$\boxed{a X^2 + b Y^2 = c}$ $\quad a \cdot b \neq 0$ $\quad$ $\boxed{\text{FRV 1985}}$ $\quad$ (1)

$\mathcal{C} = \{ P = (X, Y) \mid a X^2 + b Y^2 = c \}$

$P_0 = (x_0, y_0)$

1) $Y = \lambda \cdot (X - x_0) + y_0$ recta que pasa por P_0 de pendiente λ

$$a \cdot X^2 + b\left(\lambda^2 (X - x_0)^2 + y_0^2 + 2\lambda y_0 (X - x_0)\right) = c$$

$$a X^2 + y_0^2 \cdot b - c = a X^2 - a x_0^2 = a(X - x_0)(X + x_0)$$

$X \neq x_0$

$$\Rightarrow \lambda^2 (X - x_0) + 2\lambda y_0 b + a(X + x_0) = 0$$

$$X \cdot (b \cdot \lambda^2 + a) + 2\lambda y_0 b + (a - b\lambda^2) \cdot x_0 = 0$$

(i) $\quad b \lambda^2 + a = 0 \implies \lambda^2 = -\frac{a}{b} \Rightarrow -\frac{a}{b} \in k^2$

(ii) $\quad b \cdot \lambda^2 + a \neq 0$

$$\boxed{X = \frac{(b\lambda^2 - a) \cdot x_0 - 2\lambda y_0 b}{b\lambda^2 + a}} \quad (*)$$

$$Y = \lambda \cdot (X - x_0) + y_0$$

$$= \frac{b\lambda^2 y_0 + a y_0 + \lambda \cdot [(b\lambda^2 - a) x_0 - 2\lambda y_0 b - a x_0 - b\lambda^2 x_0]}{b \cdot \lambda^2 + a}$$

$$\boxed{Y = \frac{-(b\lambda^2 - a) y_0 - 2\lambda \cdot a \cdot x_0}{b\lambda^2 + a}} \quad (*)$$

Si $\lambda = \infty$ $\quad X = x_0 \quad Y = -y_0$

$P = (X, Y)$ pto en que la recta corta a $\mathcal{C}$.

1) La recta tangente a $\mathcal{C}$ en el p-to $P=(x,y)$ es de pendiente

$$\boxed{\lambda = -\frac{a}{b}\frac{x}{y}}$$

3) Definimos la siguiente operación

$$\mathcal{C}\times\mathcal{C} \xrightarrow{\ \cdot\ } \mathcal{C}$$
$$(P_1, P_2) \longmapsto P_1\cdot P_2$$

$P_1=(x_1,y_1)$ $\quad P_2=(x_2,y_2)$

Definimos $P_1\cdot P_2$ con las formulas (*)

y con λ

$$\boxed{\begin{array}{lll} (i) & \lambda := \dfrac{y_2-y_1}{x_2-x_1} & \text{si } P_1\neq P_2 \\ (ii) & \lambda := -\dfrac{a}{b}\cdot\dfrac{x_1}{y_1} & \text{si } P_1 = P_2 \end{array}}$$

• (i) $\lambda = \dfrac{y_2-y_1}{x_2-x_1}$ $\qquad x_2\neq x_1 \ (\Rightarrow P_2\neq P_1)$

$b\lambda^2+a=0 \iff b(y_2-y_1)^2+a(x_2-x_1)^2=0$

$\iff \underbrace{by_2^2+ax_2^2}_{c}+\underbrace{by_1^2+ax_1^2}_{c}-2(by_2y_1+ax_1x_2)=0$

$\iff \quad by_2y_1+ax_1x_2=c \qquad 2\neq 0$

(ii) $b\lambda^2+a=0 \iff b\dfrac{a^2}{b^2}\left(\dfrac{x_1}{y_1}\right)^2+a=0 \qquad y_1\neq 0$

$ab\neq 0 \qquad ax_1^2+by_1^2=0$ absurdo

(1) $(x_2,y_2)=P_2$ pertenece a la recta $by_1\cdot Y+ax_1X=c$

que es la recta tangente a $\mathcal{C}$ en $P_1 = (x_1, y_1)$ (2)
lo cual implica que $P_2 = P_1$ en contra de
lo supuesto
Usando (*) con $x_0 = x_1$, $y_0 = y_1$ y $\lambda = -\frac{a}{b}\frac{x_1}{y_1}$

$y_1 \neq 0$

$$X = \frac{\left(b\frac{a^2}{b^2}\frac{x_1^2}{y_1^2} - a\right)x_1 + 2\frac{a}{b}\frac{x_1}{y_1}\, y_1\, b}{b\frac{a^2}{b^2}\frac{x_1^2}{y_1^2} + a}$$

$$= \frac{(a^2x_1^2 - aby_1^2)\,x_1 + 2\,abx_1y_1^2}{a^2x_1^2 + bay_1^2}$$

$$= (ax_1^2 + by_1^2)\cdot x_1 / c$$

$$= x_1$$

Igualmente Y.

Entonces + está bien definida

- $P_1 \cdot P_2 = P_2 \cdot P_1$ es claro
- $P \cdot P_0 = P_0 \cdot P = P \quad \forall P \in \mathcal{C}$

Es geométricamente claro

$-P = \ell \cap \mathcal{C}$
ℓ recta $\parallel$ a la tg a $\mathcal{C}$ por P_0 que pasa por P

- $P^{-1} = (\bar{x}, \bar{y})$, $P = (x, y)$

$$\bar{x} = \frac{\left(b\cdot\frac{a^2}{b^2}\cdot\frac{x_0^2}{y_0^2} - a\right)\cdot x + 2\frac{a}{b}\, y\, b\cdot\frac{x_0}{y_0}}{b\frac{a^2}{b^2}\cdot\frac{x_0^2}{y_0^2} + a}$$

$$= \frac{(ax_0^2 - by_0^2)\cdot x + 2\,y\,y_0\,b\,x_0}{ax_0^2 + by_0^2}$$

$$= x + 2\frac{b}{c}y_0(yx_0 - xy_0)$$

$$\bar{Y} = \frac{-\left(b\frac{a^2}{b^2}\frac{x_0^2}{y_0^2} - a\right)\cdot y + 2\frac{a}{b}\frac{x_0}{y_0}\cdot a x}{b\frac{a^2}{b^2}\frac{x_0^2}{y_0^2} + a}$$

$$= \frac{-(a x_0^2 - b y_0^2)\, y_0 + 2 a x_0^2 y_0}{a x_0^2 + b \cdot y_0^2}$$

$$= \left(-(a x_0^2 + a x_0^2 - c)\cdot y + 2 a x x_0 y_0\right)/c$$

$$= y + 2\frac{a}{c} x_0 (x y_0 - y x_0)$$

$$\boxed{\begin{aligned} \bar{X} &= x + 2\frac{b}{c} y_0 (y x_0 - x y_0) \\ \bar{Y} &= y - 2\frac{a}{c} x_0 (y x_0 - x y_0) \end{aligned}}$$

Por construcción es claro que

$$P \cdot P^{-1} = P^{-1} \cdot P = P_0 \quad \forall P \in G$$

- Escribamos explícitamente las coord. del pto $P_1 \cdot P_2$

(i) $P_1 \neq P_2$

- $x_1 \neq x_2$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}$$

$$X = \frac{x_0\left[b(y_2-y_1)^2 - a(x_2-x_1)^2\right] - 2(y_2-y_1)\cdot y_0 b \,(x_2-x_1)}{b(y_2-y_1)^2 + a(x_2-x_1)^2}$$

$2(y_2-y_1)y_0 b\,(x_2-x_1)$

$$= \frac{x_0\cdot\left[b y_2^2 - a x_2^2 + b y_1^2 - a x_1^2 + 2(\overset{a}{x_1 x_2} - \overset{b}{y_1 y_2})\right] - }{2(c - a x_1 \cdot x_2 - b y_1 y_2)}$$

$$= \frac{x_0\cdot\left[c - 2 a x_2^2 + c - 2 a x_1^2 + 2 a x_1 x_2 - 2 b y_1 y_2\right] - 2(y_2-y_1) y_0 b\,(x_2-x_1)}{2(c - (a x_1 x_2 + b y_1 y_2)}$$

$$X = \frac{x_0 \left[c - a\cdot(x_2-x_1) - (a x_1 x_2 + b y_1 y_2)\right] - (y_2-y_1) y_0 b}{(c - (a x_1 x_2 + b y_1 y_2))} \quad (3)$$

$$X = x_0 - \frac{x_0\, a\, (x_2-x_1)^2 + y_0 (y_2 - y_1) b}{(c - (a x_1 x_2 + b y_1 y_2))} \qquad (x_2-x_1)$$

$$X = x_0 - (x_2 - x_1) \left[\frac{a \cdot x_0 (x_2 - x_1) + b y_0 (y_2 - y_1)}{c - (a x_1 x_2 + b y_1 y_2)} \right]$$

$$Y = \frac{-\left(b(y_2-y_1)^2 - a(x_2-x_1)^2\right) y_0 - 2 (y_2-y_1)(x_2-x_1)\, a\, x_0}{b(y_2-y_1)^2 + a(x_2-x_1)^2}$$

$$= \frac{-y_0 \left[2 b y_2^2 - c + 2 b y_1^2 - c + 2 a x_1 x_2 - 2 b y_1 y_2 \right] - 2 (y_2-y_1)(x_2-x_1)\, x_0\, a}{2 (c - (a x_1 x_2 + b y_1 y_2))}$$

$$= \frac{-y_0 \left[b(y_2-y_1)^2 + a x_1 x_2 + b y_1 y_2 - c\right] - (y_2-y_1)(x_2-x_1)\, x_0\, a}{c - (a x_1 x_2 + b y_1 y_2)}$$

$$= y_0 - (y_2 - y_1) \cdot \left[\frac{a x_0 (x_2 - x_1) + b y_0 (y_2 - y_1)}{c - (a x_1 x_2 + b y_1 y_2)} \right]$$

Definiendo $\phi : K^2 \times K^2 \longrightarrow K$

$$\phi((x_1, y_1), (x_2, y_2)) = a x_1 x_2 + b y_1 y_2$$

se puede escribir sintéticamente

$$\boxed{P_1 \cdot P_2 = P_0 - (P_2 - P_1) \cdot \frac{\phi(P_0, P_2 - P_1)}{c - \phi(P_2, P_1)} \qquad P_1 \neq P_2} \quad (\Delta)$$

- $x_1 = x_2 \quad y_2 \neq y_1 \Rightarrow \lambda = \infty \qquad -y_1 = y_2$

  En (*) se tiene $\quad X = x_0$
  $\quad Y = -y_0$

Da formula (1) segue se

$$X = x_0$$

$$Y = y_0\left(1 - \frac{b\, 4 y_2^2}{c - a x_2^2 + b y_2^2}\right) = y_0(1-2) = -y_0$$

$$b y_2^2$$

- $P_1 = P_2$

$$\lambda = -\frac{a}{b}\frac{x_1}{y_1} \qquad y_1 \neq 0$$

$$X = \frac{\left(b \frac{a^2}{b^2}\frac{x_1^2}{y_1^2} - a\right) x_0 + 2\frac{a}{b}\frac{x_1}{y_1} y_0 b}{b\frac{a^2}{b^2}\frac{x_1^2}{y_1^2} + a}$$

$$= \frac{(a x_1^2 - b y_1^2)\cdot x_0 + 2ab x_1 y_1 y_0}{a x_1^2 + b y_1^2}$$

$$= ((c - 2b y_1^2)\cdot x_0 + 2ab x_1 y_1 y_0)/c$$

$$= x_0 + 2\frac{b}{c} y_1 (y_0 x_1 - x_0 y_1)$$

$$Y = \frac{-\left(b\frac{a^2}{b^2}\frac{x_1^2}{y_1^2} - a\right) y_0 + 2\frac{a}{b}\frac{x_1}{y_1}\cdot x_0 a}{b\frac{a^2}{b^2}\frac{x_1^2}{y_1^2} + a}$$

$$= \frac{-(a x_1^2 - b y_1^2) y_0 + 2a x_1 y_1 x_0}{a x_1^2 + b y_1^2}$$

$$= (-(2a x_1^2 - c) y_0 + 2a x_1 y_1 x_0)/c$$

$$= y_0 - 2\frac{a}{c} x_1 (y_0 x_1 - x_0 y_1)$$

$$\boxed{P^2 = P_0 - \frac{2}{c}(x_0 y - y_0 x)(by, -ax)} \qquad (4)$$

De donde $P^2 = P_0 \iff x_0 y - y_0 x = 0$

$y_0 \neq 0 \qquad x = \frac{x_0}{y_0} \cdot y$

$$ax^2 + by^2 = c \Rightarrow a\frac{x_0^2}{y_0^2}y^2 + by^2 = c$$

$$\Rightarrow y^2 = \frac{y_0^2 \cdot c}{ax_0^2 + by_0^2} = y_0^2 \Rightarrow \begin{matrix} y = \pm y_0 \\ x = \pm x_0 \end{matrix}$$

$x_0 \neq 0 \qquad y = \frac{y_0}{x_0} x$

$$ax^2 + by^2 = c \Rightarrow ax^2 + b\frac{y_0^2}{x_0^2}x^2 = c$$

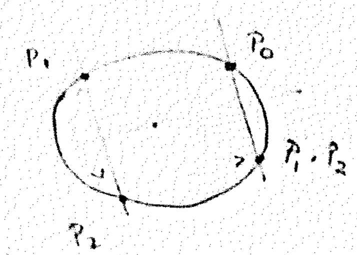
$$\Rightarrow x^2(ax_0^2 + by_0^2) = x_0^2 \cdot c \Rightarrow \begin{matrix} x = \pm x_0 \\ y = \pm y_0 \end{matrix}$$

Hay solo dos eltos de orden 2
P_0 y $-P_0$

- Otra manera de escribir la formula (Δ) es la siguiente

$$\boxed{P_1 \cdot P_2 = P_0 - 2 \cdot \frac{\phi(P_0, P_2 - P_1)}{\phi(P_2 - P_1, P_2 - P_1)} \cdot (P_2 - P_1) \qquad P_1 \neq P_2}$$

P_0 y $P_1 \cdot P_2$ son simetricos respecto de $(P_2 - P_1)^{\perp}$



• Si $-\frac{a}{b} < k^2$ la ec $aX^2 + b\cdot Y^2 = c$ se puede transformar en un del tipo

$G = \{(x,y) \mid x\cdot y = 1\}$

$\boxed{X \cdot Y = 1}$ hipérbola

Elegimos $P_0 = (1, 1)$

Recta de pend. λ que pasa por P_0

$$Y = \lambda(X-1)+1$$

$$X\cdot(\lambda(X-1)+1) = 1$$

$$\lambda(X-1)\cdot X = 1\cdot(1-X)$$

$X\neq 1, \lambda\neq 0$ $\quad X = -\frac{1}{\lambda} \qquad Y = -\lambda$

Si $\lambda = 0 \qquad X = 1$

$\lambda = -1$ da la recta tangente a G en P_0

La recta tg a G en el pto $P = (x, y)$ es

$x \neq 0$

$$Y = \lambda(X - x) + \frac{1}{x}$$

$$X\cdot\left(\lambda(X-x)+\frac{1}{x}\right) = 1$$

$$\lambda(X-x)\cdot X + 1\left(\frac{X}{x} - 1\right) = 0$$

$$\lambda x(X-x)X + 1\cdot(X-x) = 0$$

$X \neq x \Rightarrow \quad \lambda x X + 1 = 0$

$$X = -\frac{1}{\lambda x} \ ; \ Y = -\lambda x$$

$X = x$
$Y = y$
$\Rightarrow \lambda = -\frac{1}{x^2}$
Es decir la recta tangente a G por $P=(x,y)$ tiene pendiente $-\frac{1}{x^2}$

$P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ $\quad P_1 \neq P_2$

$P_1\cdot P_2 = \left(-\frac{1}{\lambda}, -\lambda\right) \qquad \lambda = \frac{y_2 - y_1}{x_2 - x_1}$

$$-\frac{1}{\lambda} = -\frac{x_2 - x_1}{y_2 - y_1} = -\frac{x_2 - x_1}{\frac{1}{x_2} - \frac{1}{x_1}} = -\frac{x_2 - x_1}{\frac{x_1 - x_2}{x_1 \cdot x_2}} = x_1 \cdot x_2$$

$$P_1 \cdot P_2 = \left(x_1 \cdot x_2 , \frac{1}{x_1 \cdot x_2} \right)$$

$$P = (x, y) \qquad P^2 = \left(-\frac{1}{\lambda} , -\lambda \right)$$

$$\lambda = -\frac{1}{x^2} \Rightarrow P^2 = \left(x^2 , \frac{1}{x^2} \right)$$

Entonces

$$\begin{aligned} \mathcal{C} &\longrightarrow (\mathbb{R}^*, \cdot) \\ (x, y) &\longmapsto x \end{aligned}$$

es un isomorfismo de grupos

- Tomemos $\mathcal{C} = \{ (x, y) \mid y = x^2 \}$

$$\boxed{Y = X^2} \qquad \text{parabola}$$

Elegimos $P_0 = (0, 0)$

Recta de pend. λ que pasa por P_0

$$Y = \lambda X$$

$$\lambda X = X^2 \qquad X \neq 0$$

$$\lambda = X \qquad \lambda^2 = Y$$

Recta tangente a $\mathcal{C}$ en $P = (x, y)$

$$Y = \lambda (X - x) + y$$

$$\lambda (X - x) + y = X^2$$

$$\lambda (X - x) = X^2 - y = X^2 - x^2 = (X + x)(X - x)$$

$X \neq x \qquad \lambda = X + x \qquad X = \lambda - x$

Si $X = x \Rightarrow \lambda = 2x$

$P_1 = (x_1, y_1) \quad P_2 = (x_2, y_2)$

$P_1 \neq P_2 \qquad \lambda = \dfrac{y_2 - y_1}{x_2 - x_1}$

$P_1 \cdot P_2 = (\lambda, \lambda^2)$

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} = \frac{x_2^2 - x_1^2}{x_2 - x_1} = \frac{(x_2 - x_1)(x_2 + x_1)}{x_2 - x_1} = x_2 + x_1$$

$P_1 \cdot P_2 = (x_1 + x_2, (x_1 + x_2)^2)$

$P = (x, y) \qquad \lambda = 2x$

$P^2 = (\lambda, \lambda^2) = (2x, 4x^2)$

Entonces

$$\mathcal{C} \longrightarrow (k, +) \quad \text{es iso de grupos}$$
$$(x, y) \longmapsto x$$

- $\boxed{x^2 + by^2 = c} \quad -b \notin k^2$

Si $-b \notin k^2$ considero $F = k(\sqrt{-b})$ extensión cuadrática de k

$\mathcal{C}_k = \{ (x, y) \in k \times k \mid x^2 + by^2 = c \}$

$\mathcal{C}_F = \{ (x, y) \in F \times F \mid x^2 + by^2 = c \}$

Es claro que $\mathcal{C}_k \hookrightarrow \mathcal{C}_F$ naturalmente

$F = \{ u + v \cdot \varepsilon \mid u, v \in k, \ \varepsilon = \sqrt{-b} \}$

$P_0 = (x_0, y_0) \qquad x_0 + by_0^2 = c$

$(x_0 + \varepsilon y_0)^{-1} = \frac{1}{c}(x_0 - \varepsilon y_0)$

$(x + \varepsilon y) \cdot \frac{1}{c} \cdot (x_0 - \varepsilon y_0) = \frac{1}{c}\left(x \cdot x_0 + b y y_0 + \varepsilon(x_0 y - y_0 y)\right)$

CamScanner

$$X' = \frac{1}{c}\cdot(x\cdot x_0 + b\,y\cdot y_0)$$ Vale sin que $-b \in k^2$

$$Y' = \frac{1}{c}(x_0\,y - y_0\,x)$$

$$X'^2 + bY'^2 = \frac{1}{c^2}(x^2 x_0^2 + b^2 y^2 y_0^2 + 2xy\,x_0 y_0 b) +$$

$$+ \frac{b}{c^2}(y^2 x_0^2 + x^2 y_0^2 - 2xy\,x_0 y_0) =$$

$$= \frac{1}{c^2}\cdot(x_0^2 + b y_0^2)(x^2 + b y^2) = 1$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \frac{1}{c}\begin{pmatrix} x_0 & b y_0 \\ -y_0 & x_0 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$P = \frac{1}{c}\cdot\begin{pmatrix} x_0 & b\,y_0 \\ -y_0 & x_0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} x_0 & -b\cdot y_0 \\ y_0 & x_0 \end{pmatrix}$$

$$X^2 + b\cdot Y^2 = (X \quad Y)\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= (X' \; Y')\begin{pmatrix} x_0 & y_0 \\ -by_0 & x_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x_0 & -by_0 \\ y_0 & x_0 \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix}$$

$$= c\cdot(X'Y')\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} X' \\ Y' \end{pmatrix} = c\cdot(X' + bY'^2)$$

$$X^2 + bY^2 = c \iff X'^2 + bY'^2 = 1$$

$G_k \longrightarrow G'_k$ es iso de grupos, con $P_0' = (1,0)$

$(x,y) \longmapsto (x',y')$

$$\boxed{X^2 + bY^2 = 1} \quad -b \notin k^2 \qquad F = k(\sqrt{-b})$$

$$\mathcal{C}_k = \{ (x,y) \in k \times k \mid x^2 + by^2 = 1 \} \qquad P_0 = (1,0)$$

$$\mathcal{C}_F = \{ (x',y') \in F \times F \mid x'^2 + by'^2 = 1 \}$$

$$\mathcal{C}_k \hookrightarrow \mathcal{C}_F \qquad\qquad \varepsilon = \sqrt{-b}$$

$$X^2 + bY^2 = X^2 - (\varepsilon Y)^2 = (X+\varepsilon Y)(X - \varepsilon Y) = 1$$

$$X' = X + \varepsilon Y \qquad P = \begin{pmatrix} 1 & \varepsilon \\ 1 & -\varepsilon \end{pmatrix} \qquad P^{-1} = \frac{1}{2\cdot\varepsilon}\begin{pmatrix} \varepsilon & \varepsilon \\ 1 & -1 \end{pmatrix}$$

$$Y' = X - \varepsilon Y$$

$$X^2 + bY^2 = 1 \iff X' Y' = 1$$

$$\mathcal{C}'_F = \{ (x', y') \in F \times F \mid x' \cdot y' = 1 \}$$

$$\mathcal{C}_F \xrightarrow{\simeq} \mathcal{C}'_F \xrightarrow{\simeq} (\dot{F}, \cdot) \quad \text{iso de grupos}$$

Se ve que

$$\mathcal{C}_k \simeq \{ \omega \in \dot{F} \mid N_{F/k}(\omega) = 1 \}$$

$$P_1 = (x_1, y_1) \qquad P_2 = (x_2, y_2) \quad \in k \times k \qquad P_1 \neq P_2$$

$$P_1 \cdot P_2 = (1,0) - (P_2 - P_1)\,\frac{\phi((1,0), P_2 - P_1)}{1 - \phi(P_1, P_2)}$$

$$\phi(P, P') := x \cdot x' + b\, y \cdot y'$$

$$P_1 \cdot P_2 = \left(\frac{b(y_2-y_1)^2 - (x_2-x_1)^2}{b(y_2-y_1)^2 + (x_2-x_1)^2} \,,\, \frac{-2(y_2-y_1)(x_2-x_1)}{b(y_2-y_1)^2 + (x_2-x_1)^2} \right)$$

Considerando $\quad z_1 = x_1 + \varepsilon y_1 \quad z_2 = x_2 + \varepsilon y_2$ en $k(\varepsilon) = F$

$$P_1 \cdot P_2 \rightsquigarrow -\frac{(z_1-z_2)^2}{N_{F/k}(z_1-z_2)} = -\frac{(z_1-z_2)}{(z_1-z_2)^\sigma} = -\frac{(z_1-z_2)}{(z_1^\sigma - z_2^\sigma)}$$

$$(z_1^\sigma - z_2^\sigma)\cdot z_1 z_2 = z_1^\sigma z_1 z_2 - z_2^\sigma z_1 z_2 = z_2 - z_1$$

Es decir $P_1 P_2$ corresponde con $z_1 \cdot z_2$ es decir

$$P_1 \cdot P_2 = (x_1 \cdot x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

($P = (x,y)$

$$P^2 = (1,0) - 2y\cdot(by, -x)$$
$$= (1 - 2by^2,\ 2yx)$$
$$= (x^2 - by^2,\ 2yx)$$

$$P \rightsquigarrow z = x + \varepsilon y$$
$$P^2 \rightsquigarrow (x^2 - by^2) + 2xy\cdot\varepsilon = z^2$$

(

Otra demostración de que $P_1 \cdot P_2 \mapsto z_1 \cdot z_2$

Basta ver que las rectas que unen z_1 con z_2 y $z_1 \cdot z_2$ con 1 son paralelas. Esto es equivalente a que $\dfrac{z_1 - z_2}{z_1 \cdot z_2 - 1} \in k$. Aplicando σ

$$\left(\frac{z_1 - z_2}{z_1 z_2 - 1}\right)^\sigma = \frac{z_1^\sigma - z_2^\sigma}{z_1^\sigma \cdot z_2^\sigma - 1} = \frac{\frac{1}{z_1} - \frac{1}{z_2}}{\frac{1}{z_1 \cdot z_2} - 1} = \frac{(z_2 - z_1)/z_1 \cdot z_2}{(1 - z_1 z_2)/z_1 z_2} =$$

$$= \frac{z_2 - z_1}{1 - z_1 z_2}$$