t)\, g'(\gamma(t)) \\
&= f(c(t)) \cdot g'(f(c(t))) \\
&= \frac{f(c(t))}{f'(c(t))}
\end{aligned}
$$

$$\rightarrow \quad n b_n = \frac{1}{2\pi i}\int_C \frac{1}{f(z)^n}\, dz$$

$$= \sum_{m\geq 0} C_m^{(n)} \underset{\substack{\shortparallel \\ 0 \\ \text{unless } m-n+1=0}}{\frac{1}{2\pi i}\int_C z^{m-n+1}\frac{dz}{z}}$$

$$= C^n_{n-1} \qquad \square$$

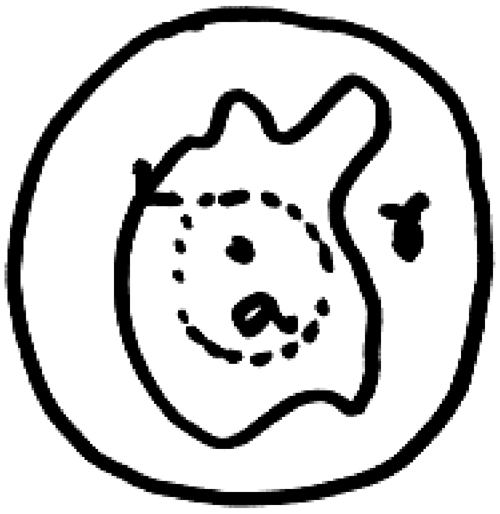
March 20, 2006

Residues

f analytic on a disk
and γ is closed path then

$$\int_{\gamma} f(z)\,dz = 0$$

f analytic on a disk centered
at a but not necessarily at a.



$$\alpha := \frac{1}{2\pi i} \int_{\gamma} f(z)\,dz$$

$$n(\gamma, a) = 1$$

Residue of f at a

$$\alpha =: \operatorname{Res}_{z=a} f$$

E.g. $$\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 1$$

$$\operatorname*{Res}_{z=a} \frac{1}{z-a} = 1$$

$$\operatorname*{Res}_{z=b} \frac{1}{z-a} = 0$$

$a \neq b$

E.g $$\operatorname*{Res}_{z=a} \frac{1}{(z-a)^2} = 0$$

$$\frac{1}{2\pi i}\int_C \frac{1}{(z-a)^2}\, dz = 0$$

$$\operatorname*{Res}_{z=a} \frac{1}{(z-a)^k} = 0$$

$k > 1$

$$f(z) = \frac{c_k}{(z-a)^k} + \frac{c_{k-1}}{(z-a)^{k-1}} + \cdots$$

$$+ \frac{c_{-1}}{(z-a)} + g(z)$$

g analytic

f pole of order k at $z=a$

$$\boxed{\operatorname*{Res}_{z=a} f = c_{-1}}$$

If f is ~~meromorphic~~ analytic on a disk about a and has possibly a pole at $z=a$

$$g = \frac{f'}{f}$$

$$\operatorname*{Res}_{z=a} g = k$$

④

$$f(z) = (z-a)^{K}\, h(z)$$

$h(a) \neq 0$ h analytic

$K \in \mathbb{Z}$ order of zero/pole

$$g = \frac{f'}{f} = \frac{K}{z-a} + \frac{h'(z)}{h(z)}$$

$$\operatorname*{Res}_{z=0} \frac{\cos z \cdot e^{z^2}}{\sin z \cdot (z-2)} =: f(z)$$

$$f(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + \dots$$

$$c_{-1} = \lim_{z \to 0} z\, f(z)$$

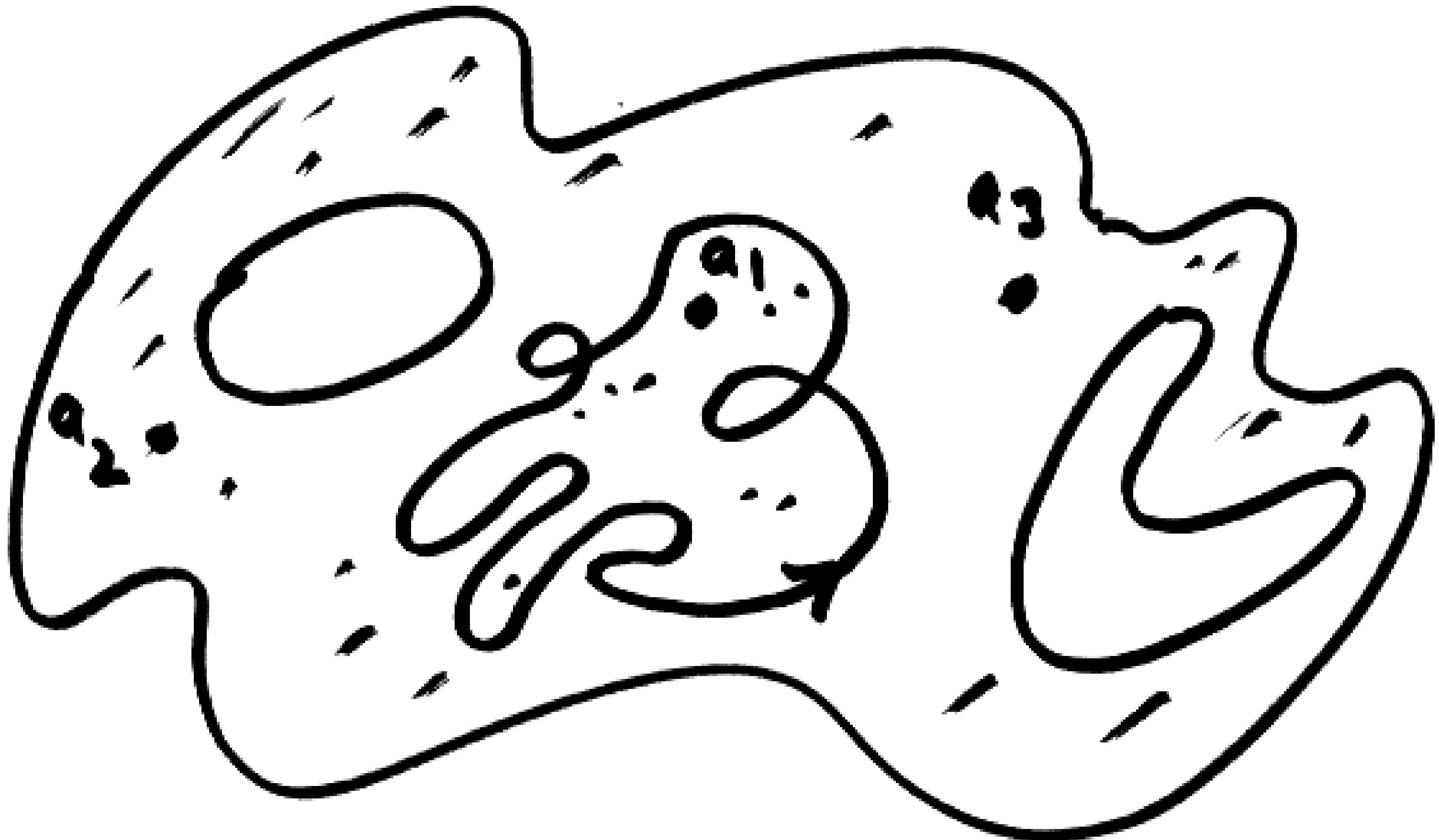
$$= \lim_{z \to 0} \frac{\cos z \cdot e^{z^2}}{\frac{\sin z}{z} \cdot (z-2)} = -\frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} (z-2) f(z)$$

$$= \frac{\cos 2 \cdot e^{4}}{\sin 2}$$

Residue Theorem

U region



γ closed path in U

$n(\gamma, a) = 0 \quad a \notin U$

$\gamma \sim 0$ in U

f analytic in U with ⑥

possibly exceptions $a_1, a_2 \ldots, a_N$

(γ not going through the a_i's)

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\,dz = \sum_{j=1}^{N} n(\gamma, a_j) \operatorname*{Res}_{z=a_j} f$$

March 22, 2006

f meromorphic

$$\operatorname*{Res}_{z=a} \frac{f'}{f} = \text{order of } f \text{ at } a = k$$

$$f(z) = (z-a)^k g(z)$$

$g(a) \neq 0$

g analytic about a

$k \in \mathbb{Z}$

Apply residue theorem to $\frac{f'}{f}$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f}(z)\, dz = \sum_{j=1}^{n} n(\gamma, a_j) \operatorname*{ord}_{z=a_j} f$$

Typical situation

f meromorphic

disk D

$\gamma = \partial D$

a_j zero/pole of f

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,d(z) = \#\text{zeros of } f - \#\text{poles of } f$$

(counting w/ multiplicity)

Argument principle

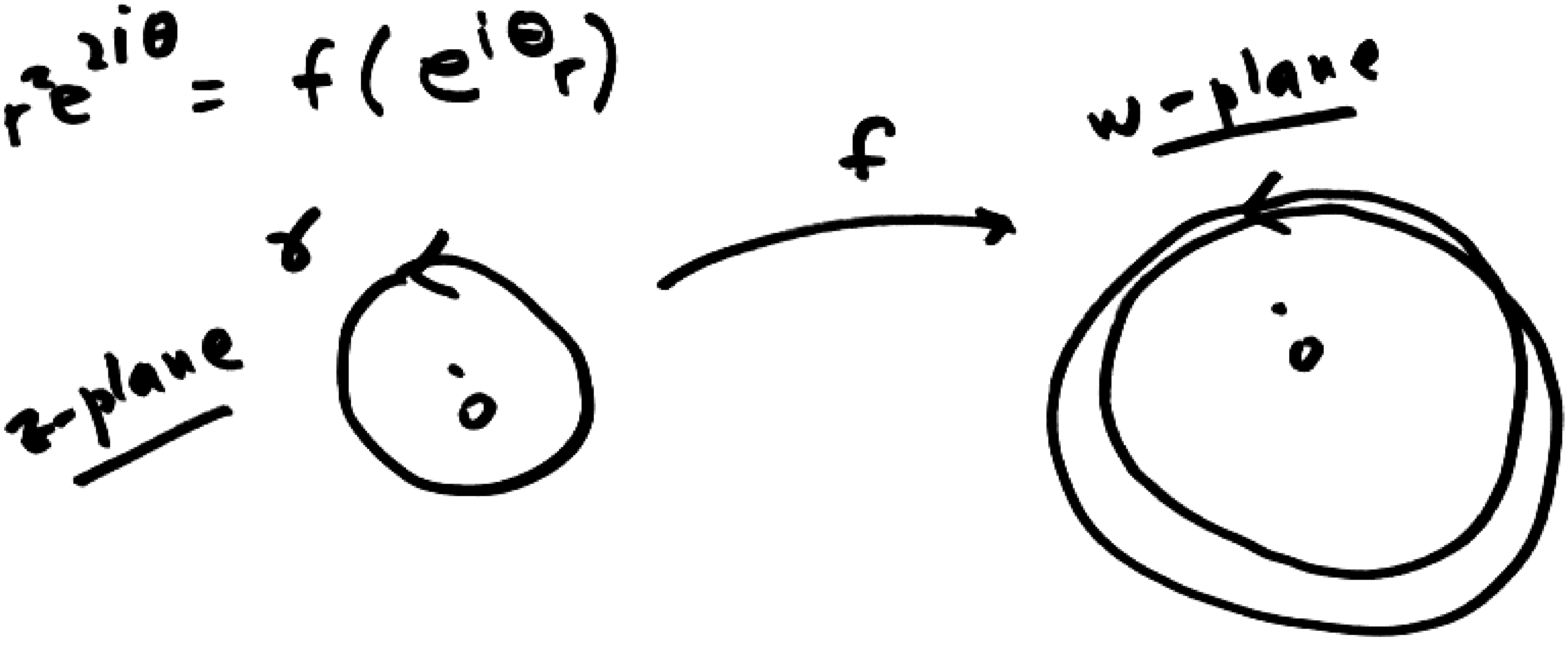
$$f(z) = z^2$$

$$\frac{f'}{f} = \frac{2}{z}$$

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{f'}{f}(z)\,dz = 2$$

$$z = r \cdot e^{i\theta}$$

$$r^2 e^{2i\theta} = f(e^{i\theta} r)$$



$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\int_{\sigma}\frac{\sigma'(t)dt}{\sigma(t)}$$

$$\sigma = f\circ\gamma$$

$$\sigma' = f'\circ\gamma\cdot\gamma'$$

$$= \frac{1}{2\pi i}\int_{\sigma}\frac{dw}{w}$$

$$= n(\sigma, 0)$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f}(z)\,dz = \frac{1}{2\pi i}\int_{a}^{b}\frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt$$

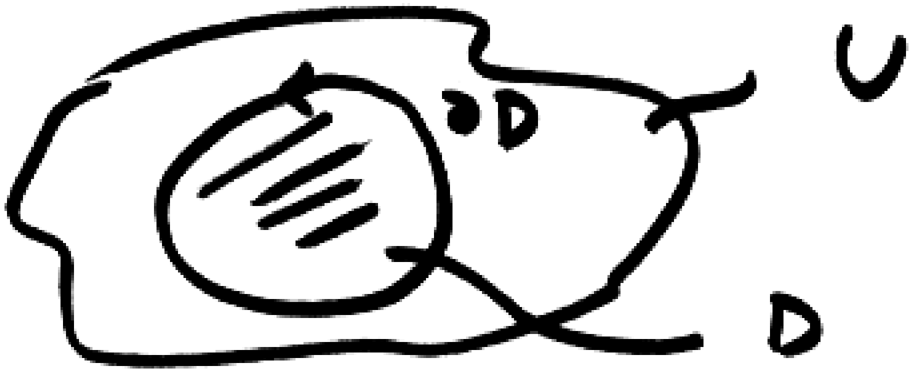
Rouché's theorem

$U \supseteq \overline{D}$

f, g analytic on disk

If $|f(z) - g(z)| < |f(z)|$

for $z \in \partial D$



then f, g have the same number of zeros on D

Pf $h(z) := \dfrac{g(z)}{f(z)}$

meromorphic on $U \supseteq \overline{D}$

~~[illegible]~~

$\#$ zeros of g $-$ $\#$ zeros of f

$= \#$ zeros of h $- \#$ poles of h

$= \dfrac{1}{2\pi i} \displaystyle\int \frac{h'}{h}(z)\, dz$

$= n(\sigma, 0)$ $\qquad \gamma = \partial D$

$\sigma = h \circ \gamma$

Need to show $n(\sigma, 0) = 0$

$|1 - h(z)| < 1 \qquad z \in \partial D$

$\Rightarrow \sigma : [a, b] \longrightarrow$

$\square$

$\Rightarrow \quad n(\sigma, 0) = 0$

March 24, 2006

Rouche's theorem

f, g analytic on $U \geq D$

on ∂D: $|f(z)-g(z)| < |f(z)|$

then f, g have same number of zeros on D.

Example

$g(z) = z^9 + 5z^3 + 2z + 1$

How many zeros does it have with $|z| \geq 1$ $\quad \bar{D} = \{|z| \leq 1\}$

$f(z) = 5z^3$

$|f(z) - g(z)| = |z^9 + 2z + 1| \leq 4$
$< |f(z)|$

on $|z| = 1$

f & g have the same number of zeros in $\bar{D}$ i.e. 3

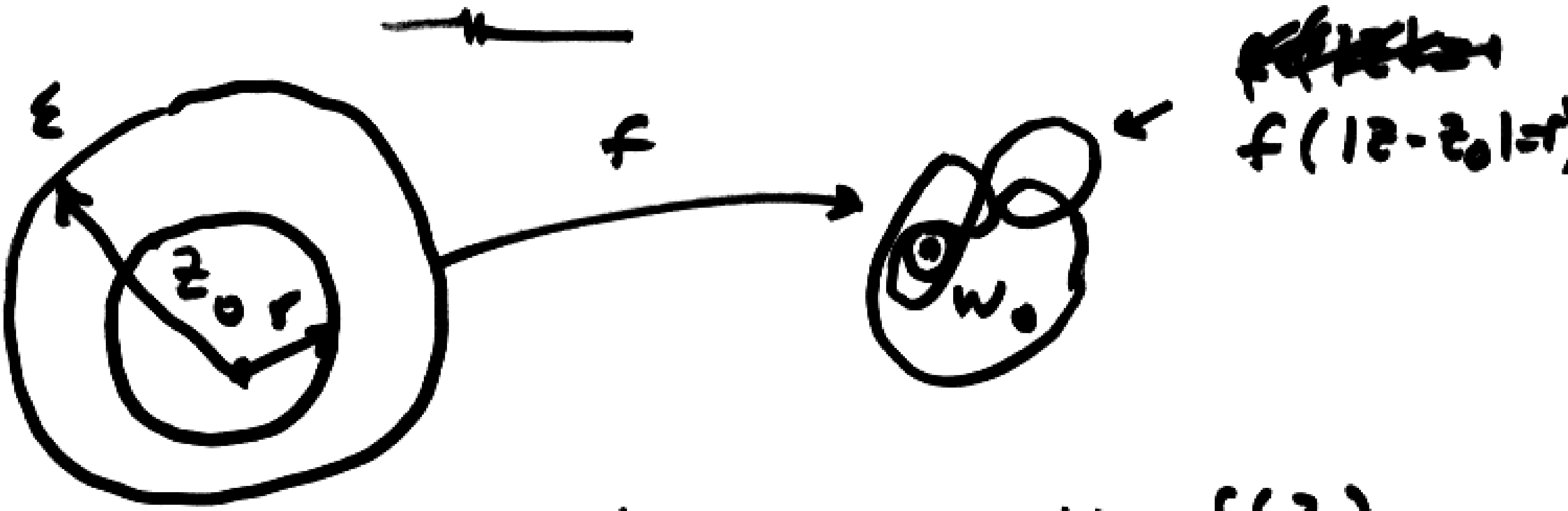
zeros of g on $|z|>1$ is 6

In general

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

If $|a_k| > \sum_{j \neq k} |a_j|$

then p has k zeros in $|z| \leq 1$



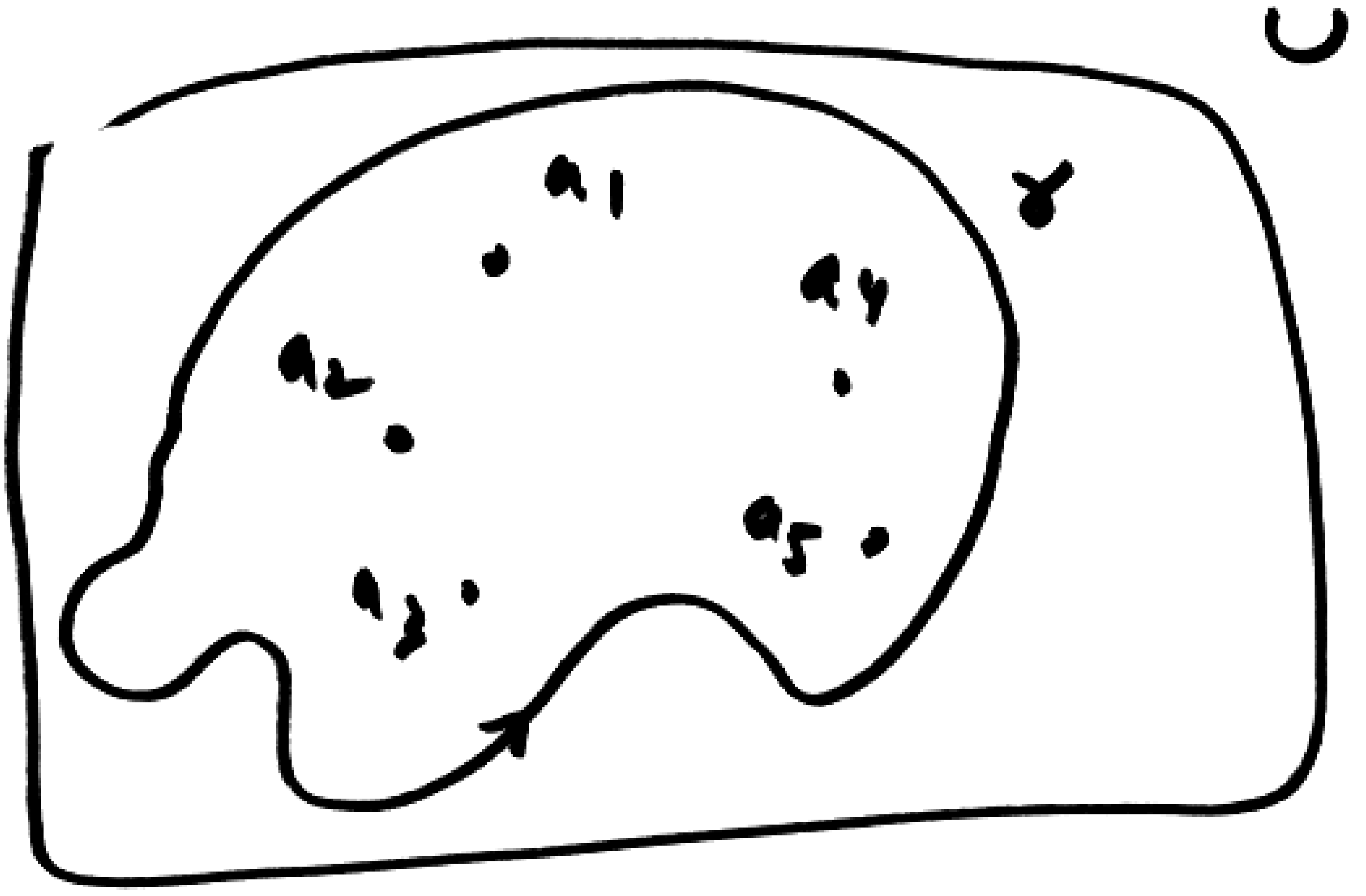
$w_0 = f(z_0)$ order $\underset{\geq 1}{n}$ $w_0 - f(z)$

given $\varepsilon > 0$ choose $0 < r < \varepsilon$

$f(z) - w_0 \neq 0$ $0 < |z - z_0| < r$

Pick $|w - w_0| < |f(z) - w_0|$, $|z - z_0| = r$

Residue theorem



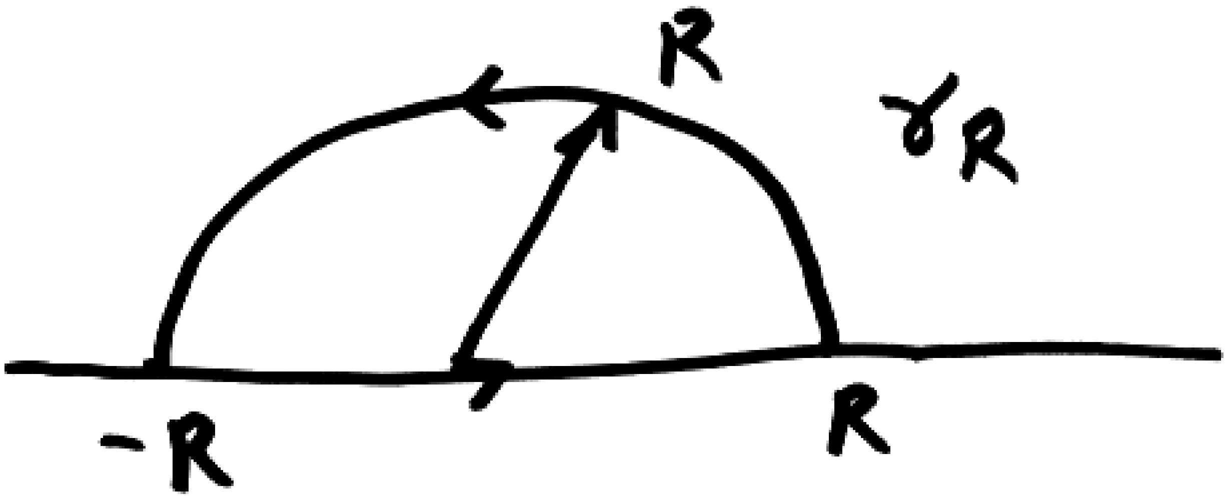
f analytic on U except for (possibly) singularities at $a_1, \dots, a_N$ (not on γ) in the interior of γ

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum_{i=1}^{N} \mathrm{Res}(f, a_i)$$

Application to real integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx = \frac{\pi}{e}$$

$$f(z) = \frac{e^{iz}}{z^2+1}$$



$$\int_{\gamma_R} f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f,i)$$

$$= 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$$

$R > 1$

$$\lim_{z \to i} \frac{(z-i)\,e^{iz}}{z^2+1} = \left.\frac{e^{iz}}{z+i}\right|_{z=i}$$

$$= \frac{e^{-1}}{2i}$$

Claim

$$\int f(z)\,dz \longrightarrow 0$$

as $R \longrightarrow \infty$

$$z = R e^{i\theta}, \qquad 0 \leq \theta \leq \pi$$

$$|e^{iz}| = |e^{Ri\cos\theta}|\,|~~R~~\, e^{-R\sin\theta}|$$

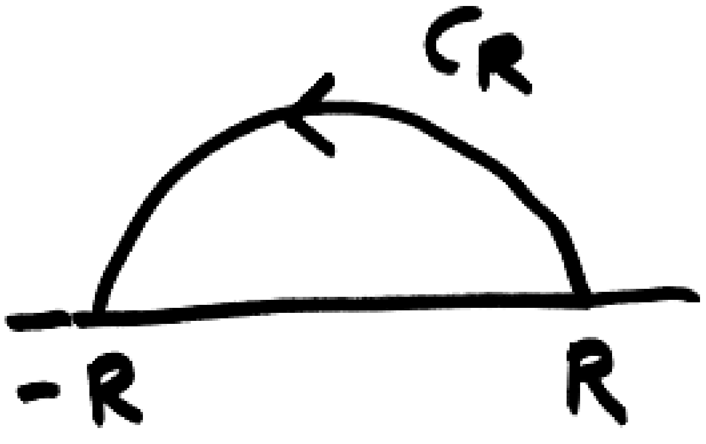
$$iz = R(i\cos\theta - \sin\theta)$$

$$= |e^{-R\sin\theta}|$$

Lemma (Jordan)

~~g(z) analytic on Im z ≥ 0~~

$$M_R := \max_{\substack{|z|=R \\ \operatorname{Im} z \geq 0}} |g(z)|$$



$$M_R \to 0$$

then

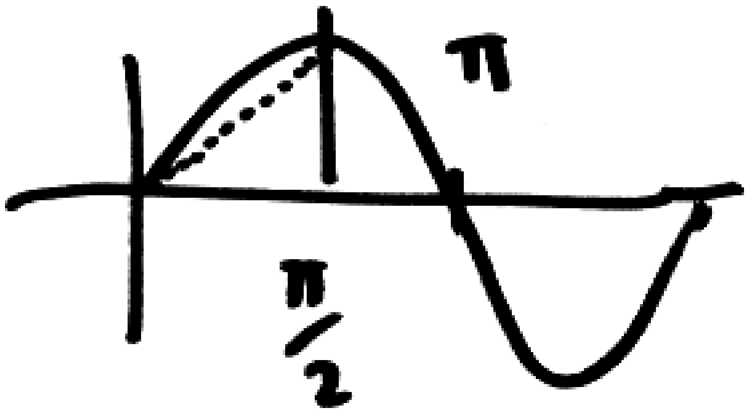
$$\lim_{R\to\infty} \int_{C_R} g(z) e^{i\alpha z}\, dz = 0$$

$$\alpha > 0$$

pf

$$\left| \int_{C_R} g(z)\, e^{i\alpha z}\, dz \right| \leq M_R\, 2 \int_{0}^{\pi/2} e^{-\alpha R \sin\theta}\, R\, d\theta$$

$$z = R e^{i\theta}, \quad 0 \leq \theta \leq \pi$$

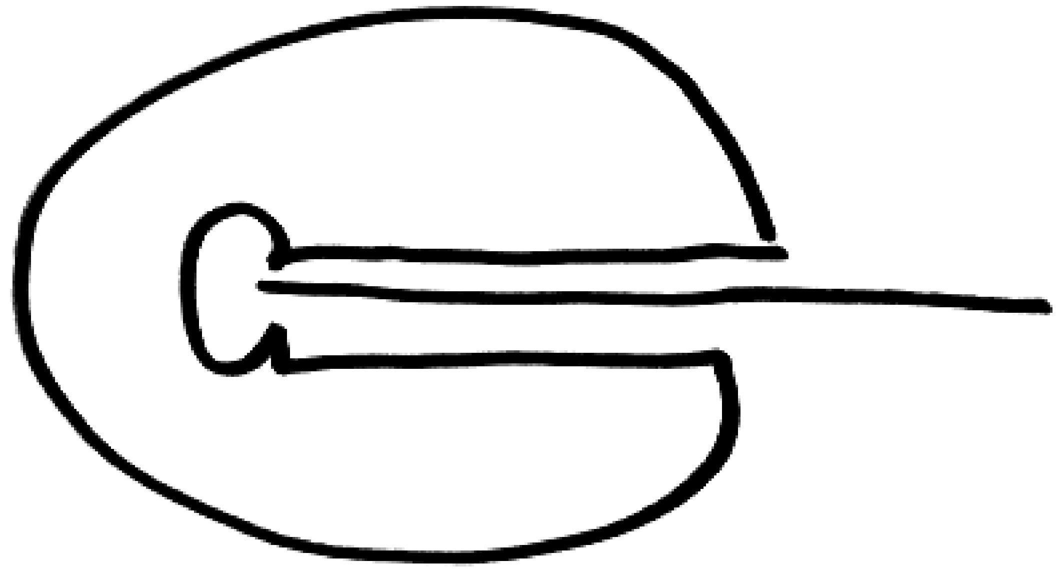


$$\sin\theta \geq \frac{2}{\pi}\theta \quad \text{on } 0 \leq \theta \leq \frac{\pi}{2}$$

because graph is convex

$$\int_0^{\pi/2} e^{-\alpha R \sin\theta}\, d\theta \leq \int_0^{\pi/2} e^{-\alpha \frac{2}{\pi} R\theta}\, d\theta$$

$$= \frac{-1}{\alpha \frac{2}{\pi} R}\, e^{-\alpha \frac{2}{\pi} R\theta} \Big|_0$$



$\square$

$$|(f(z)-w_0)-(f(z)-w)|$$

$$= |w_0 - w| < |f(z)-w_0|$$

$\Rightarrow$ $f(z)-w$ has n zeros in $|z-z_0|<r$

If we pick ε small enough s.t $f'(z)\neq 0$ on $0<|z-z_0|<\varepsilon$ then the previous roots of $f(z)-w$ are all distinct because they all are simple zeros.

March 27, 2006

$$I_n := \int_0^{2\pi} \sin^{2n}\theta \, d\theta$$

$n = 0, 1, 2 \ldots$

$$z = e^{i\theta} \qquad dz = ie^{i\theta} d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\frac{1}{2}(z + z^{-1}) = \cos\theta$$

$$\frac{1}{2i}(z - z^{-1}) = \sin\theta$$

$$I_n = \frac{1}{(2i)^{2n}} \int_{|z|=1} (z - z^{-1})^{2n} \frac{dz}{iz}$$

$$J_n := \int_0^{2\pi} \cos^{2n}\theta \, d\theta$$

$$= \frac{1}{i\,2^{2n}} \int_{|z|=1} (z + z^{-1})^{2n} \frac{dz}{z}$$

$$R_1(z) = \frac{(z + z^{-1})^{2n}}{z}$$

In general

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

R ratnl function

$$\rightsquigarrow \quad I = \int_{|z|=1} R_1(z)\, dz$$

for another ratnl function $R_1(z)$.

Assume R_1 has no poles on $|z|=1$

By residue theorem

$$I = \sum_j \operatorname{Res}_{z=a_j} R_1$$

where a_j are the poler of R_1 ~~at~~ in $|z|<1$.

$$\frac{1}{2\pi i}\int_{|z|=1} z^{k}\,\frac{dz}{z}$$

$$= \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$$

$$\frac{1}{2\pi i}\int_{|z|=1} (z+z^{-1})^{2n}\,\frac{dz}{z} = \binom{2n}{n}$$

= constant term of $(z+z^{-1})^{2n}$

By the binomial theorem

$$\binom{2n}{k}\underbrace{z^{k}z^{-(2n-k)}}_{=1} \iff k=n$$

$$\boxed{J_n = \frac{2\pi}{2^{2n}}\binom{2n}{n}}$$

$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2} \qquad |a|<1$$

$$2\cos\theta = z + z^{-1}, \qquad z = e^{i\theta}$$

$$I = \int_{|z|=1} \frac{dz}{(1 - a(z+z^{-1}) + a^2)\, i z}$$

$$= \frac{1}{i}\int_{|z|=} \frac{dz}{-az^2+(1+a^2)z - a}$$

poles are at $z = a, \frac{1}{a}$

$$= 2\pi \operatorname*{Res}_{z=a} \frac{1}{-az^2+(1+a^2)z-a}$$

$$-a(z-a)(z-\tfrac{1}{a})$$

$$= \frac{2\pi}{1-a^2}$$

$$\left.\frac{1}{-a(z-1/a)}\right|_{z=a}$$

$$= \frac{1}{-a(a-1/a)} = \frac{1}{1-a^2}$$

II $$I = \int_{-\infty}^{\infty} Q(x)\,dx$$

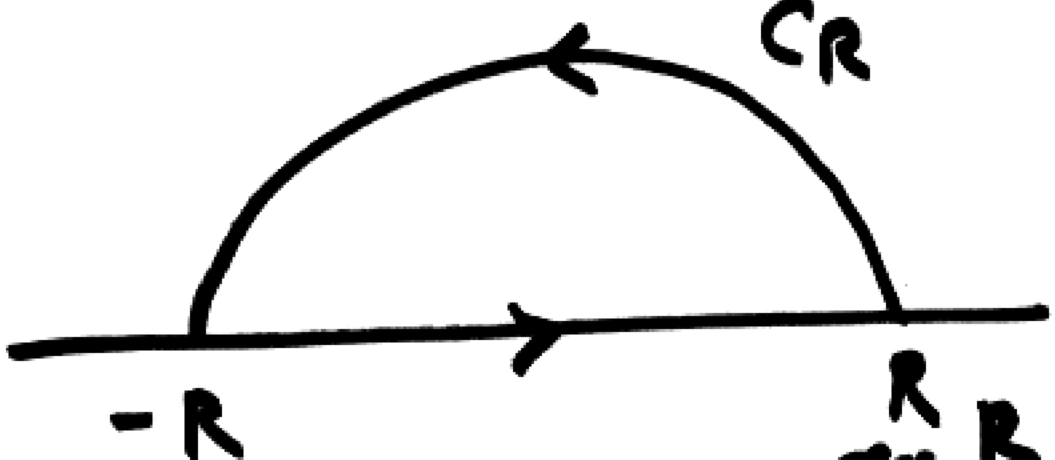
Q rational function of x

$$Q(x) \sim \frac{c}{X^K} \qquad K \geq 0$$

$$\int_0^N \frac{dx}{x^K} = \frac{X^{1-K}}{1-K} \sim \frac{N^{1-K}}{1-K}$$

$K > 1$

$K \geqslant 2$, Q no poles on $\mathbb{R}$



$$I = \lim_{R \to \infty} \int_{-R}^{R} Q(x)\,dx$$

$$\left| \int_{C_R} Q(z)\,dz \right| \leq c\,\frac{1}{R^K} \cdot R = \frac{c}{R^{K-1}}$$

$R \to \infty$

$z = Re^{i\theta}$

$dz = Rie^{i\theta}d\theta$

since $k \geq 2$, $k-1 \geq 1$

$\to 0$ with $R \to \infty$

$$I = 2\pi i \sum_j \mathrm{Res}(Q, a_j)$$

a_j poles of Q on $\mathrm{Im}\, z > 0$

Example

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^4}$$

$$= 2\pi i \cdot \mathrm{Res}_{x=i} \frac{1}{(x^2+1)^4}$$

$$f(z) = \frac{c_k}{z^k} + \dots + \frac{c_{-1}}{z} + c_0 + c_1 z + \dots$$

$$z^k f(z) = c_k + c_{k-1} z + \dots + c_{-1} z^{k-1} + \dots$$

$$\left(\frac{d}{dz}\right)^{k-1} (z^k f(z)) \Big|_{z=0} = (k-1)!\, c_{-1}$$

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

$$\left(\frac{1}{z^2+1}\right)^4 = \frac{1}{16}\Big(\underbrace{-\binom{4}{3}\frac{1}{(z+i)^3}}_{\cdots}\cdot\frac{1}{z-i}+\dots\Big)$$

(III) $$I = \int_{-\infty}^{\infty} e^{iax}\, Q(x)\, dx$$

$a > 0$, Q rational function

- Q no poles on $\mathbb{R}$.
- Q has a pole at ∞

if pole at ∞ is of order $k \geq 2$ then I is convergent by (II)

Why does it when $k = 1$?

Jordan's lemma will guarantee this. We'll show

$$\lim_{R\to\infty} \int_{-R}^{R} e^{iax} Q(x)\, dx$$

exists.

March 29, 2006

$$f(z) = \frac{h(z)}{(z-a)^k} \qquad k \in \mathbb{N}$$

h analytic about a $\quad h(a) \neq 0$

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(k-1)!} h^{(k-1)}(a)$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^4} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^4}$$

$$h(z) = \frac{(z-i)^4}{(z^2+1)^4} = \frac{1}{(z+i)^4}$$

$$\frac{1}{(1-z)^n} = \sum_{k\geq 0} \binom{n+k-1}{k} z^k$$

$$\underset{|z|<1}{(1+z)^\alpha} = \sum_{k\geq 0} \binom{\alpha}{k} z^k$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$\alpha = -n$

$$\binom{\alpha}{k} = \frac{(-n)(-n-1)\dots(-n-k+1)}{k!}$$

$$= (-1)^k \binom{n+k-1}{k}$$

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

$$\frac{1}{(z^2+1)^n} = \frac{1}{(z-i)^n} \cdot \frac{1}{(z+i)^n}$$

$$z+i = 2i + z - i$$

$$\frac{1}{(z+i)^n} = \frac{1}{(2i+z-i)^n} = \frac{1}{(2i)^n} \cdot \frac{1}{(1+(\frac{z-i}{2i})}^{n}$$

$$= \frac{1}{(2i)^n} \cdot \sum_{k \geq 0} \binom{n+k-1}{k} \underbrace{\left(-\left(\frac{z-i}{2i}\right)\right)^k}_{\left(\frac{-1}{2i}\right)^k (z-i)^k}$$

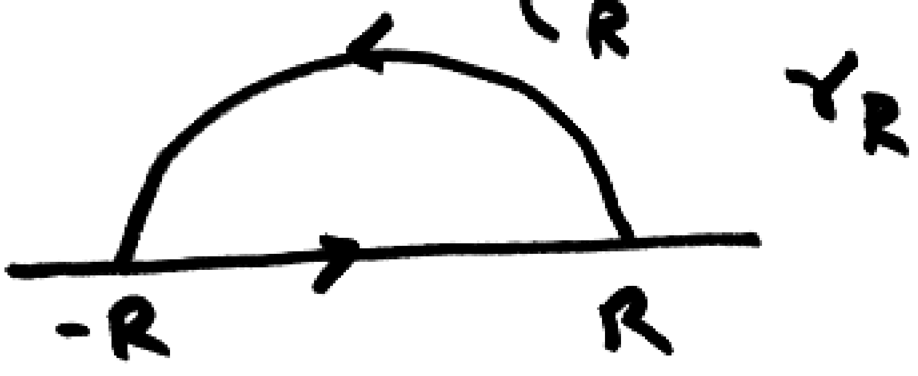
$$\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^n} = \frac{1}{(2i)^n} \cdot \frac{1}{(-2i)^{n-1}} \cdot \binom{2n-2}{n-1}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^n} = 2\pi i \cdot \frac{-2i}{4^n} \cdot \binom{2n-2}{n-1}$$

III $\displaystyle\int_{-\infty}^{\infty} e^{iax} Q(x)\,dx$, $a>0$

- Q no poles on $\mathbb{R}$
- Q pole at ∞.

(Fourier transform of $Q(x)$)



$$\lim_{R\to\infty} \int_{-R}^{R} e^{iax}\, Q(x)\,dx$$

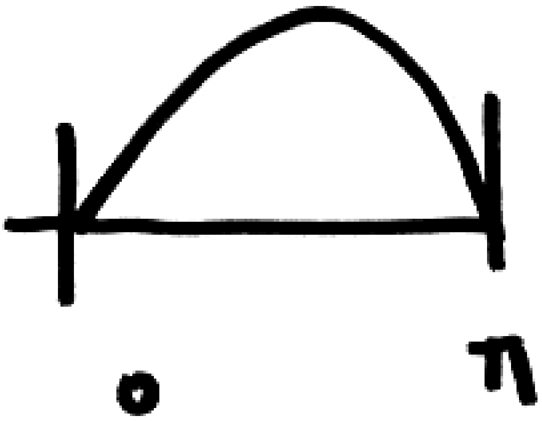
Claim $\displaystyle\int_{C_R} e^{iax} Q(x)\,dx \to 0$ $R\to\infty$

all R sufficiently large

$$\int_{\gamma_R} e^{iaz} Q(z)\,dz = 2\pi i \sum \underset{\substack{z=a \\ \operatorname{Im} a > 0}}{\operatorname{Res}} \ldots$$

$$z = R e^{i\theta}, \quad 0 \le \theta \le \pi$$

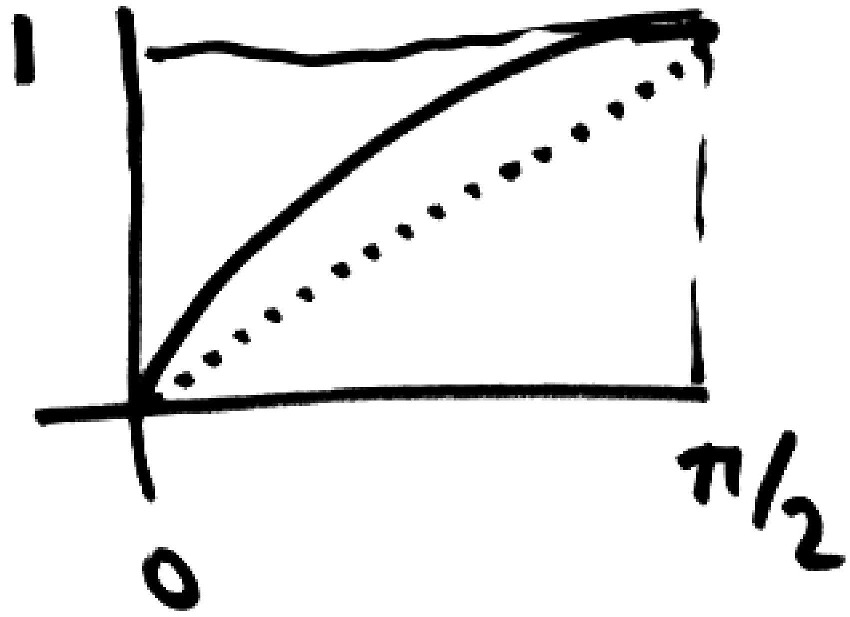
$$|e^{iaz}| = e^{-\overset{R}{a}\sin\theta}$$



$$\left| \int_{C_R} e^{iaz} Q(z)\,dz \right| \le M_R \cdot R \cdot \int_0^{\pi} e^{-\overset{R}{a}\sin\theta}\,d\theta$$

$$M_R = \max_{\substack{|z|=R \\ \operatorname{Im} z \geq 0}} |Q(z)|$$

$$\int_0^{\pi} e^{-aR\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta$$



$$\sin\theta \geq \frac{2}{\pi}\theta \quad , \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta \leq \int_0^{\pi/2} e^{-aR\frac{2}{\pi}\theta}\,d\theta$$

$$= \frac{\pi}{2aR}\left(1 - e^{-aR}\right)$$

This extra R

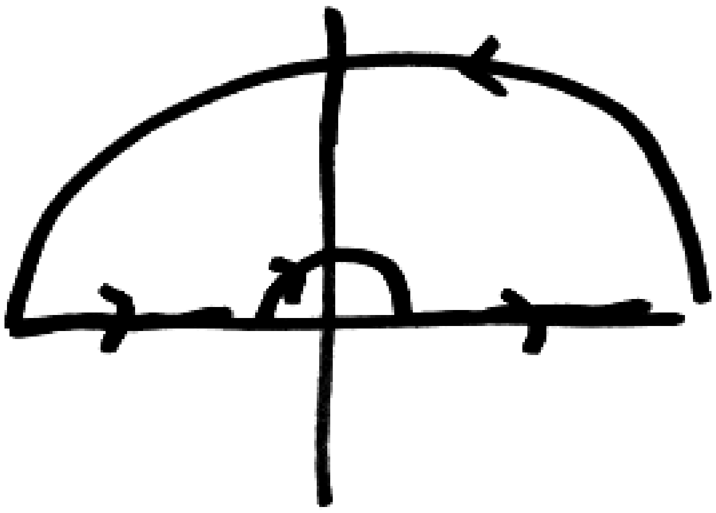
gives

$$M_R \cdot R \cdot \frac{\text{const}}{R} \to 0$$

since M_R is at least $\frac{\text{const}}{R}$

IV

$$\int_0^\infty \frac{\sin x}{x}\, dx$$



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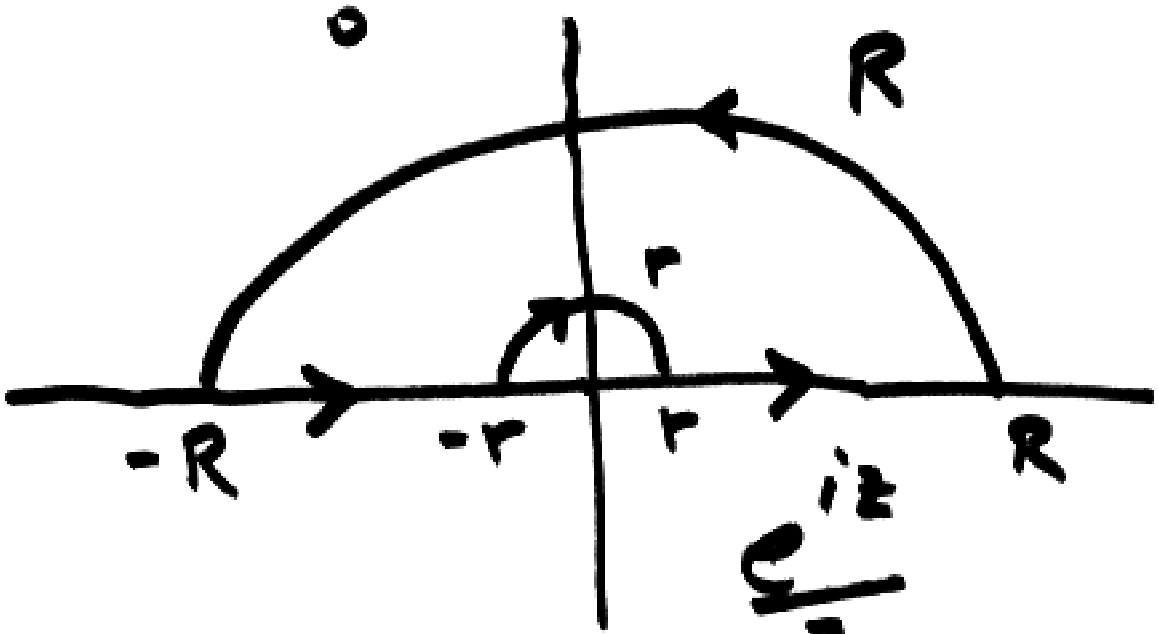
$$\int_0^\infty \frac{\sin x}{x}\,dx$$



$$0 = \int_\gamma \frac{e^{iz}}{z}\,dz$$

$$= \int_{C_R} + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx$$

$$+ \int_r^R \frac{e^{ix}}{x}\,dx$$

$$+ \int_{C_r}$$

[illegible]

$$f(z) = \frac{e^{iz}}{z} = \frac{\cos z + i \sin z}{z}$$

$$\int_{C_R} f(z)\,dz \longrightarrow 0$$

$R \to \infty$

by Jordan's lemma

$$\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx = -\int_{r}^{R} \frac{e^{-ix}}{x}\,dx$$

$$0 = \underset{\overset{\swarrow}{0}}{\int_{C_R}} + \int_{C_r} + \int_{r}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx$$

$$\int_{C_r} \frac{e^{iz}}{z}\,dz$$

$$e^{iz} = 1 + z\,h(z)$$

h analytic at 0

$$\frac{e^{iz}}{z} = \frac{1}{z} + h(z)$$

$$\int_{C_r} \frac{e^{iz}}{z}\,dz = \int_{C_r} \frac{dz}{z} + \underset{\substack{\searrow\\ 0}}{\int_{C_r} h(z)\,dz}$$

$$z = r e^{i\theta} \qquad 0 \le \theta \le \pi$$

$$\int_{C_r} \frac{dz}{z} = -i\int_0^{\pi} d\theta = -\pi i$$

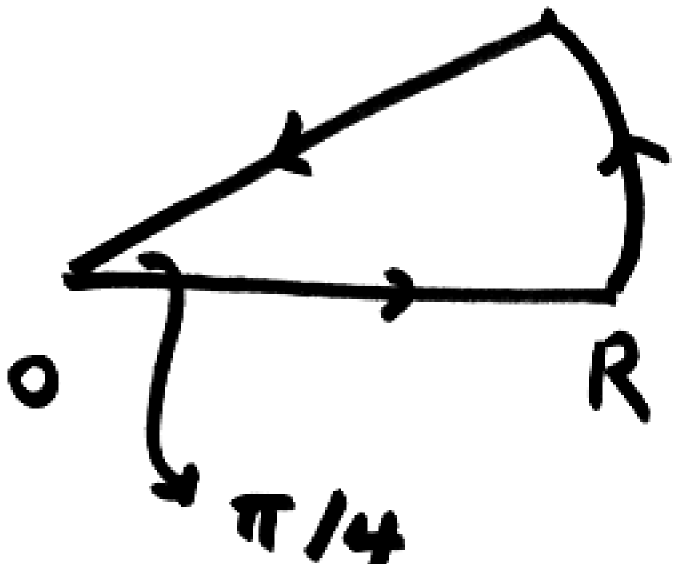
$$\underset{\substack{\downarrow\\ \\}}{2i\int_r^R \frac{\sin x}{x}\,dx} = -\underset{\substack{\downarrow\\ \pi i}}{\int_{C_r}} - \underset{\substack{\searrow\\ 0}}{\int_{C_R}}$$

$$\int_r^R \frac{\sin x}{x}\,dx \to \boxed{\frac{\pi}{2}}$$

Fresnel integrals

$$\int_0^\infty \cos x^2 \, dx$$

$$f(z) = e^{iz^2}$$



$$\int_\gamma f(z)\, dz = 0$$

[redacted] $z = t\, e^{\frac{\pi i}{4}}$

$dz = e^{\frac{\pi i}{4}} dt$

$$iz^2 = i t^2 i = -t^2$$

[redacted] $f(z) = e^{-t^2}$

$$0 = \int_0^R e^{ix^2} dx + \int_{C_R} e^{iz^2} dz$$

$$-\int_0^R e^{-t^2} dt$$

$$e^{\pi i/4}$$

$$\left| \int_{C_R} e^{iz^2} dz \right| \le R \int_0^{\pi/4} e^{-R^2 \sin 2\theta} d\theta$$

$$z = R e^{i\theta}$$

$$z^2 = R^2 e^{i2\theta}$$

$$|e^{iz^2}| = e^{-R^2 \sin 2\theta}$$

$$0 \le \theta \le \pi/4, \quad 0 \le 2\theta \le \pi/2$$

$$\sin(2\theta) \ge \frac{2}{\pi} 2\theta$$

$$R \int_0^{\pi/4} e^{-R^2 \frac{4}{\pi}\theta} d\theta$$

$$= R \cdot \frac{\pi}{R^2 4} \cdot e^{-R^2 \frac{4}{\pi}\theta} \Big|_0^{\pi/4}$$

$$(1 - e^{-R^2})$$

$\to 0$ as $R \to \infty$

$$\int_0^\infty e^{ix^2}\,dx = e^{\pi i/4}\int_0^\infty e^{-t^2}\,dt$$

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx =$$

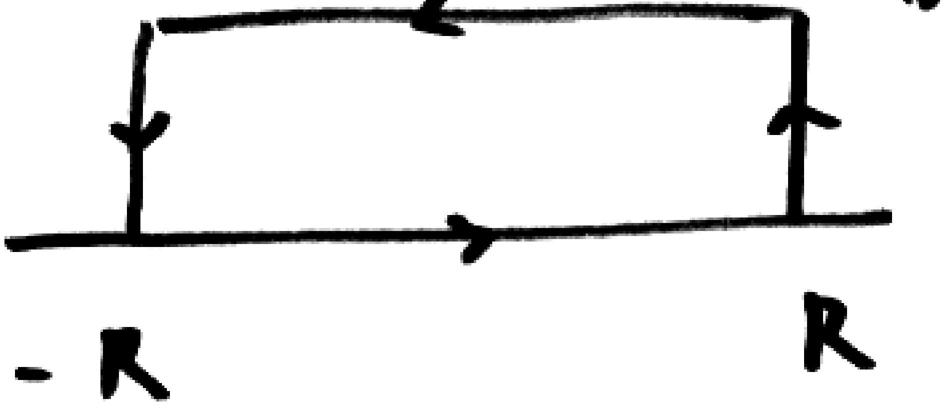
$$e^{\pi i/4} = \frac{1+i}{\sqrt{2}}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}}$$

$$= \boxed{\frac{\sqrt{2\pi}}{4}}$$

$$=$$

$$\int_{-\infty}^{\infty} e^{-x^2}e^{2\pi a i x}\,dx$$



$$e^{-z^2}$$

$$z = x + ib$$

$$z^2 = x^2 + 2ib^{x} - b^2$$

$$e^{-z^2} = e^{-x^2} e^{-2ibx} e^{b^2}$$

on top $$-\int_{-R}^{R} e^{-x^2} e^{-2ibx} e^{b^2} dx$$

on bottom $$\int_{-R}^{R} e^{-x^2} dx$$

vertical segments

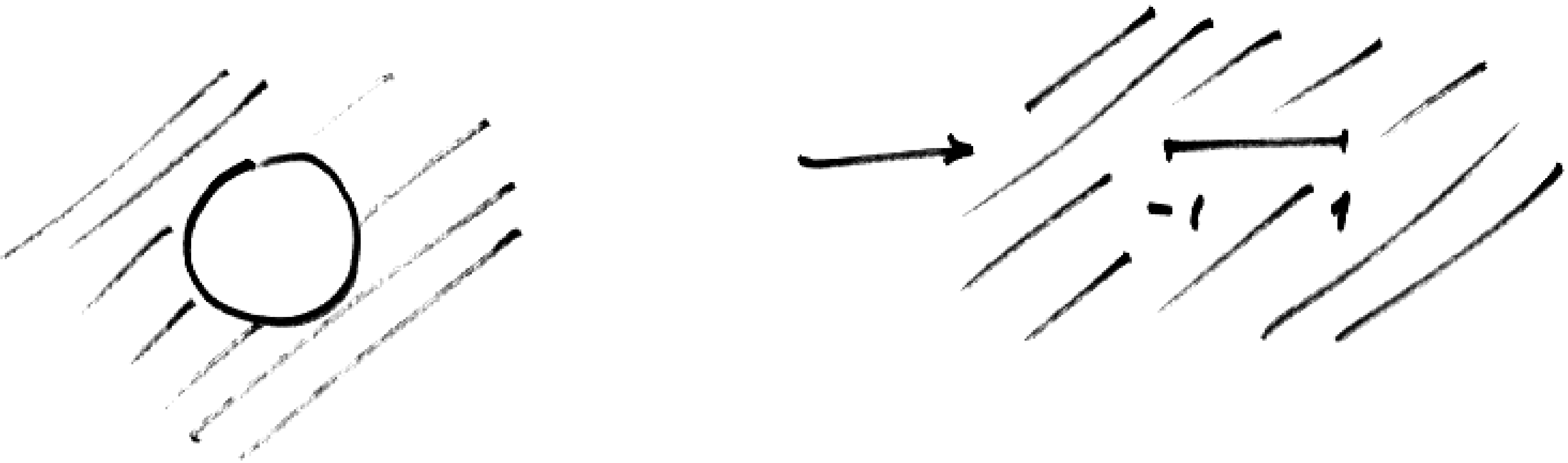
$$z = R + it \qquad 0 \leq t \leq b$$

$$e^{-z^2} = e^{-R^2} \cdot e^{2Rit} \cdot e^{-t^2}$$

$$\int_{\substack{\text{vert} \\ \text{segment}}} \rightarrow 0$$

April 17, 2006

1) $$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$



- singularities?
  simple poles at $z = 0, \infty$.

- generically 2 - 1

$$\frac{1}{2} \; \frac{z^2+1}{z}$$

$$R(z) = \frac{P(z)}{Q(z)}$$

P, Q no common factor

generically: n to 1 map

$w = R(z)$, $\quad P(z) - w\,Q(z) = 0$

$n = \max\{\deg P, \deg Q\}$

②

For what w's do we have < n roots?

- Roots repeated

$$P'(z) - w\,Q'(z) = 0$$

$$R'(z) = \frac{P'(z)\,Q(z) - P(z)\,Q'(z)}{Q(z)^2} = 0$$

$$P(z) - w\,Q(z) = 0$$

$$\begin{pmatrix} P(z) & Q(z) \\ P'(z) & Q'(z) \end{pmatrix}\begin{pmatrix} 1 \\ -w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \det = 0$$

- Degree drops $\qquad z = \infty$ is preimage

$$|z| = 1 \quad \to \quad \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$z = e^{i\theta} \longmapsto \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$= \cos\theta$$

$$R(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$R'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

$$0 = R'(z) \leftrightarrow z = \pm 1$$

Injective on $|z| > 1$

since $z \leftrightarrow 1/z$ interchanges preimages

surjective: $|z|>1 \longrightarrow \mathbb{C} \setminus [-1,1]$

$$w = \frac{1}{2}(z + 1/z)$$

has two solutions one in $|z|>1$ the other in $|z|<1$.

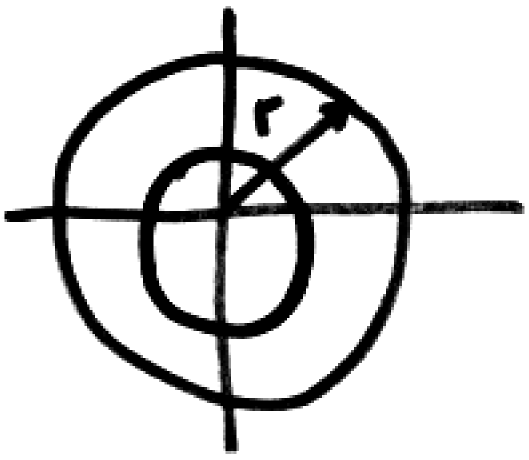
$$z = re^{i\theta} \longmapsto \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right)$$

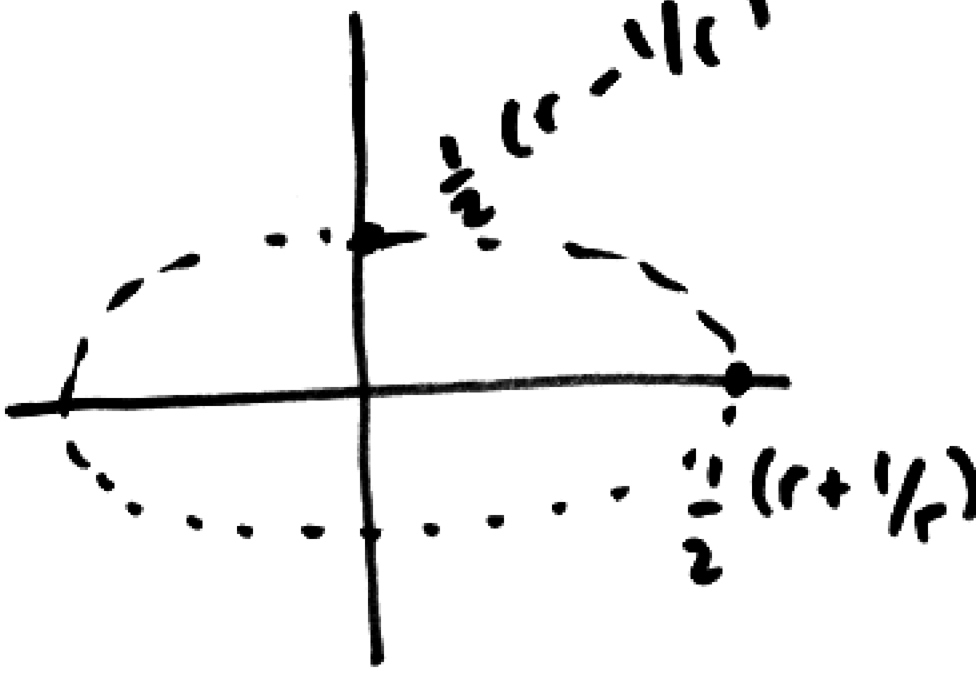
$$\frac{1}{2}\left(r\cos\theta + \frac{1}{r}\cos\theta\right)$$

$$+ \frac{i}{2}\left(r\sin\theta - \frac{1}{r}\sin\theta\right)$$

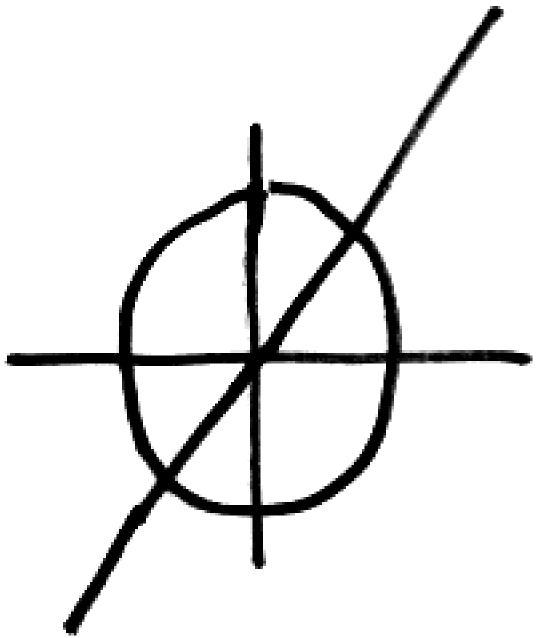
$$= \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta + \frac{i}{2}\left(r-\frac{1}{r}\right)\sin\theta$$

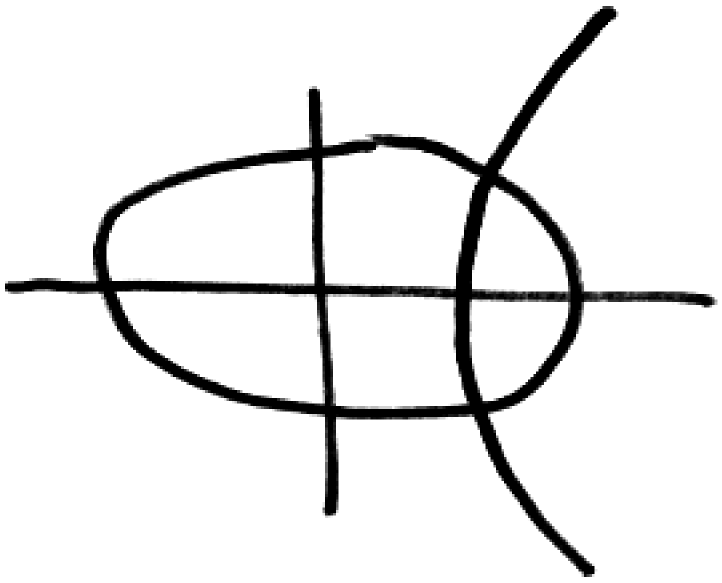
ellipse





$r > 0$



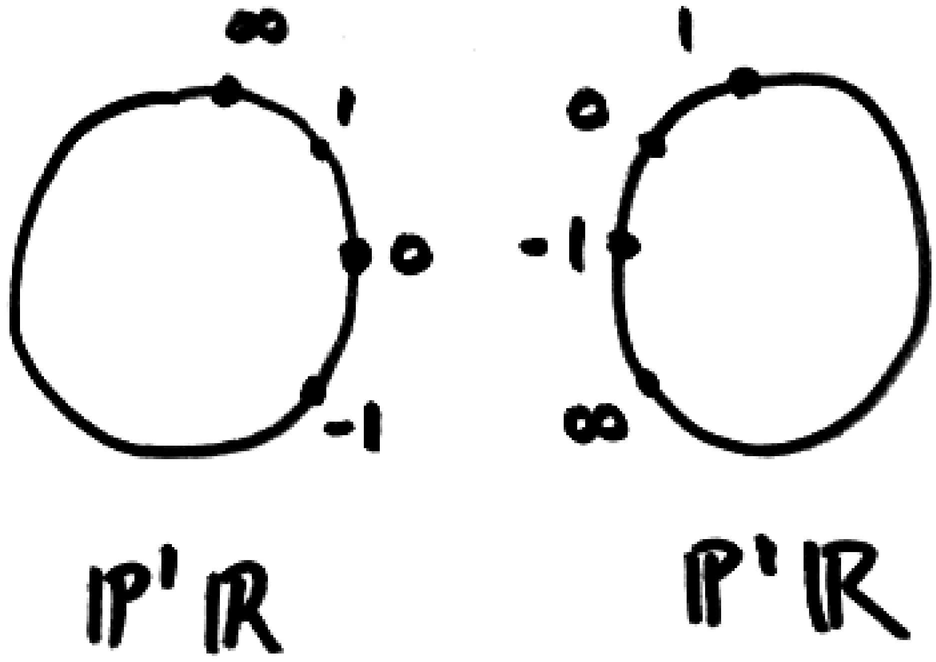


$z = r e^{i\theta}$ fixed θ

2. $\mathbb{C} \setminus [-1, 1]$

$$f(z) = \log\left(\frac{z-1}{z+1}\right)$$

$\frac{z-1}{z+1}$:



$\{(x_0 : x_1)\}/\sim$

$(x_0 : x_1) = (y_0 : y_1)$

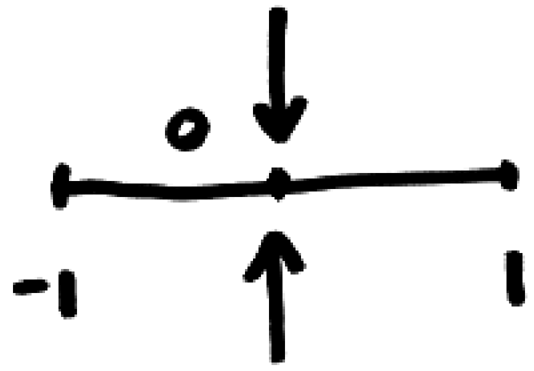
$(x_0 , x_1) = \lambda (y_0 , y_1)$ $\lambda \neq 0$

$(x_0 : x_1) = (\frac{x_0}{x_1} : 1)$ $x_1 \neq 0$

$(x_0 : 0) = (1 : 0)$ $x_0 \neq 0$

$$\begin{tikzcd}
{[-1, 1]} \arrow[r, leftrightarrow] & {[-\infty, \overset{0}{\cancel{\infty}}]} \\
\mathbb{C} \setminus [-1, 1] \arrow[r, "\frac{z-1}{z+1}"] \arrow[rd, dotted] & \mathbb{C} \setminus [0, -\infty) \arrow[d, "\log"] \\
 & {}
\end{tikzcd}$$

- $$\log\left(\frac{i-1}{i+1}\right) = \log(i)$$
  $$= \frac{\pi}{2} i$$

- 

it

$t \to 0^{\pm}$

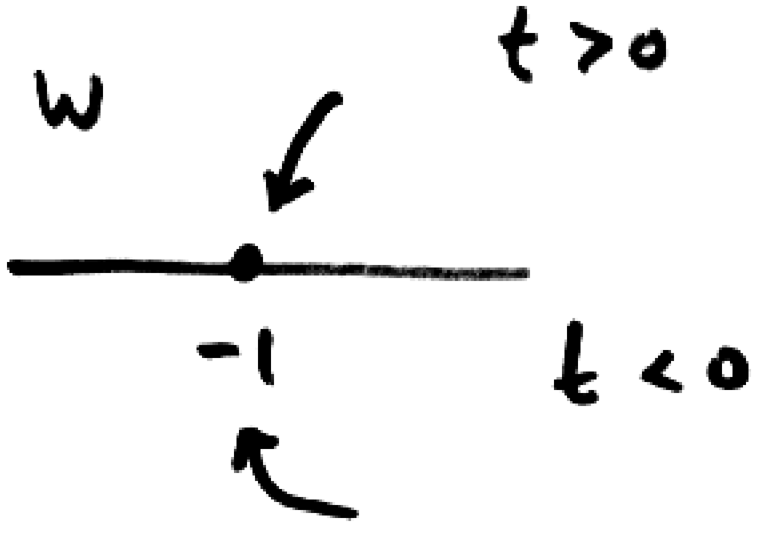
$$\left.\frac{z-1}{z+1}\right|_{z=0} = -1$$

$$w = \frac{z-1}{z+1}$$

$z = it$

$$w = \frac{it-1}{it+1} = \frac{-(it-1)^2}{t^2+1}$$

$$= \frac{t^2-1+2it}{t^2+1}$$



$$\lim_{t \to 0^{\pm}} f(it) = \pm \pi i$$

April 19, 2006

$$f(z) = \log\left(\frac{z-1}{z+1}\right)$$

coeff of $(z-i)^{100}$ in the Taylor expansion about i.

$$f'(z) = \frac{1}{z-1} - \frac{1}{z+1}$$

$$\frac{f^{(100)}(i)}{100!}$$

$$f''(z) = \frac{-1}{(z-1)^2} + \frac{1}{(z+1)^2}$$

$$f^{(n)}(z) = (-1)^{n+1}\frac{(n-1)!}{(z-1)^n} + (-1)^n\frac{(n-1)!}{(z+1)^n}$$

$n = 100$
$z = i$

$$f^{(100)}(i) = 0$$

$$\frac{f^{(100)}(i)}{100!} = \frac{1}{100}\left(-\frac{1}{(i-1)^{100}} + \frac{1}{(i+1)^{100}}\right.$$

$$(1 \pm i)^2 = \pm 2i$$

$$\Big(\qquad \Big(\underset{\underset{e^{\frac{2\pi i}{8}}}{\shortparallel}}{\frac{1+i}{\sqrt{2}}}\Big)^8 = 1 \qquad \Big)$$

$$(1 \pm i)^4 = -4$$

3. Rouché's thm ⇒ FTA

wlog $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

$n > 0$

$g(z) = z^n$

$|f(z) - g(z)| < |g(z)|$

$|z| = R$

$$\left| \frac{f(z) - g(z)}{g(z)} \right| < 1 \qquad |z| = R$$

$$\frac{f(z)-g(z)}{g(z)} \to 0$$

$z \to \infty$

for all R $|z| \geq R$

$$\left| \frac{f(z)-g(z)}{g(z)} \right| < 1$$

f & g have same # zeros in $|z|<R$
w/ multiplicities i.e. n.

$$|f(z)-g(z)| = \left| \sum_{k=0}^{n-1} a_k z^k \right|$$

$$\leq \sum_{k=0}^{n-1} |a_k| \, |z|^k$$

If $|z| = R > 1$

$$\leq \left(\sum_{k=0}^{n-1} |a_k| \right) R^{n-1}$$

$$\overset{?}{\leq} |g(z)| = R^n$$

true if
$$R \geq \left(\sum_{k=0}^{n-1} |a_k| \right)$$

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Its zeros are in

$$|z| \leq \max \left\{ 1, \sum_{k=0}^{n-1} |a_k| \right\}$$

gershgorin's Lemma

$$A = (a_{ij}) \in \mathbb{C}^{n \times n}$$

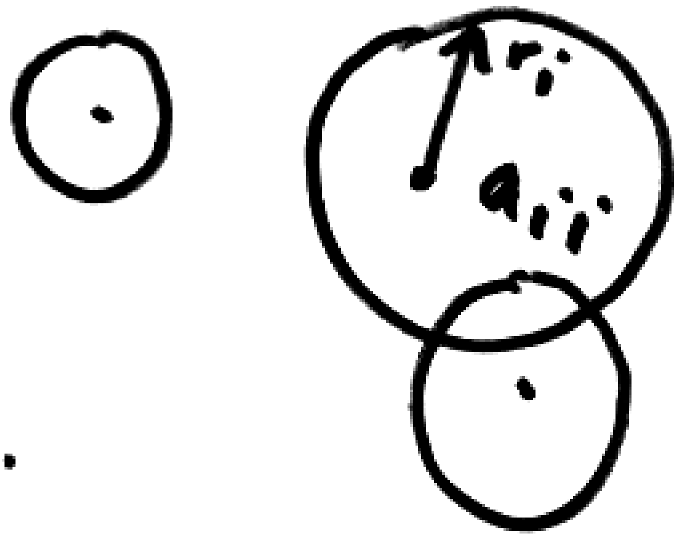
$$r_i := \sum_{j \neq i} |a_{ij}| \qquad \begin{pmatrix} \cdots\cdots \bullet \cdots\cdots \end{pmatrix}$$

Eigenvalues of A are all in

$$\bigcup_i \{ |z - a_{ii}| \leq r_i \}$$

Cor $|a_{ii}| > r_i$

$\Rightarrow$ A is invertible.



companion matrix

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

$$\begin{pmatrix} 0 & 0 & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & & \vdots \\ & & \ddots \; 1 & -a_{n-1} \end{pmatrix}$$

$$\mathbb{C}[z]/(f) \circlearrowleft z$$

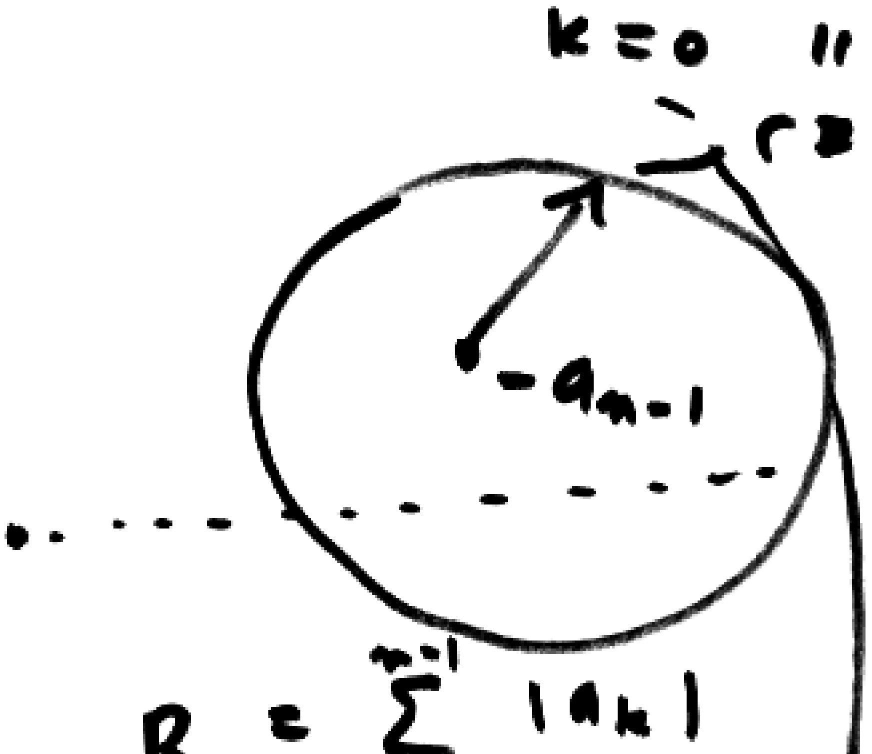
basis: $1, z, z^2, \ldots, z^{n-1}$

$$z^n = -a_{n-1} z^{n-1} \cdots - a_1 z - a$$

Gershgorin lemma applied to transpose

$$|-a_{n-1} - z| \leq \sum_{k=0}^{n-2} |a_k|$$

$$|z| \leq 1$$



$$R = \sum_{k=0}^{n-1} |a_k|$$

4. f meromorphic $U \subseteq \mathbb{C}$

$\mathcal{P}$ = poles of f in U

$$m := \sum_{p \in \mathcal{P}} m_p \geqslant 2$$

⇒ i) at least two distinct poles

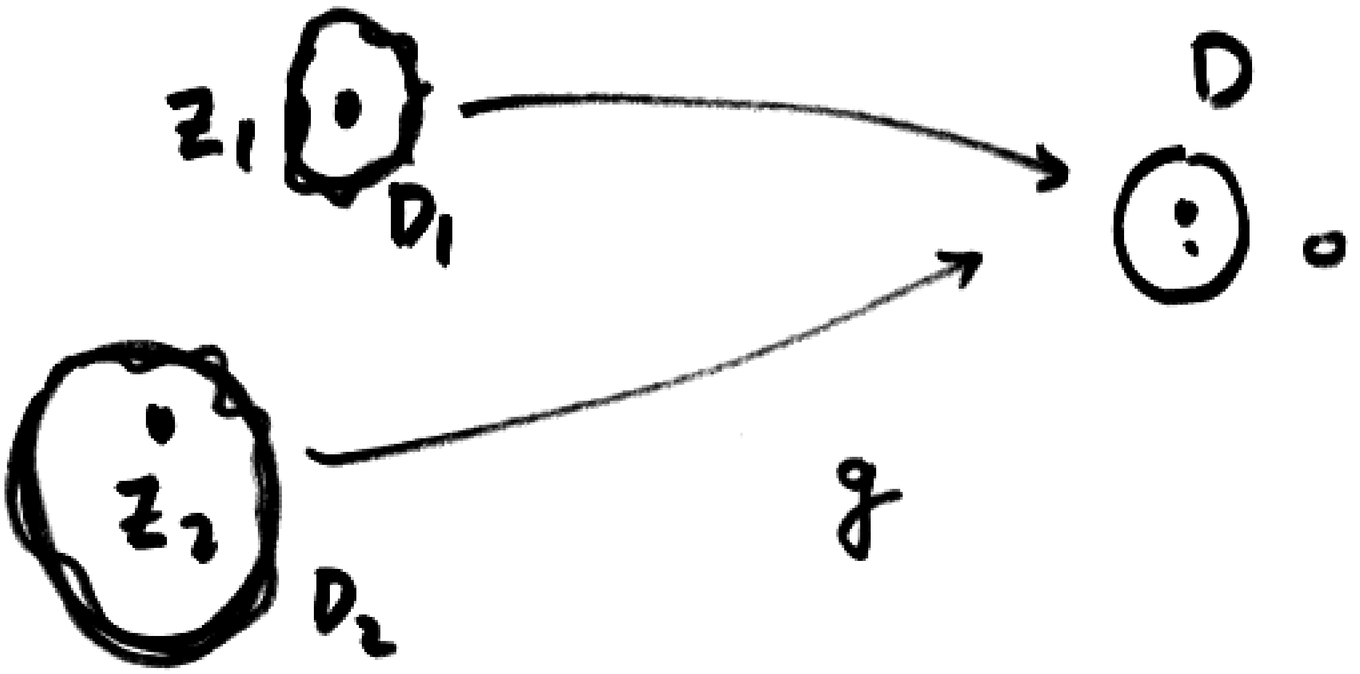
ii) at least one pole not single.

$$g(z) = \frac{1}{f(z)}$$

meromorphic in U

zeros of g ⟷ poles of f.

i)



$0 \in D \subseteq g(D_1) \cap g(D_2)$

~~local mapping~~ $z \neq 0$ in D

has at least one preimage in

D_1 & D_2

ii) $g(z_0) = 0$ multiplicity at least 2

local mapping there are two

preimages for $z \neq 0$ $z \in D$

disk s.t. $g'(z) \neq 0$

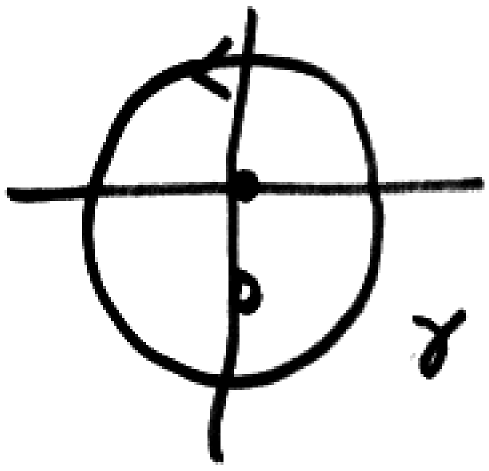
preimages of $z \neq 0$ have

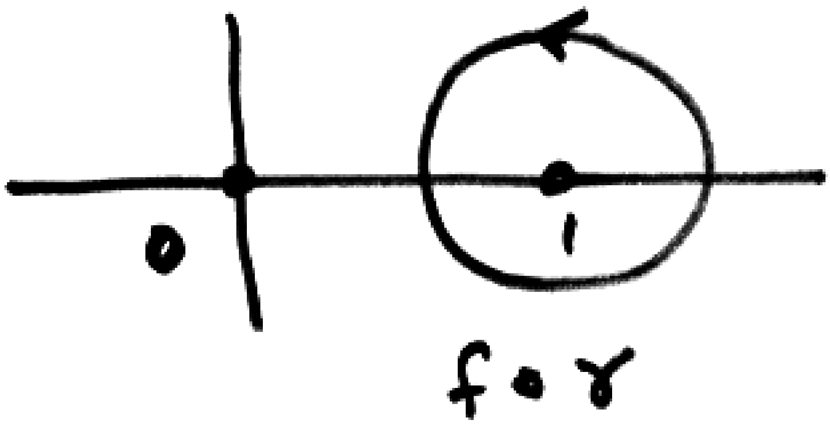
multiplicity 1 ⇒ there are

at least two distinct preimages

April 21, 2006

$f(0)=0$





not analytic

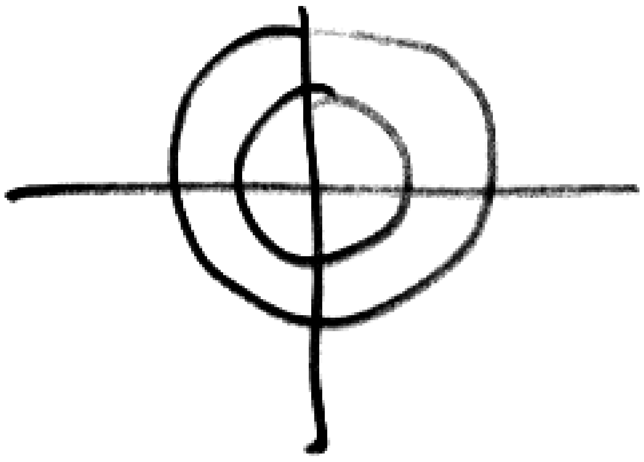
$$n(f\circ\gamma, 0) = \#Z - \#P$$

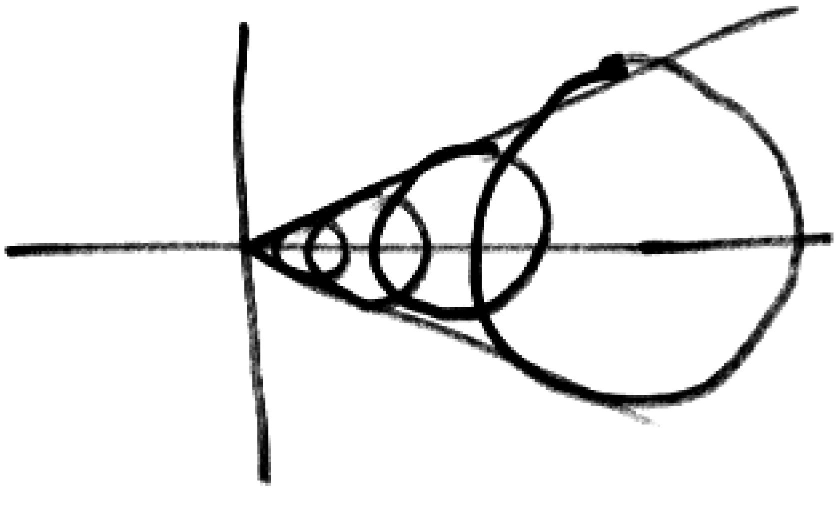
$$\geq 1 \qquad *$$

continuous

$$f(z) = R\left(1 + \tfrac{1}{2}e^{i\theta}\right)$$

$$z = Re^{i\theta}$$





f two distinct poles

$\Rightarrow$ not injective

Local mapping theorem $\quad \frac{1}{f} = g$

$$\int_0^1 \left(\frac{x}{1-x} \right)^{\alpha} \frac{dx}{x-a}$$

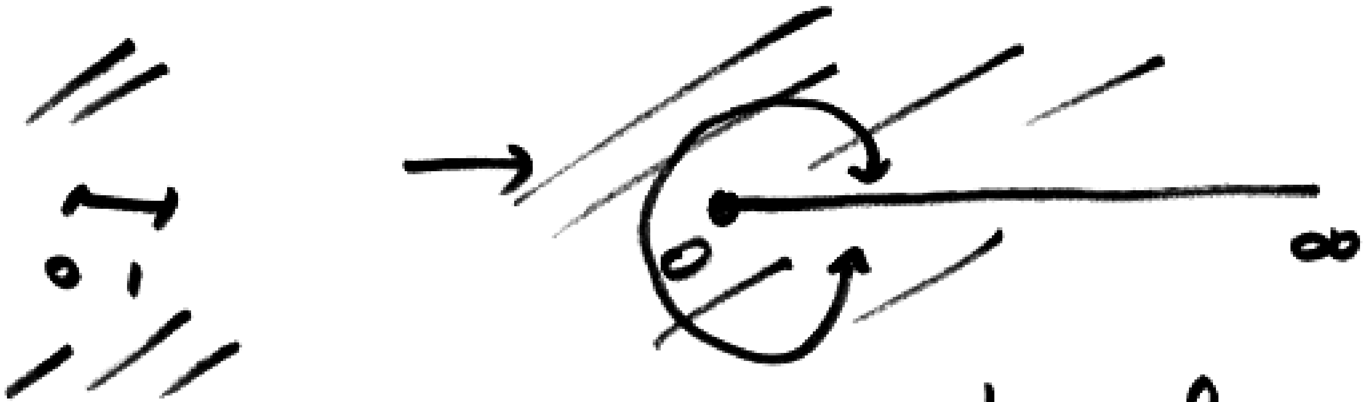
$a > 1 \qquad |\alpha| < 1$

$$f(z) = \left(\frac{z}{1-z} \right)^{\alpha} \frac{1}{z-a}$$

$U :=$

$z \in \mathbb{C} \setminus [0, 1]$

$$\frac{z}{1-z} : \quad \begin{array}{l} 0 \longmapsto 0 \\ 1 \longmapsto \infty \\ \infty \longmapsto -1 \end{array} \qquad [0,1] \longrightarrow [0,\infty]$$



$\downarrow$ log

$$\log z = \log|z| + i \arg z$$

$$0 < \arg z < 2\pi$$

$\rightsquigarrow \quad \log\left(\frac{z}{1-z}\right) \quad$ in U

$\rightsquigarrow \quad \left(\frac{z}{1-z}\right)^{\alpha} := \exp\left(\alpha \log \frac{z}{1-z}\right)$

$\sqrt{z}$ Cannot be defined as analytic function on any nbhd of 0

"branch point"

$$\frac{f(z)'}{f(z)} = \frac{1}{2} \cdot \frac{1}{z}$$

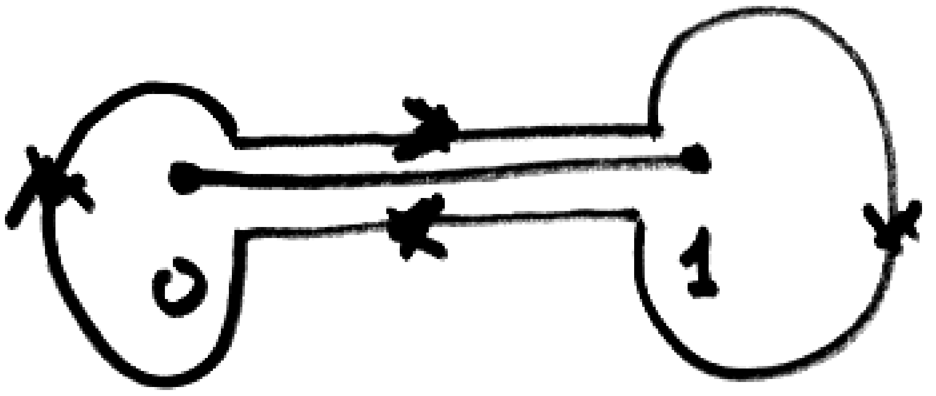
$$f'(z) = \frac{1}{2} \frac{1}{\sqrt{z}}$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2} \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}$$

$$= 1/2$$

Frobenius method for solving linear diff eqns.

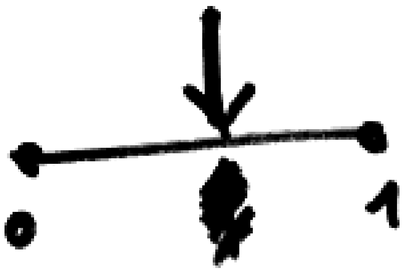
$$z^{\alpha} \cdot (a_0 + a_1 z + a_2 z^2 + \dots)$$

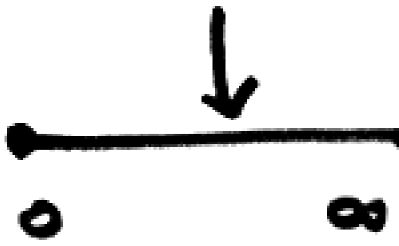


$$\lim_{\varepsilon \to 0} f(t \pm i\varepsilon)$$

$$=$$

$$\frac{z}{1-z} :$$





$$\frac{t+i\varepsilon}{1-t-i\varepsilon} = \frac{(t+i\varepsilon)(1-t+i\varepsilon)}{(1-t)^2+\varepsilon^2}$$

$$= \frac{i(\varepsilon(1-t)+\varepsilon t)}{(1-t)^2+\varepsilon^2} + \text{real}$$

$$= \frac{i\varepsilon}{(1-t^2)+\varepsilon^2} + \text{real}$$

$$\lim_{\varepsilon \searrow 0} \arg f(t \pm i\varepsilon) = \begin{cases} 0 & + \\ 2\pi & - \end{cases}$$

$$\lim_{\varepsilon \searrow 0} f(t + i\varepsilon) = \left(\frac{t}{1-t}\right)^{\alpha} \frac{1}{t-a}$$

$$\lim_{\varepsilon \searrow 0} f(t - i\varepsilon) = \left(\frac{t}{1-t}\right)^{\alpha} \frac{1}{t-a} \cdot e^{2\pi i\alpha}$$

$$\lim_{\varepsilon \searrow 0} \int_{\substack{\longrightarrow \\ \longleftarrow}} f(z)\,dz = I \cdot (1 - e^{2\pi i\alpha})$$

April 24, 2006

$p \in \mathbb{C}[z]$ deg n

$L = \{ z \mid |p(z)| = 1 \}$

$\mathbb{C} \setminus L$ has at most $n+1$ components.

$\gamma : [0,1] \longrightarrow U$

$\sigma = f \circ \gamma$

$|\sigma(t)| = 1$ all t

~~[illegible]~~ $\Rightarrow$ f has a zero inside γ

By argument principle equiv to

$\# Z = n(\sigma, 0) > 0$

If $\sigma'(t) \neq 0$ then we are done

$$\sigma'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

f' could vanish on γ !

If f' does vanish on γ it does 'so (at finitely many points and we can deform γ) so that f' does not vanish.

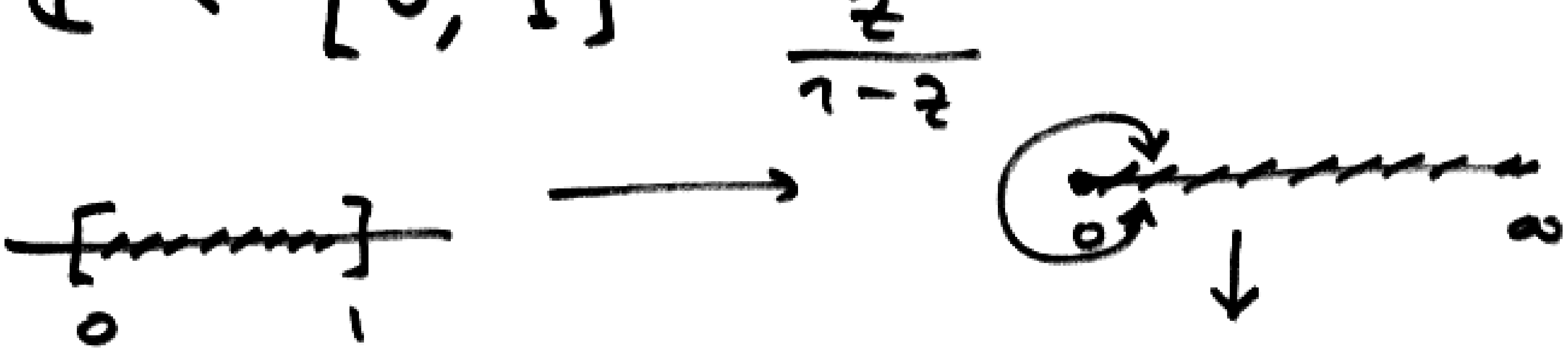
winding number of σ won't change.

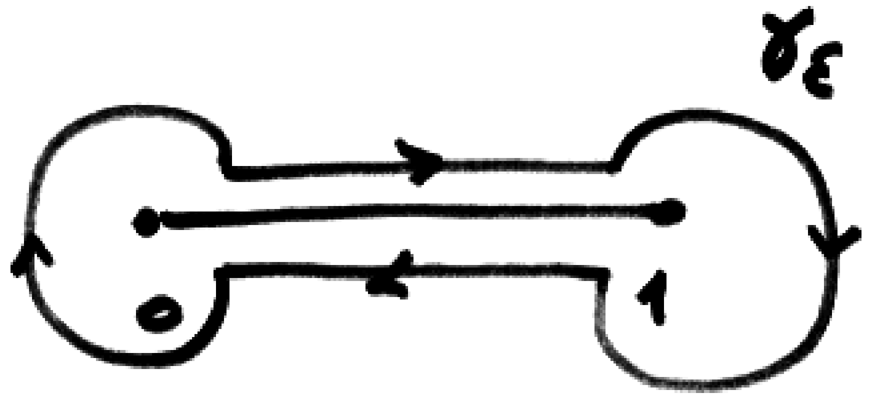
to be completed...

* * *

$$f(z) = \left(\frac{z}{1-z}\right)^{\alpha} \frac{1}{z-a}$$

$\mathbb{C} \setminus [0, 1]$

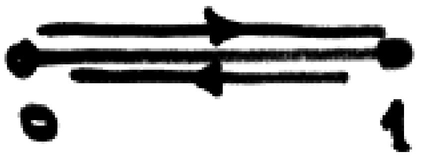




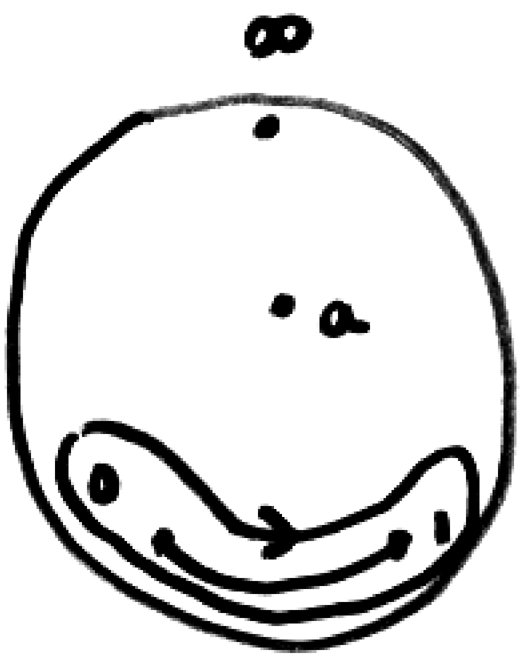
$$\int_{\rightleftarrows} f(z)\,dz = (1-e^{2\pi i\alpha})\, I$$

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^{\alpha} \frac{dx}{x-a}$$

Shrink path $\xrightarrow[\varepsilon\to 0]{}$



$$\int_{\gamma_\varepsilon} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=a} f + \operatorname*{Res}_{z=\infty} f\right)$$



$$w = \frac{1}{z}$$

$$\operatorname*{Res}_{z=a} f = \left(\frac{a}{1-a} \right)^{\alpha}$$

$$f(z) = \left(\frac{z}{1-z} \right)^{\alpha} \frac{1}{z-a}$$

$$z = 1/w$$

$$g(w) = f\left(\frac{1}{w}\right) = \left(\frac{1/w}{1-1/w} \right)^{\alpha} \frac{1}{1/w - a}$$

$$= \left(\frac{1}{w-1} \right)^{\alpha} \frac{w}{1-aw}$$

~~$\operatorname*{Res}_{z=\infty} f \quad \operatorname*{Res}_{w=0} g$~~

$$\frac{1}{2\pi i} \int_{|z|=R} f(z)\,dz = \operatorname*{Res}_{z=\infty} f \qquad ⊙$$

large R

$$= -\frac{1}{2\pi i} \int_{|z|=1/R} g(w) \frac{dw}{w^2} \qquad ⊙$$

$$2\pi i\Big[\underset{\substack{\uparrow\\ \operatorname{Res}\limits_{z=a} f}}{\left(\frac{a}{1-a}\right)^{\alpha}} - \underset{\substack{\uparrow\\ \operatorname{Res}\limits_{z=\infty} f}}{e^{\pi i\alpha}}\Big] = (1-e^{2\pi i\alpha})\,I$$

$\alpha \neq 0$ $$I = \frac{2\pi i}{1-e^{2\pi i\alpha}}\cdot\left[e^{\pi i\alpha}\left(\frac{a}{a-1}\right)^{\alpha} - e^{\pi i\alpha}\right]$$

$$\underset{\substack{\uparrow\\ a>1 \quad \text{negative real}}}{\left(\frac{a}{1-a}\right)^{\alpha}} = \left(\frac{a}{a-1}\right)^{\alpha} e^{\pi i\alpha}$$

$$\log\frac{a}{1-a} = \log\left|\frac{a}{1-a}\right| + i\,\underset{\substack{\shortparallel\\ \pi}}{\arg\Big(\frac{a}{1-a}}$$

$$\exp(\overset{\alpha\times}{\cdots}) = \left|\frac{a}{1-a}\right|^{\alpha} e^{\pi i\alpha}$$

$$= \left(\frac{a}{a-1}\right)^{\alpha} e^{\pi i\alpha}$$

$\underline{z=0}$ $\quad \varepsilon e^{i\theta}$

$$|f(\varepsilon e^{i\theta})| \le \varepsilon^{\alpha}\cdot C$$

$$\Big|\int_{\odot} f(z)\,dz\Big| \le C\,\varepsilon^{\alpha}\cdot\varepsilon$$

$\alpha+1>0$

$\boxed{\alpha>-1}$

$$\xrightarrow[\varepsilon\to 0]{} 0$$

$$\int_0^{*} x^{\alpha}\,dx \quad < 0 \quad \text{if} \quad \alpha>-1$$

$$\int_{\varepsilon}^{*} x^{\alpha}\,dx = -\frac{1}{\alpha+1}\varepsilon^{\alpha+1} + *$$

$\alpha+1>0$ $\quad \searrow 0$

same argument around 1

use $\boxed{\alpha<1}$

$$\frac{g(w)}{w^2} = \left(\frac{1}{w-1}\right)^\alpha \cdot \frac{1}{w(1-aw)}$$

$$-\operatorname*{Res}_{w=0} \frac{g(w)}{w^2} = -\left(\frac{1}{-1}\right)^\alpha \cdot \frac{1}{1.}$$

$$= -e^{\pi i \alpha}$$

For our branch

$$\log(-1) = i \arg(-1)$$

$$= i\pi$$

[redacted]

$$\int \left(\frac{z}{1-z}\right)^\alpha \frac{1}{z-a} \, dz \qquad \to 0$$

$$\varepsilon \to 0$$

$\to 0$

because $|\alpha| < 1$

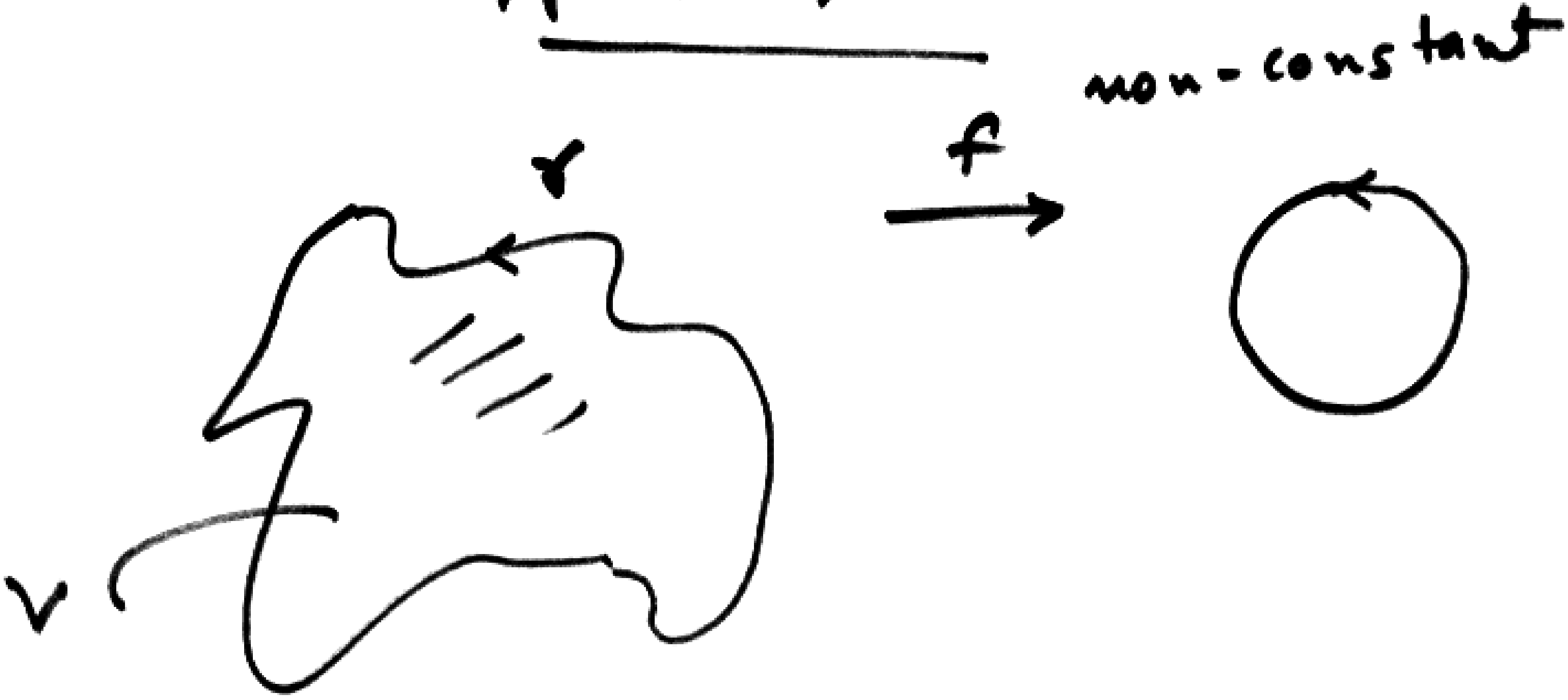
$$= \frac{2\pi i \, e^{\pi i \alpha}}{1-e^{2\pi i \alpha}} \cdot \left(\left(\frac{a}{a-1}\right)^{\alpha} - 1 \right)$$

$$=$$

$$2\pi i \, \frac{e^{\pi i \alpha}}{1-e^{2\pi i \alpha}} = \frac{2\pi i}{e^{-\pi i \alpha} - e^{\pi i \alpha}} = \frac{-\pi}{\sin \pi\alpha}$$

$$\boxed{I = \frac{\pi}{\sin \pi \alpha} \left(1 - \left(\frac{a}{a-1}\right)^{\alpha} \right)}$$

April 26, 2006



$$|f(z)| = 1 \qquad z \in \gamma$$

$\Rightarrow$ f vanishes inside $\gamma =: V$

max modulus principle

$$|f(z)| \leq 1 \quad \text{on } V$$

$f \neq 0$ on V

$$g(z) = \frac{1}{f(z)} \qquad \text{analytic}$$

on some open set containing $\overline{V}$

$$|g(z)| \leq 1 \qquad z \text{ in } V$$

$$\Rightarrow |f(z)| = 1 \quad \text{in } V$$

$\Rightarrow$ f is constant.

$$|g(z)| = 1 \qquad z \in \gamma$$

Pblm p $\deg n \geq 1$

$$L = \{ z \in \mathbb{C} \mid |p(z)| = 1 \}$$

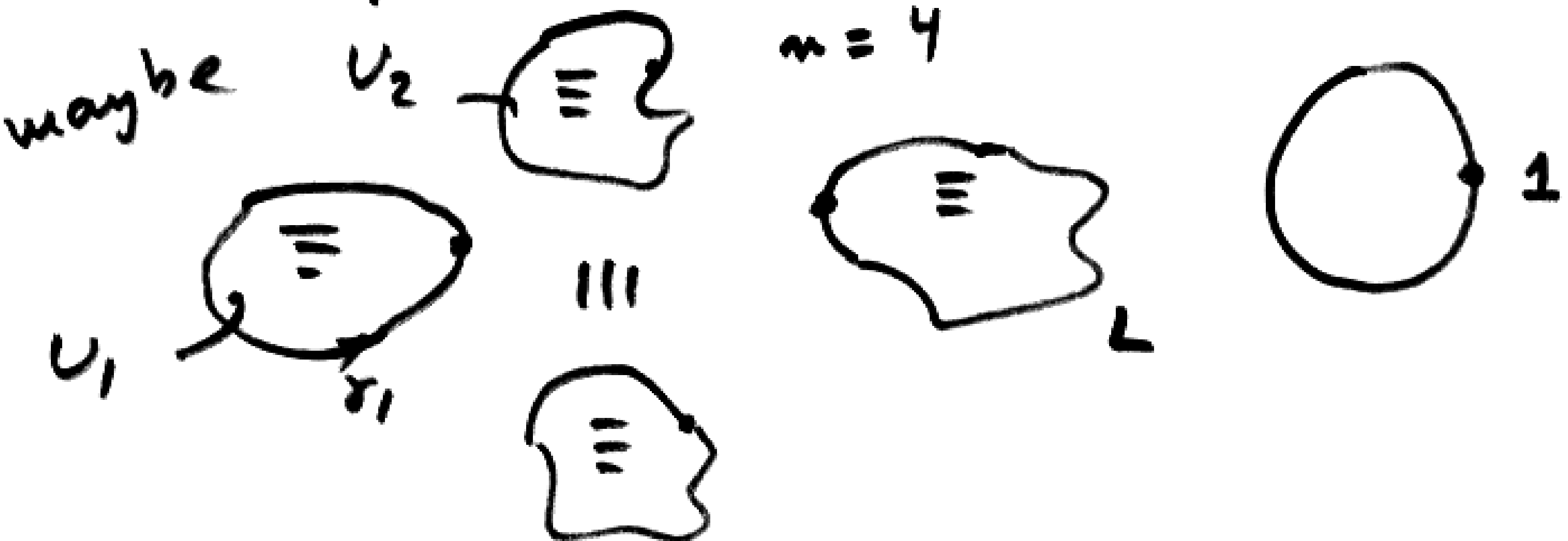
$\mathbb{C} \setminus L$ has at most $n+1$ components.

- $p(z) = z^n$ $\quad n \geq 1$

$L = S^1$

$\mathbb{C} \setminus L$

- $p(z) - 1$ $\quad n$ distinct roots



L is compact

$\mathbb{C} \setminus L$ open $= U_\infty \cup \bigcup_i U_i$

$\gamma_i = \partial \overline{U_i}$ $\quad \gamma_i \subset L$

$|p(z)| = 1$ on γ_i

Lemma $\quad p$ vanishes on U_i

p has at most n roots

$\# i$'s $\leq n$

$\Rightarrow$ # components $\leq n+1$

$$I = \int_0^1 \overbrace{\left(\frac{x}{1-x}\right)}^{\alpha} \frac{dx}{x-a}$$

$$t = \frac{x}{1-x}, \quad (1-x)t = x$$
$$t = x(1+t)$$

$$dt = dx\,\frac{1-x+x}{(1-x)^2} = \frac{dx}{(1-x)^2}$$

$$x = 0 \longleftrightarrow t = 0$$

$$x = 1 \longleftrightarrow t = +\infty$$

$$I = \int_0^\infty t^\alpha \frac{dt}{(1+t)^2 \left(\frac{a+(1-a)t}{(1+t)}\right)}$$

$$t = x(1+t)$$

$$\frac{t}{1+t} = x \qquad 1 - x = 1 - \frac{t}{1+t}$$

$$= \frac{1}{1+t}$$

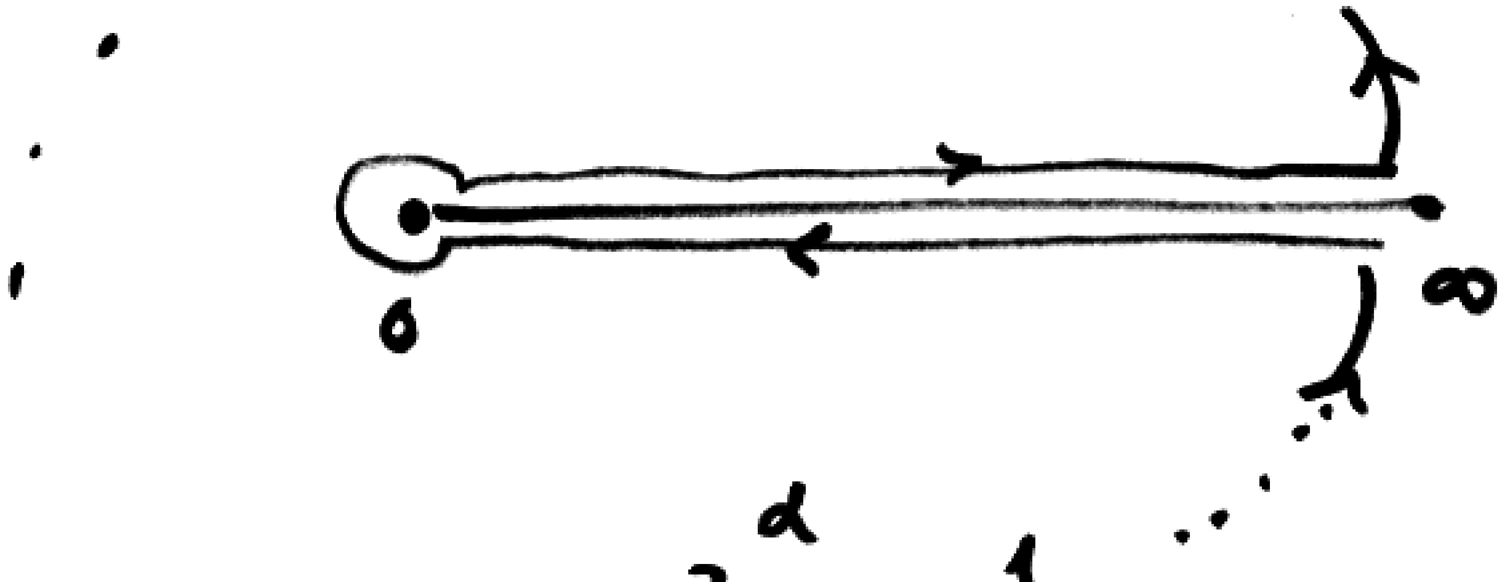
$$dx = (1-x)^2 dt$$

$$= \frac{dt}{(1+t)^2}$$

$$x - a = \frac{t}{1+t} - a = \frac{-a+(1-a)t}{1+t}$$

$$I = \int_0^\infty t^\alpha \frac{dt}{(1+t)(-a+(1-a)t)}$$

finite $\alpha > -1$



$$f(z) = z^{\alpha} \frac{1}{(1+z)(-a+(1-a)z)}$$

z^{α} defined in $\mathbb{C} \setminus [0,\infty]$

use $0 < \arg(z) < 2\pi$

$$\int_0^{\infty} t^{\alpha} R(t)\, dt$$

R no poles on $(0,\infty)$

R

~~$\int_*^* x^{\beta} dx <$ finite~~

locally around $\int_0^{*} \frac{1}{x^{\beta}}\, dx$ finite iff $\beta < 1$

$\beta \geq 0$

$$\int x^{-\beta} dx = \frac{1}{-\beta+1} x^{-\beta+1}$$

$\beta \neq 1$

$$1 - \beta > 0$$

$$1 > \beta$$

$$\int_0^* \frac{f(x)}{x^\beta} dx$$

$f \neq 0, \infty$
at $x = 0$

$$\int_*^\infty \frac{1}{x^\beta} dx$$

$$\int x^{-\beta} dx = \frac{1}{-\beta+1} x^{-\beta+1}$$

$x = N$ $\qquad \frac{1}{-\beta+1} N^{1-\beta}$

$$1 - \beta < 0$$

$$1 < \beta$$

$$\int_*^\infty \frac{f(x)}{x^\beta} dx$$

$f \to c \neq 0, \infty$
as $x \to \infty$

$$\int_0^\infty t^\alpha \frac{dt}{(1+t)(-a+(1-a)t)}$$

$$\alpha - 2 < -1$$

$$\alpha < 1$$

$$\int_0^\infty t^\alpha . \log t . R(t)\, dt$$

$$\int_0^* t^\alpha \log t \, dt \qquad \alpha > -1$$

$$I(\alpha) = \int_0^\infty t^\alpha R(t)\, dt$$

$$\frac{d}{d\alpha} I(\alpha) = \int_0^\infty t^\alpha . \log t . R(t)\, dt$$

$$(e^{\log t \alpha})' = e^{\log t \alpha} . \log t.$$

May 1, 2006

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k \geq 1} \frac{2z}{z^2 - k^2}$$

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = i \, \frac{e^{i\pi z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}$$

$$= i + \frac{2i}{e^{2\pi i z} - 1}$$

$$= \frac{1}{z} \sum_{n \geq 0} B_n \frac{(2iz)^n}{n!}$$

$$+ i$$

$$\zeta(s) = \sum_{n \geq 1} n^{-s}, \quad Re(s) > 1$$

$$s = 2k \qquad k = 1, 2, \ldots$$

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} (-1)^{k+1} B_{2k} \pi^{2k}$$

(Euler)

$$\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2}$$

$$= \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(12) = \frac{691 \; \pi^2}{\cdots}$$

$\zeta(3)$, $\zeta(5)$, ... ??

Gamma function

Euler interpolate $n!$

$$\Gamma(n+1) = n!$$

$$\Gamma(z) := \int_0^\infty e^{-t}\, t^z \,\frac{dt}{t}$$

$$\int_0^\infty f(t) \,\frac{dt}{t}$$

$u = a t$ $\qquad$ $a > 0$

$$\frac{dt}{t} = \frac{du}{u}$$

$$\int_0^\infty f\left(\frac{u}{a}\right) \frac{du}{u}$$

$$\Gamma(z) = \underbrace{\int_0^1 e^{-t}\, t^z \,\frac{dt}{t}} + \underbrace{\int_1^\infty e^{-t}\, t^z \,\frac{dt}{t}}_{\substack{\text{well defined} \\ \text{for any } z \\ \text{analytic.}}}$$

$$|t^z| = t^{\operatorname{Re} z}$$

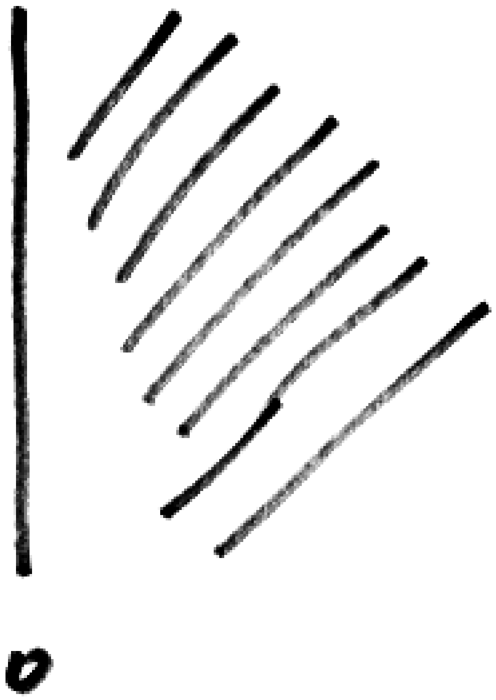
$$\operatorname{Re}(z) > 0$$

$$\mathrm{Re}(z) \geqslant \delta > 0$$

$$\int_0^1 e^{-t} t^{z} \frac{dt}{t}$$

converges uniformly $\rightsquigarrow$ analytic

$\Gamma(z)$ analytic in $\mathrm{Re}(z) > 0$



Integration by parts

$$u = t^{z} \qquad dv = e^{-t} dt$$

$$du = z t^{z-1} dt \qquad v = -e^{-t}$$

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{z} dt$$

$$= -e^{-t} t^{z}\Big|_{0}^{\infty} + z\int_{0}^{\infty} e^{-t} t^{z} \frac{dt}{t}$$

$\mathrm{Re}(z) > 0$

$$= z\,\Gamma(z)$$

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

$$\Gamma(1) = \int_{0}^{\infty} e^{-t}\,dt = 1$$

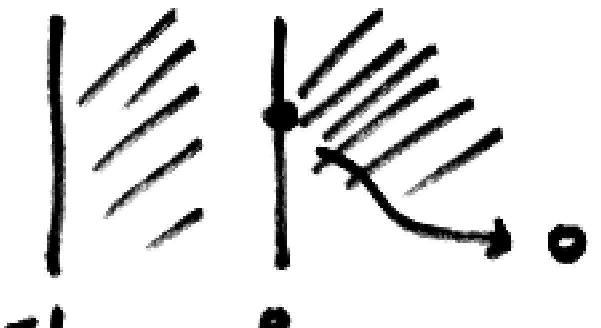
$$\begin{aligned} \Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\,\Gamma(n-2) \\ &= n(n-1)\cdots 1 = n! \end{aligned}$$

Define

$$\Gamma(z) := \frac{\Gamma(z+1)}{z} \leftarrow \text{meromorphi}$$

$$\mathrm{Re}(z+1) > 0$$

$$\mathrm{Re}(z) > -1$$



In fact since $\Gamma(1) = 1$

$\Gamma(z)$ has a simple pole at $z = 0$ w/ residue $= 1$.

Repeat process

[illegible] : d

$$\Gamma(z+2) = (z+1)\Gamma(z+1)$$
$$= (z+1)\, z\, \Gamma(z)$$

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}$$

$\text{Re}(z) > -2$

simple pole at $z = -1$

Repeat $\rightsquigarrow$ $\Gamma(z)$ gets extended to a meromorphic function in $\mathbb{C}$ with simple poles at

$$z = 0, -1, -2, -3, \ldots$$

$$\Gamma(z) = \underbrace{\int_0^1 e^{-t}\, t^z \frac{dt}{t}} + \int_1^\infty e^{-t} t^z \frac{dt}{t}$$

$$e^{-t} = \sum_{k\geq 0} (-1)^k \frac{t^k}{k!}$$

$$= \int_0^1 t^z \sum_{k\geq 0} (-1)^k \frac{t^k}{k!} \frac{dt}{t}$$

$$= \sum_{k\geq 0} \frac{(-1)^k}{k!} \frac{1}{z+k}$$

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}$$

$k = 0, 1, 2, \dots$

- $$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\text{poles}\quad \underset{\substack{\uparrow \\ z=0,-1,-2,\dots}}{} \qquad \underset{\substack{\uparrow \\ z=1,2,3,\dots}}{}$$

lhs simple poles at all $\mathbb{Z}$

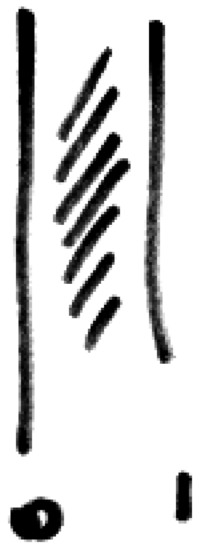
rhs ,, ,,

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty\int_0^\infty e^{-(s+t)} s^{1-z}\, t^{z} \frac{dt}{t}\frac{ds}{s}$$

$$\Gamma(1-z) = \int_0^\infty e^{-s}\, s^{1-z} \frac{ds}{s}$$

$$\mathrm{Re}\,(1-z) > 0$$

$$1 > \mathrm{Re}(z) > 0$$



$u = s+t$

$v = t/s$

$$\Gamma(z)\,\Gamma(1-z) = \int_0^\infty\int_0^\infty e^{-u}\, v^{z} \,\frac{du\,dv}{v(1+v)}$$

$$= \int_0^\infty \frac{v^z\,dv}{(v+1)v} = \frac{\pi}{\sin \pi z}$$

$$\Gamma(1/2)^2 = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}} = \int_0^\infty e^{-t} \frac{dt}{\sqrt{t}}$$

May 3, 2006

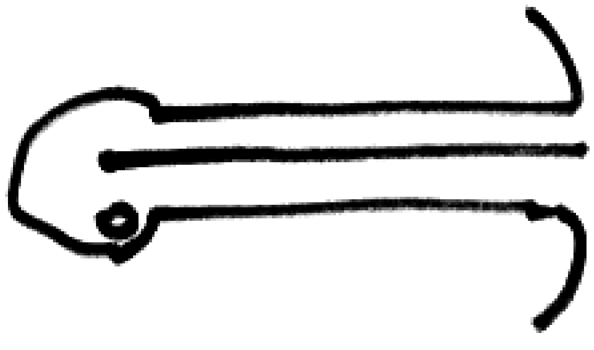
$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$I = \int_0^\infty \int_0^\infty e^{-(s+t)}\, t^z\, s^{1-z}\, \frac{dt}{t}\,\frac{ds}{s}$$

$$\begin{cases} u = s+t \\ v = t/s \end{cases}$$

$$I = \int_0^\infty \frac{v^z}{1+v}\,\frac{dv}{v} = \frac{\pi}{\sin \pi z}.$$

Use Residues !



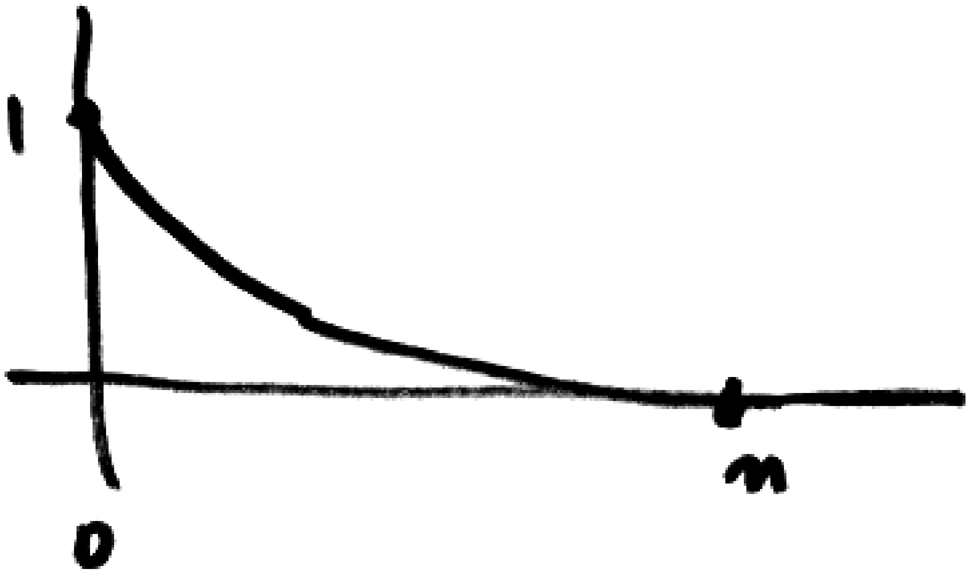
$\Rightarrow$ $\Gamma(z)$ has no zeros

Courant

Claim: $\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^z \frac{dt}{t}$

$$\left(1+\frac{z}{n}\right)^n \longrightarrow e^z$$

$n \longrightarrow \infty$ unif. in cpt sets.



$$\left(1-\frac{t}{n}\right)^n$$

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^z \,\frac{dt}{t}$$

$$= \int_0^n e^{-t}\, t^z \,\frac{dt}{t} + \int_n^\infty e^{-t}\, t^z \,\frac{dt}{t}$$

$$= \int_0^n \left(1-\frac{t}{n}\right)^n t^z \,\frac{dt}{t}$$

$$+ \int_0^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^z \,\frac{dt}{t}$$

$$+ \int_n^\infty e^{-t}\, t^z \,\frac{dt}{t}$$

....

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^z \,\frac{dt}{t}$$

Integrate by parts.

$$= \frac{1}{n^n}\cdot \int_0^n (n-t)^n t^z \frac{dt}{t}$$

$u=(n-t)^n \quad dv = t^z\frac{dt}{t}$ ~~ch $du=(n-t)^n dt$~~

$du = -n(n-t)^{n-1} \quad v = \frac{t^{z}}{z}$

$$= \frac{1}{n^n}\cdot\frac{n}{z}\cdot\int_0^n (n-t)^{n-1}\, t^{z+1}\frac{dt}{t}$$

$$= \frac{1}{n^n}\cdot\frac{n}{z}\cdot\frac{n-1}{z+1}\cdots\cdots\frac{1}{z+n}$$

$$= \frac{n!\;\; n^z}{z(z+1)\cdots(z+n)}$$

last integral $\quad \int_0^n t^{z+n-1}\,dt = n^{z+n}$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$$

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n!\, n^z}$$

$$= \lim_{n\to\infty} e^{-z\log n} z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{n}\right.$$

$$=$$

$$e^{-z\log n} z\left(1+\frac{z}{1}\right)\cdots\left(1+\frac{z}{n}\right)$$

$$= e^{-z\log n}\, z \prod_{k=1}^{n} e^{-z/k}\left(1+\frac{z}{k}\right)$$

$$\cdot\, e^{z\sum_{k=1}^{n}\frac{1}{k}}$$

$$= e^{-z\left(\log n - \sum_{k=1}^{n}\frac{1}{k}\right)} z \prod_{k=1}^{n} e^{-z/k}\left(1+\frac{z}{k}\right)$$

$$-\log n + \sum_{k=1}^{n}\frac{1}{k} \longrightarrow \gamma = 0.5772\ldots$$

Euler's constant

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{k \geq 1} e^{z/k}\left(1+\frac{z}{k}\right)^{-1}$$

$\Rightarrow$ Γ does not vanish

$$\Gamma(1-z) = -z\,\Gamma(-z)$$

$$\Gamma(z).\Gamma(1-z) = -\,z\,\Gamma(z)\,\Gamma(-z)$$

$$= -\frac{z}{z}\cdot\frac{1}{(-z)}\cdot\prod_{k \geq 1}\left(1-\frac{z^2}{k^2}\right)^{-1}$$

$$= \frac{1}{z}\cdot\prod_{k \geq 1}\left(1-\frac{z^2}{k^2}\right)^{-1}$$

$$= \frac{\pi}{\sin \pi z}$$

Stirling's fmla.

Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$Re(x) > 0$

$Re(y) > 0$

$$B(x,y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)}$$

e.g.

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4}\int_0^1 \frac{1}{(1-u)^{1/2}}\frac{du}{u^{3/4}}$$

$u = t^4$

$du = 4t^3\,dt$

$$dt = \frac{1}{4}\ \frac{du}{t^3} = \frac{1}{4}\ \frac{du}{u^{3/4}}$$

$$= \frac{1}{4} B(1/4, 1/2) = \frac{1}{4}\ \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4}\ \frac{\Gamma(1/4)^2}{\frac{\pi}{\frac{\sqrt{2}}{2}}}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$=$$

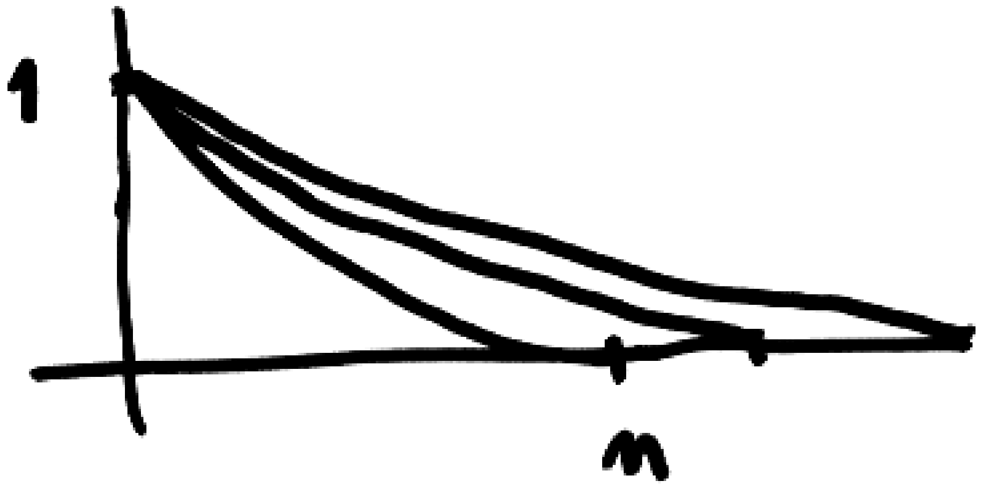
May 5, 2006

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^z \frac{dt}{t}$$

$$\varphi_n(t) := \begin{cases} \left(1-\frac{t}{n}\right)^n & 0 \le t \le n \\ 0 & t > n \end{cases}$$

$$\int_0^\infty \varphi_n(t)\, t^z \frac{dt}{t}$$

$$\varphi_n(t) \underset{n\to\infty}{\longrightarrow} e^{-t}$$ for given t



φ_n monotone

$$\varphi_n(t) \le \varphi_{n+1}(t)$$

Beppo-Levi

$$\gamma(x) = \int_x^\infty e^{-t} \frac{dt}{t}$$

incomplete gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

$$t \rightsquigarrow nt \qquad n \in \mathbb{N}$$

$$\Gamma(s) = n^s \int_0^\infty e^{-nt} t^s \frac{dt}{t}$$

$$\Gamma(s)\, n^{-s} = \int_0^\infty e^{-nt} t^s \frac{dt}{t}$$

$$\Gamma(s) \sum_{n \geq 1} n^{-s} = \int_0^\infty \sum_{n \geq 1} e^{-nt} t^s \frac{dt}{t}$$

$$\Gamma(s)\, \zeta(s) = \int_0^\infty \frac{e^{-t} t^s}{1 - e^{-t}} \frac{dt}{t}$$

$$= \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t}$$

$\rightsquigarrow$ analytic continuation

Better

$$\Gamma(s/2)\,\zeta(s) = \Gamma(s/2)\sum_{n\geq 1} n^{-s}$$

$$= \int_0^\infty \underbrace{\sum_{n\geq 1} e^{-n^2 t}}_{\rightsquigarrow\ \frac{\theta(t)-1}{2}} t^s \,\frac{dt}{t}$$

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$$

$$= 1 + 2e^{-\pi t} + 2e^{-\pi 4 t} \dots$$

$$\theta(1/t) = \sqrt{t}\,\theta(t)$$

$$\rightsquigarrow \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \Lambda(s)$$

extends meromorphic function ~~at~~ all s

$$\Lambda(1-s) = \Lambda(s)$$

functional equation

$$\zeta(s) = \prod_p \underbrace{\left(1 - \frac{1}{p^s}\right)^{-1}}_{\text{Euler factor}}$$

$$\underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\text{Euler factor } p=\infty.}\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \Lambda(s)$$

* * *

$$\gamma(x) = \int_x^\infty e^{-t}\,\frac{dt}{t}$$

$$\gamma(x)\,e^x = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} + \dots \frac{(-1)^n n!}{x^{n+1}}$$

$$+ \quad (-1)^{n+1}(n+1)! \int_x^\infty e^{x-t}\,t^{-n-1}\underbrace{\frac{dt}{t}}_{=: R_n(x)}$$

$$e^{x-t} \leq 1 \quad \text{for } x \leq t$$

$$|R_n(x)| \leq (n+1)! \int_x^\infty t^{-n-1}\frac{dt}{t}$$

$$= (n+1)!\,\frac{1}{(n+1)\,x^{n+1}}$$

$$\leq \frac{n!}{x^{n+1}}$$

$$\gamma(x)\, e^{x} = \frac{1}{x} - \frac{1}{x^2} + \cdots + \frac{(-1)^n n!}{x^{n+1}}$$

$$+ O\left(\frac{n!}{x^{n+1}} \right)$$

Fix n x large

we can use to compute

$\gamma(x)$

asymptotic series

$$\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$+ O\left(\frac{2n!}{x^{n+1}}\right)$$

$$\gamma(x)e^{x} \approx \sum_{n\geq 0} \frac{(-1)^n n!}{x^{n+1}}$$

$$\sum_{n\geq 0} (-1)^n n! z^n$$

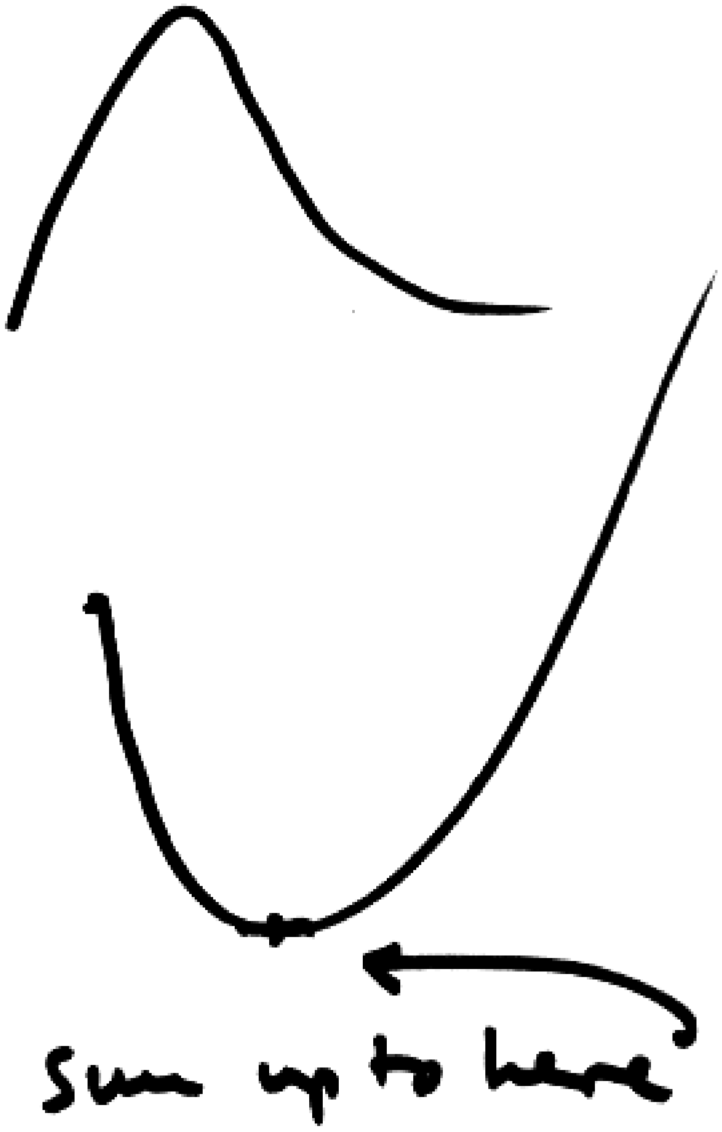
Radius of convergence $= 0$

Sum does not converge for a single x

$$e^{x} = \sum_{n\geq 0} \frac{x^n}{n!}$$

$x = 10^6$

$$\sum_{n\geq 0} (-1)^n \frac{n!}{x^{n+1}}$$



Lagrange

Formal calculation

$$\Delta f(x) := f(x+1) - f(x)$$

Taylor's theorem about x

$$f(x+1) = f(x) + \frac{f'(x)}{1!}1 + \frac{f''(x)}{2!}1^2 + \cdots$$

$$f(x+1) - f(x) = \sum_{n \geq 1} \frac{f^{(n)}(x)}{n!}$$

$$\Delta = \sum_{n \geq 1} \frac{D^n}{n!} = e^D - 1$$

$$Df := f'$$

$$\Delta^{-1} = \frac{1}{e^{D}-1}$$

$$= \frac{1}{D} \cdot \frac{D}{e^{D}-1}$$

$$= \frac{1}{D} \cdot \sum_{k \geqslant 0} B_k \frac{D^k}{k!}$$

$$\Delta^{-1} = \frac{1}{D} - \frac{1}{2} I + \frac{1}{6} D + \cdots$$

Euler - Maclaurin summation fmla

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)\,dx - \frac{1}{2}\left(f(n)+f(0)\right)$$

$$+ \sum_{k \geqslant 1} \frac{B_{2k}}{(2k)!} \cdot \left[f^{(2k-1)}(n) - f^{(2k-1)}(0) \right]$$

$$\log n! = \sum_{k=1}^{n} \log k$$

$$\sim \int_1^n \log x \, dx$$

$$= n \log n - n + 1$$

$$\sim n \log n - n$$

$$n! \sim e^{-n} n^n$$ not quite there yet

$$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

Stirling's fmla

Asymptotic expansion for

$\log \Gamma(z)$

involves Bernoulli numbers