Topics in K-theory and L-functions

May 9, 2002

1 Relation with Basic Toric Varieties

1.1 Introduction

Consider $P(x,y) \in \mathbb{C}[x,y,x^{-1},y^{-1}]$. We construct its Newton polytope Δ. Take, for example, $P(x,y) = 1 + x + x^2 + x^3 y - x^2 y^3 + xy^3 + y^2 + 17xy$. See Figure 1.

Take Y to be the zero locus of $P(x,y)$ in $\mathbb{C}^\times \times \mathbb{C}^\times$. We will complete Y. Rather than completing it using the projective space, we will complete it to an X in a surface X_Δ, which will carry divisors, one for each face, which will intersect in the same way as the faces. (Usually we add one line to $\mathbb{C} \times \mathbb{C}$ to get $\mathbb{P}^2(\mathbb{C})$, instead, we will add six lines to $\mathbb{C}^\times \times \mathbb{C}^\times$).

Fact: $\text{genus}(X) \leq \#\{\text{interior points of}\,\Delta\}$ and it is equal generically. Then the Newton polytope of a generic degree 3 polynomial has one interior point. For a generic polynomial in two variables of degree d, it is a rectangular triangle which has sides of length d. It will have $\frac{(d-1)(d-2)}{2}$ points, as we could expect for the genus of a generic polynomial of degree d. See Figure 2.

The case of hyperelliptic curves gives another example where we know how to compute genus. An hyperelliptic curve has an equation of the form $y^2 = f(x)$ with $\deg f = d$. The Newton polytope will have $\left[\frac{d-1}{2}\right]$ interior points and this number is equal to the genus. See Figure 2 again.

So we have $Y \subset \mathbb{C}^\times \times \mathbb{C}^\times$, and x, y are rational functions on this curve.

Theorem 1 $(x,y)_w \in \mu_\infty$ *for all* $w \in Y \iff$ *the roots of* P_τ *are zero or elements in* μ_∞.

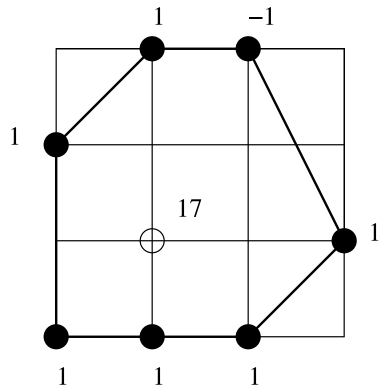


Figure 1: Newton polytope for $P(x,y) = 1 + x + x^2 + x^3y - x^2y^3 + xy^3 + y^2 + 17xy$.

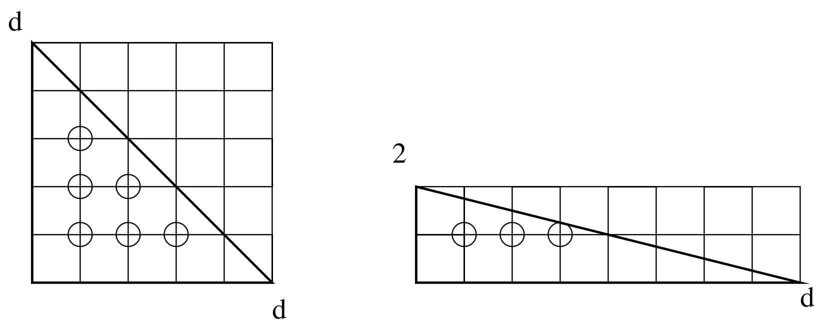


Figure 2: The genus is in general equal to the number of interior points. First case for a polynomial of degree d in two variables. Second case for a degree d hyperelliptic curve.

Ultimately, we will want to consider $\{x, y\} \in K_2(\mathbb{C}(Y))$. We will be looking for pairs $\{\varphi_1, \varphi_2\}$, which are in the kernel of all the Tame symbols, this is analogous to the case of $K_2(\mathcal{O}_F)$.

1.2 Basic Toric Varieties

A good reference for what follows can be found in [F].

Definition 2 *A toric variety is a normal variety X containing a torus $\mathbb{T}$, $\mathbb{T} \cong (\mathbb{C}^\times)^n$, as a dense Zariski open subset, together with an action $\mathbb{T} \times X \to X$, of $\mathbb{T}$ on X, that extends the natural action of $\mathbb{T}$ on itself.*

Essentially, one completes $\mathbb{T}$ by adding various objects at ∞ in a coherent way.

Following, we will show a way of constructing toric varieties. Let N be a lattice, i.e. $N \cong \mathbb{Z}^n$, and let $M = Hom(N, \mathbb{Z}) \cong \mathbb{Z}^n$. We have the pairing:

$$\langle\,,\rangle : M \times N \longrightarrow \mathbb{Z} : (u, v) \to u(v)$$

Definition 3 *A strongly convex rational polyhedral cone, or more simply, a cone, $\sigma \subset N_\mathbb{R} = N \otimes \mathbb{R}$, is a convex polyhedral cone, i.e. a set of the form*

$$\{r_1 v_1 + \ldots + r_s v_s \in N_\mathbb{R} \,|\, r_i \geq 0\}$$

with vertex at the origin, spanned by a finite number of vectors in the lattice and such that it contains no line through the origin. See Figure 3.

The dual cone $\check{\sigma} \subset M_\mathbb{R}$ is defined as

$$\check{\sigma} = \{u \in M_\mathbb{R} \,|\, \langle u, v\rangle \geq 0 \,\forall\, v \in \sigma\}$$

Given σ, $\check{\sigma}$ is spanned by the primitive inward pointing normals to facets (faces of codimension 1) with coordinates in $\mathbb{Z}$. See Figure 3.

Then $S_\sigma = \check{\sigma} \cap M$ is a semigroup.

Lemma 4 (Gordon) *S_σ is finitely generated.*

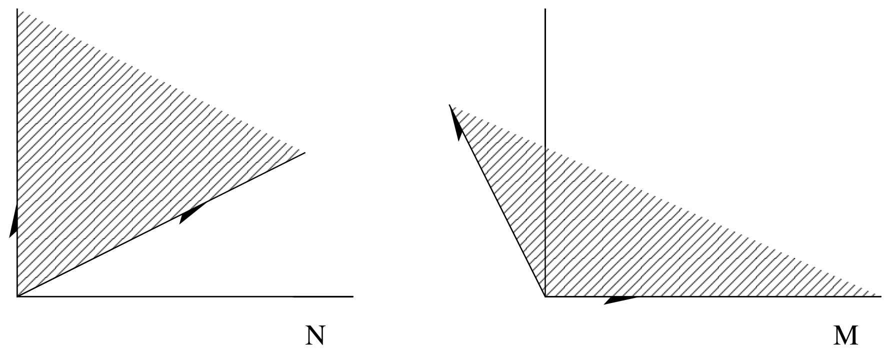


Figure 3: Cone $\sigma \subset N$ and dual cone $\check{\sigma} \subset M$.

Proof. Take $u_1, \dots u_s \in \check{\sigma} \cap M$ generating $\check{\sigma}$ as a cone. Consider $K = \{\sum t_i u_i \,|\, 0 \leq t_i \leq 1\}$. K is compact and M is discrete, hence $K \cap M$ is finite. Claim: $K \cap M$ generates the semigroup. Indeed, if $u \in \check{\sigma} \cap M$, then $u = \sum r_i u_i$, $r_i \geq 0$. Write $r_i = m_i + t_i$, $m_i \in \mathbb{Z}_{\geq 0}$, $t_i \in [0,1]$. Then $u = \sum m_i u_i + u'$, with $u_i \in K \cap M$. Also $u' \in K$. Since $u \in M$ and $\sum m_i u_i \in M$, then $u' \in M$, hence $u' \in K \cap M$, and $K \cap M$ generates the semigroup. $\square$

We will build an S_σ-algebra

$$A_\sigma = \mathbb{C}[S_\sigma] = \left\{ \sum_{m \in S_\sigma} a_m x^m \right\}$$

where $m \in S_\sigma$ corresponds to $x^m \in A_\sigma$. (Here we are using the multi index notation).

For example if σ is spanned by $(1,0),(0,1)$, then $\check{\sigma}$ is also spanned by $(1,0),(0,1)$ and $A_\sigma \cong \mathbb{C}[x_1, x_2]$ naturally.

Finally, we will associate to σ the affine toric variety

$$U_\sigma = Spec(A_\sigma)$$

Note that we could have vectors in $\check{\sigma}$ which are not integral linear combination of the generators, (this will imply that U_σ is singular). Take for instance, σ spanned by $(0,1),(2,-1)$. Then $\check{\sigma}$ is spanned by $(1,0),(1,2)$. Note that $(1,1)$ is in the interior of σ but is not an integral linear combination of $(0,1)$ and $(2,-1)$. See Figure 4.

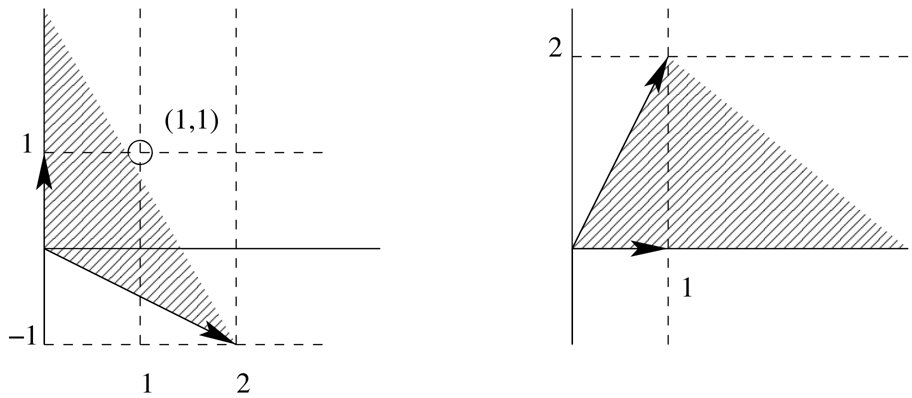


Figure 4: The point $(1,1)$ belongs to σ but is not an integral linear combination of $(0,1)$ and $(2,-1)$.

Definition 5 *σ is smooth if it is generated by part of a $\mathbb{Z}$-basis for N. Think of*

$$\sigma = \{\lambda_1 v_1 + \ldots + \lambda_n v_n \,|\, \lambda_i \in \mathbb{R}_{\geq 0}\}$$

Then σ is smooth if we can add some elements to $\{v_i\}$ in such a way that we get a basis for N.

Definition 6 *σ is simplicial if the v_i are part of a basis for $N_{\mathbb{R}}$.*

Proposition 7 *U_σ is smooth iff σ is smooth. U_σ is orbifold iff σ is simplicial.*

Note that in $\mathbb{R}^2$ smooth and simplicial coincide.

Continuing with our second example, we get

$$A_\sigma = \mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[u, v, w]/(v^2 - uw)$$

A_σ is then the coordinate ring for a cone, which certainly has a singularity.

Toric varieties in general are obtained by gluing affine toric varieties. For example, we can obtain $\mathbb{P}^1$ by gluing two copies of $\mathbb{C}$ having $\mathbb{C}^\times$ in common (Figure 5).

Definition 8 *A fan Σ is a collection of cones $\sigma \in N_{\mathbb{R}}$ such that*

- $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 \in \Sigma$,

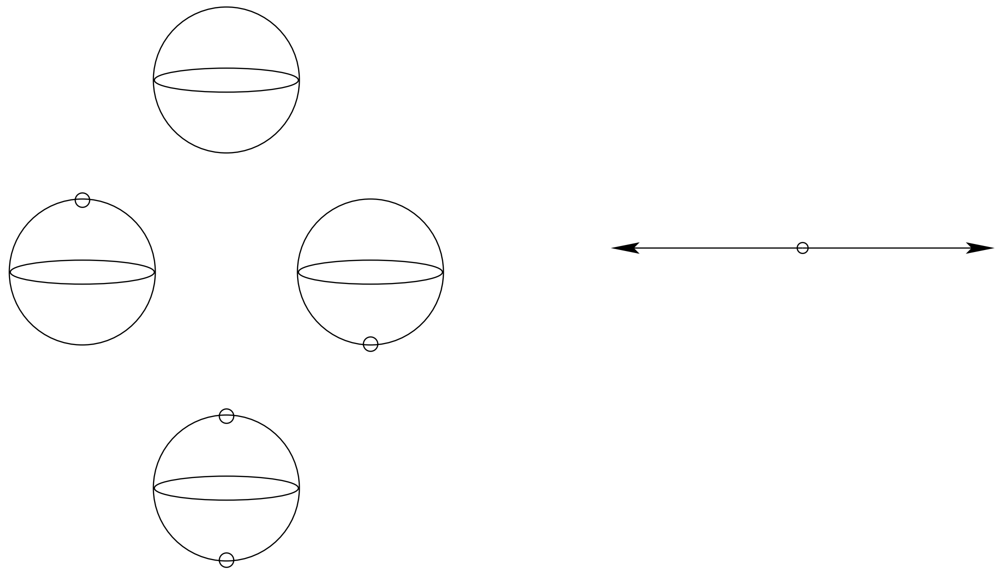


Figure 5: $\mathbb{P}^1$ is homeomorphic to a sphere which is the union of two copies of $\mathbb{C}$ (homeomorphic to a sphere minus a point). The two copies of $\mathbb{C}$ have $\mathbb{C}^\times$ in common which is homeomorphic to a sphere minus two points. The fan for $\mathbb{P}^1$ is composed by the cones $\mathbb{R}_{\leq 0}$, $\{0\}$ and $\mathbb{R}_{\geq 0}$.

- $\tau < \sigma$ *(a face)*, $\sigma \in \Sigma \Rightarrow \tau \in \Sigma$

The fan Σ gives rise to X_Σ, a toric variety. Basically, given $\sigma_1, \sigma_2 \in \Sigma$, let $\tau = \sigma_1 \cap \sigma_2$ and glue U_{σ_1} along U_{σ_2} by identifying the images of $U_\tau \hookrightarrow U_{\sigma_1}$ and of $U_\tau \hookrightarrow U_{\sigma_2}$.

In order to get $\mathbb{P}^1$ we take Σ a 1-dimensional fan which is union of $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, and $\{0\}$. See Figure 5. The algebras are: $\mathbb{C}[x]$, $\mathbb{C}[x^{-1}]$, and $\mathbb{C}[x, x^{-1}]$ respectively. The affine varieties are: $\mathbb{C}$, $\mathbb{C}$, and $\mathbb{C}^\times$ with a patching isomorphism given by $x \mapsto x^{-1}$ on the overlap.

Let Δ be a polytope in a lattice $M_\mathbb{R}$. Say, for instance, that Δ is the triangle with vertices $(0,0), (3,0)$ and $(0,3)$ (Figure 6). Then $\mathbb{T} = (\mathbb{C}^\times)^2$; we will add three divisors and get $\mathbb{P}^2$.

We get X_Δ by making the fan Σ_Δ. We take n_τ the primitive inward normal vector to each $\tau < \Delta$ facet. So a face $\eta < \Delta$ yields σ_η = cone spanned by n_τ, $\eta \leq \tau$. This is a complete fan that gives rise to X_Δ which is always projective. (Not all the fans are of the form Σ_Δ and not all these varieties are projective).

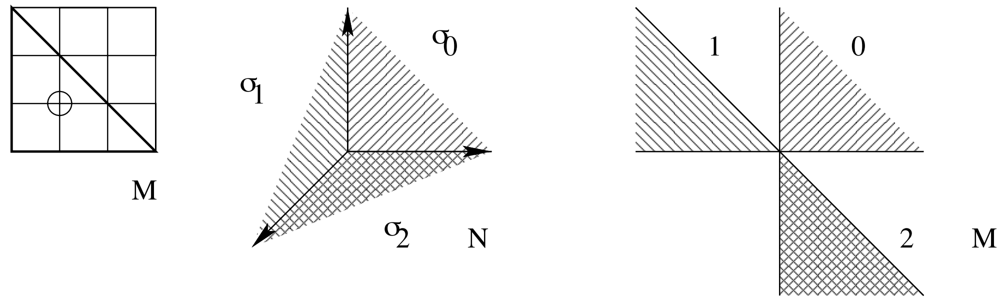
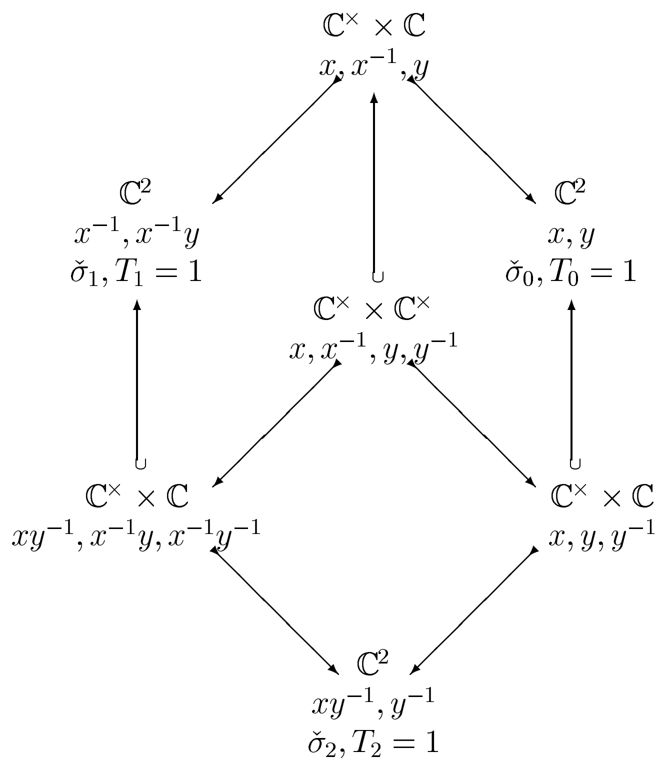


Figure 6: Starting from a polytope $\subset M_{\mathbb{R}}$ we get a fan in N and the fan of dual cones in M.

In our example, Σ_Δ is composed by seven cones: three 2-dimensional, three 1-dimensional and 0. Now we look at the duals of the σ_i. See Figure 6. We get the following diagram (the arrows are inclusions):



All of them glue together to give $\mathbb{P}^2$ with homogeneous coordinates $[T_0 : T_1 : T_2]$

$$x = \frac{T_1}{T_0}, y = \frac{T_2}{T_0}, \quad \text{so} \quad \begin{cases} x^{-1} = \frac{T_0}{T_1} & x^{-1}y = \frac{T_2}{T_1} \\ y^{-1} = \frac{T_0}{T_2} & xy^{-1} = \frac{T_1}{T_2} \end{cases}$$

Another way of constructing X_Δ is: set

$$A_\Delta = \bigoplus_{m \in M \cap nA,\ n=0,1,\ldots} \mathbb{C}[x^m t^n]$$

This is a graded ring where $\deg(x^m t^n) = n$. Then take $X_\Delta = Proj(A_\Delta)$, the standard construction from algebraic geometry.

Back to the first method, what happens with Δ = square of vertices $(\pm 1, \pm 1)$? This will yield $\mathbb{P}^1 \times \mathbb{P}^1$. We get the diagram:

$$\begin{array}{ccccc}
\begin{matrix}\mathbb{C}^2 \\ x^{-1}, y\end{matrix} & \hookleftarrow & \begin{matrix}\mathbb{C}^\times \times \mathbb{C} \\ x, x^{-1}, y\end{matrix} & \hookrightarrow & \begin{matrix}\mathbb{C}^2 \\ x, y\end{matrix} \\
\uparrow & & \uparrow & & \uparrow \\
\begin{matrix}\mathbb{C}^\times \times \mathbb{C} \\ x^{-1}, y, y^{-1}\end{matrix} & \hookleftarrow & \begin{matrix}\mathbb{C}^\times \times \mathbb{C}^\times \\ x, x^{-1}, y, y^{-1}\end{matrix} & \hookrightarrow & \begin{matrix}\mathbb{C}^\times \times \mathbb{C} \\ x, y, y^{-1}\end{matrix} \\
\downarrow & & \downarrow & & \downarrow \\
\begin{matrix}\mathbb{C}^2 \\ x^{-1}, y^{-1}\end{matrix} & \hookleftarrow & \begin{matrix}\mathbb{C}^\times \times \mathbb{C} \\ x, x^{-1}, y^{-1}\end{matrix} & \hookrightarrow & \begin{matrix}\mathbb{C}^2 \\ x, y^{-1}\end{matrix}
\end{array}$$

If we start with Δ = triangle of vertices $(0,0)$, $(2,1)$, and $(1,2)$, we get a singular variety which we will have to desingularize. To do that, one can refine the fan, so that the cones are smooth, by adding vectors. See Figure 7.

Now, given a Laurent polynomial

$$P \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$$

we get

$$P \rightsquigarrow \Delta \rightsquigarrow X_\Delta \rightsquigarrow \tilde{X}_\Delta$$

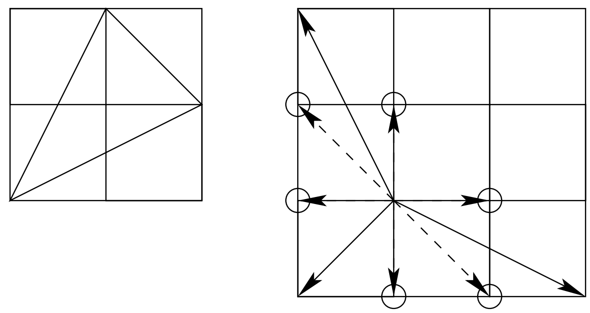


Figure 7: One can desingularize the resulting variety by refining the fan.

(the last term is the desingularized variety). Consider the curve $Y : P(x,y) = 0$, $Y \subset (\mathbb{C}^\times)^2$, completed to $\tilde{Y}$ in $X_\Delta \supset (\mathbb{C}^\times)^2$. Each side in the polytope corresponds to a divisor:

$$\tau < \Delta \longleftrightarrow D_\tau \text{ on } X_\Delta$$

which is itself a copy of $\mathbb{P}^1$, a 1-dimensional toric variety.

We will view x, y as rational functions on X_Δ. Now each D_τ gives a valuation on the function field of the toric variety, which is ord_{D_τ}, and can be read off the polytope. In fact

$$\mathrm{ord}_{D_\tau}(x^m) = \langle v, m \rangle$$

where v is the primitive normal determining the facet τ. Here, (Figure 8), $\mathrm{ord}_{D_\tau}(x^1 y^0) = \langle (0,1), (1,0) \rangle = 0$. The idea is that τ corresponds to $y = 0$ and the order of vanishing of x should be 0.

Take another example: $P(x,y) = x^2y + xy^2 + x^2 + x + y + k$. The set $\{P(x,y) = 0\}$ is generally an elliptic curve. The Newton polytope has five faces. Each face has a linear polynomial associated to it, so it corresponds to exactly one point, then each D_i corresponds to a point in the variety. See Figure 8. We have the following:

	v (normal)	$\langle v, (1,0) \rangle$	$\langle v, (0,1) \rangle$
D_1	$(-1, 0)$	-1	0
D_2	$(-1, -1)$	-1	-1
D_3	$(1, -1)$	1	-1
D_4	$(1, 1)$	1	1
D_5	$(0, 1)$	0	1

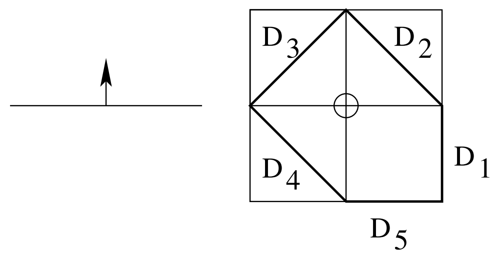


Figure 8: Reading divisors from the Newton polytope.

Then $(x) = -D_1 - D_2 + D_3 + D_4 \qquad (y) = -D_2 - D_3 + D_4 + D_5$

Δ is an intersection of half spaces, so it is given by data

$$\Delta = \{m \in M_{\mathbb{R}} | \langle m, n_\tau \rangle \geq -a_\tau\} \quad a_\tau \in \mathbb{Z}$$

Each $\tau < \Delta$ corresponds to a 1-dimensional cone in Σ_Δ generated by n_τ which corresponds to a divisor $D_\tau \subset X_\Delta$. We have

$$X_\Delta \setminus \mathbb{T} = \bigcup_{\tau < \Delta} D_\tau$$

(the divisors make up the rest of the variety outside the torus $\mathbb{T}$). Define

$$D_\Delta = \sum_{\tau < \Delta} a_\tau D_\tau$$

a_τ and D_Δ depend on the location of the origin. However, changing the origin will replace D_Δ with an equivalent divisor. Then

$$H^0(X_\Delta, \mathcal{O}_{X_\Delta}(D_\Delta)) = \bigoplus_{m \in \Delta} \mathbb{C} x^m$$

(as vectorial spaces). The monomials x^m, with $m \in M$, yield rational functions on X_Δ:

$$x^m : \mathbb{T} \longrightarrow \mathbb{C}^\times$$

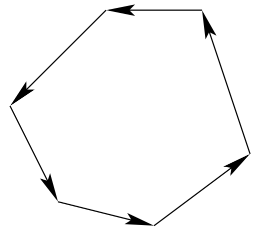


Figure 9: Choose the counterclockwise orientation in the Newton polytope.

In the example of $\mathbb{P}^2$, the sections are just homogeneous polynomials of degree 3. (Placing the origin in the interior point). We have

$$\mathrm{ord}_{D_\tau}(x^m) = \langle m, n_\tau \rangle$$

For the $n = 2$ case, there is only one direction which is orthogonal to n_τ, so the only nonvanishing x^m are in the direction of τ.

We will stay in the plane from now on. Let Δ be as before. Choose counterclockwise orientation on Δ. See Figure 9. There is a unique monomial t_τ such that all monomials not vanishing on D_τ are of the form t_τ^k for $k \in \mathbb{Z}$. So $t_\tau : D_\tau \xrightarrow{\sim} \mathbb{P}^1$.

Now take P a Laurent polynomial in x, y, with polytope Δ. So P is a section of this line bundle. We will let $Z = Z(P) \subset X_\Delta$ be the zero locus of P. We have essentially just studied plane curves, but on a toric variety, this is not the usual projective space. Here is the scenario:

$$\begin{array}{ccc} \{P = 0\} & \subset & \mathbb{C}^\times \times \mathbb{C}^\times \\ \downarrow & & \downarrow \\ Z & \subset & X_\Delta \end{array}$$

How does Z intersect the divisors? We claim:

$$D_\tau \cap Z \xrightarrow{t_\tau} \{P_\tau = 0\} \subset \mathbb{C}^\times \subset \mathbb{P}^1$$

Recall that P_τ is the polynomial in one variable corresponding to the face τ.

Let's revisit Tame Symbols. C will be used to denote a smooth projective curve over $\mathbb{C}$. Now if w is a point of C, it gives a valuation of $\mathbb{C}(C)$. The tame symbol is

$$(x, y)_w = (-1)^{w(x)w(y)} \left. \frac{x^{w(y)}}{y^{w(x)}} \right|_w \in \mathbb{C}^\times$$

The term $\frac{x^{w(y)}}{y^{w(x)}}$ is a monomial in x, y and it is $\neq 0, \infty$ by construction. Here x, y are rational functions on $Z \subset X_\Delta$. We will assume that P is irreducible, and denote $\Delta = \Delta(P)$ its Newton polygon as usual.

Note that $(x, y)_w \equiv 1$ when $w \notin$ divisors of x, y, so the only relevant points $w' \in C$ are those above points $w \in D_\tau \cap Z$.(If w is singular, we are considering w' above it). Recall that

$$t_\tau : D_\tau \xrightarrow{\sim} \mathbb{P}^1 \quad \text{takes} \quad D_\tau \cap Z \longrightarrow \{P_\tau = 0\} \subset \mathbb{C}^\times$$

Claim: $(x, y)_{w'} = \pm$ value of a monomial at $w \in D_\tau \cap Z$, hence it must be a power of t_τ, say t_τ^k. Thus $(x, y)_{w'} = \pm t_\tau^k(w)$. But $t_\tau(w)$ is a root of P_τ, which is what we wanted.

Finally we prove the

Theorem 9 $(x, y)_{w'} \in \mu_\infty$ *for all* $w' \Longleftrightarrow$ *the roots of* P_τ *are in* $\mu_\infty \cup \{0\}$ *for all* τ*.*

Proof. If $k \neq 0$ the assertion is clear. We have to argue that $k \neq 0$. If $k = 0$, then $(x, y)_w$ would be constant. It is also $= \pm \frac{x^{w(y)}}{y^{w(x)}}$. If $w \in D_\tau \cap Z$, then $(w(y), -w(x)) \neq 0$, so this quotient is not constant and it restricts to a non constant monomial on D_τ and we could not possibly have $k = 0$. □

For our purposes we will define

$$K_2(C) \otimes \mathbb{Q} = \bigcap_w \{\ker(\text{tame symbol at } w)\} \otimes \mathbb{Q}$$

Corollary 10 $\{x, y\} \in K_2(C) \otimes \mathbb{Q} \Longleftrightarrow$ *roots of* P_τ *are in* $\mu_\infty \cup \{0\}$*.*

1.3 An example with a family of elliptic curves

We will work with a family of elliptic curves with a 5-torsion point. Consider the following family of polynomials

$$P_k(x, y) = (x+y+1)(x+1)(y+1) - kxy = x^2y + xy^2 + x^2 + y^2 + (3-k)xy + 2x + 2y + 1$$

We see that it has one interior point, which is coherent with the fact that it has genus 1. Note that the P_τ are either $(t + 1)$ or $(t + 1)^2$. Taking $k \in \mathbb{Q} \neq 0$ gives a family of elliptic curves E_k and $\{x, y\} \in K_2(E_k) \otimes \mathbb{Q}$.

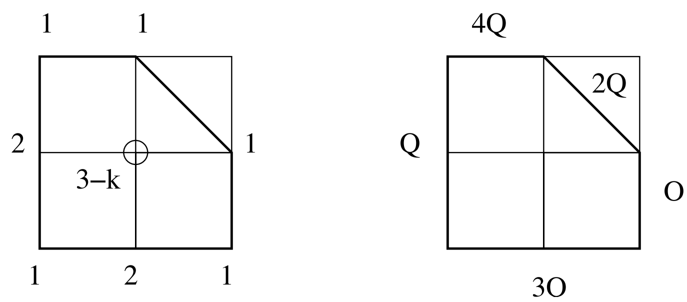


Figure 10: For this family of elliptic curves we can see clearly the correspondence between the sides of the Newton polytope and the divisors of x and y.

The curve E_k is isogenous to

$$F_k : v^2+(1-k)uv-kv = u^3-ku^2 \qquad \text{via} \qquad \begin{cases} x = \frac{-u^2+ku-kv}{u(u-k)} = -1 - \frac{kv}{u(u-k)} \\ y = \frac{v}{u-k} \end{cases}$$

We claim that $Q = (0,0)$ is a point of order 5 on F_k. We have $2Q = (k,k^2)$, $3Q = (k,0)$, $4Q = (0,k)$, $5Q = \mathbf{O}$. We would like to compute the divisors of x, y. Let's find the divisors of $u, v, u-k$ first.

	u	v	$u-k$	$-u^2+ku-kv$
$\mathbf{O}$	-2	-3	-2	-4
$Q=(0,0)$	1	2	0	1
$2Q=(k,k^2)$	0	0	1	0
$3Q=(k,0)$	0	1	1	3
$4Q=(0,k)$	1	0	0	0

We can then compute the divisors:

$$(x) = -(2Q) + 2(3Q) - (4Q)$$
$$(y) = -\mathbf{O} + 2(Q) - (2Q)$$

It turns out that this can be read off Δ. To do that, one checks the motion in the direction x as one transverses the perimeter counterclockwise. We get: $2, 0, -1, -1, 0$, this shows (y). One does the same with the y direction and gets $0, -1, -1, 0, 2$, which corresponds to (x).

The $\{nQ\}$ occur where $Z(P_k)$ intersects $\cup_{\tau<\Delta} D_\tau$ (one per face in this instance). See Figure 10.

We claim $(x,y)_w \in \mu_\infty$ ($\Rightarrow \{x,y\} \in K_2(E_k) \otimes \mathbb{Q}$). Let's compute

$$(x,y)_Q = (-1)^{0\cdot 2} \left.\frac{x^2}{y^0}\right|_Q = x(Q)^2$$

since $\operatorname{ord}_Q(x) = 0$ and $\operatorname{ord}_Q(y) = 2$. Recall that $Q = (0,0)$ in u, v coordinates. We have

$$x = -1 - \frac{kv}{u(u-k)}$$

Since $\operatorname{ord}_Q(u) = 1$, $\operatorname{ord}_Q(v) = 2$ and $\operatorname{ord}_Q(u-k) = 0$, then the second term is zero when evaluated at Q. So $x(Q) = -1$ and then

$$(x,y)_Q = 1$$

Analogously, we get: $(x,y)_{2Q} = (x,y)_{3Q} = 1, (x,y)_{4Q} = (x,y)_{\mathbf{O}} = -1$.

2 Regulator map on curves

For the content of this section, we refer to [RV].

Let C be a smooth projective curve over $\mathbb{C}$. We will construct a function

$$\operatorname{reg}: K_2(C) \otimes \mathbb{Q} \longrightarrow H^1(C, \mathbb{R})$$

whose image will be a differential form.

Let $x, y \in \mathbb{C}(C)^\times$. Define

$$\eta(x,y) = \log|x| d \arg y - \log|y| d \arg x$$

Where $d \arg x := \Im\left(\frac{dx}{x}\right)$ is well defined in spite of the fact that arg cannot be continuously defined in $\mathbb{C} \setminus \{0\}$.

Let $S = \{\text{zeros and poles of } x \text{ or } y\} \subset C(\mathbb{C})$ where η is not defined.

Proposition 11 *η is a closed differential form on $C \setminus S$.*

Proof. We have,

$$\frac{dx}{x} = d\log|x| + i d \arg x \qquad \frac{dy}{y} = d\log|y| + i d \arg y$$

$$\Im\left(\frac{dy}{y}\wedge\frac{dx}{x}\right) = d\log|x|d\arg y - d\log|y|d\arg x = d\eta(x,y)$$

Now look at the dimension of the cohomology group to conclude that

$$\Im\left(\frac{dy}{y}\wedge\frac{dx}{x}\right) = 0$$

□

Remark 12 *$\eta(x,y)$ is a cup product of $\log|x|$ and $\log|y|$ in the Deligne cohomology.*

Some properties of η are:

- $\eta(x_1x_2,y) = \eta(x_1,y)+\eta(x_2,y)$
- $\eta(y,x) = -\eta(x,y)$
- $\eta(x,1-x) = 0$ "in H^1"

Later on we will make the last statement more precise.

Now, since $\eta(x,y)$ is closed, it can be regarded as an element of $H^1(C\setminus S,\mathbb{R})$. Here we think of $H^1(C\setminus S,\mathbb{R})$ as the dual of $H_1(C\setminus S,\mathbb{Z})$. More precisely, $\eta(x,y)$ corresponds to

$$[\gamma]\longmapsto\int_\gamma\eta(x,y)$$

So far, $[\gamma]\in H_1(C\setminus S,\mathbb{Z})$. The picture we have in mind is to extend η to a map on $H_1(C,\mathbb{Z})$. In order to get that, we need that the integral over an element of $H_1(C,\mathbb{Z})$ does not depend on the path that represents it. What we need is that

$$\oint\eta(x,y) = 0$$

around any point $w\in S$.

Lemma 13

$$\frac{1}{2\pi}\oint\eta(x,y) = \log|(x,y)_w|$$

Corollary 14 *η will extend to all of C iff $|(x,y)_w| = 1$. In particular, this is so if $\{x,y\} \in K_2(C) \otimes \mathbb{Q}$.*

Proof of Lemma. Since both sides are bimultiplicative and skew-symmetric, we are reduced to the cases $(w(x), w(y)) = (0,0), (0,1), (1,1)$.

The first case is trivial. Consider the second. It is clear that $|(x,y)_w| = |x(w)|$. Now this is a local question and we can take local coordinates. Let (U, ϕ) be a coordinate system with $0 \in U$, $\phi : U \longmapsto C(\mathbb{C})$, $\phi(0) = w$. Write $f = x \circ \phi$. Observe that $f(0) \neq 0$ since $x(w) \neq 0$. Assume that the pullback of y is the parameter t (i.e. $\phi(t) = y$). Then what we have to prove is

$$\log|f(0)| = \frac{1}{2\pi i}\oint \log|f|\frac{dt}{t} - \frac{1}{2\pi}\oint \log|t| d\arg f$$

If we take the integration loop close enough to zero, $\arg f$ is a well defined function and the second integral must be zero. Then we get the equality with the first integral by direct application of Jensen's formula. The third case is similar. $\square$

As we mentioned in the Corollary, if $|(x,y)_w| = 1 \; \forall w \in S$ then we have a well defined function from

$$H_1(C, \mathbb{Z}) \longmapsto \mathbb{R} \qquad [\gamma] \mapsto \int_\gamma \eta(x,y)$$

For illustration only. Let's call

$$K_{2,S}(C) \otimes \mathbb{Q} = \bigcap_{w \notin S} \{\ker(\text{tame symbol at } w)\} \otimes \mathbb{Q}$$

Then η can be defined over $K_{2,S}(C) \otimes \mathbb{Q}$. We have the following commutative diagram:

$$\begin{array}{ccc} K_{2,S}(C) \otimes \mathbb{Q} & \xrightarrow{\eta} & H^1(C \setminus S, \mathbb{R}) \\ \downarrow{\scriptstyle (\;,\;)_w} & & \downarrow{\scriptstyle \mathrm{Res}_w} \\ \mathbb{C}^\times & \xrightarrow{\log|\cdot|} & \mathbb{C} \end{array}$$

In order to define the η over $K_{2,S}(C) \otimes \mathbb{Q}$ properly we still need to prove:

Proposition 15 *$\eta(x, 1-x) = 0$ in $H^1(C, \mathbb{R})$. (This is a differential form on the whole curve, since all the tame symbols are $= 1$ on $(x, 1-x)$).*

Proof. If $C = \mathbb{P}^1$, then $\eta = 0$ at once, since $\mathbb{P}^1$ is simply connected and η is closed. In the general case, $x : C \longmapsto \mathbb{P}^1$, then it is easy to see that $\eta(x, 1-x) = x^*\eta(t, 1-t)$ where t is a parameter in $\mathbb{P}^1$, and from this, the statement follows. □

As Tate has pointed out, $\eta(x, 1-x) = 0$ means that η must be the differential of some function on $\mathbb{C}$. This function turns out to be the Bloch – Wigner dilogarithm:

Definition 16 *The Bloch – Wigner dilogarithm is defined by:*

$$D(t) := \Im(Li_2(t)) + \log|t| \arg(1-t) \ \textit{where} \ \arg(1-t) \in (-\pi, \pi)$$

Proposition 17 *If $C = \mathbb{P}^1$ with parameter t, then*

$$\eta(t, 1-t) = dD(t)$$

Proof. Recall that

$$\mathrm{Li}_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^2} \quad (\text{for } |t| < 1) \qquad = -\int_0^t \log(1-u)\frac{du}{u}$$

We choose the branch of $\log(1-u)$ defined on $\mathbb{C}\setminus[1, \infty)$ for which $\log(1-0) = 0$. Then $-\int_0^t \log(1-u)\frac{du}{u}$ extends $\mathrm{Li}_2(t)$ to this domain.

We have seen that $\int_\epsilon^t \eta(u, 1-u)$ is independent of the path in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Define

$$f(t) := \lim_{\epsilon \to 0} \int_\epsilon^t \eta(u, 1-u)$$

Then $f(t) = \int_0^t \log|u| d\arg(1-u) - \int_0^t \log|1-u| d\arg u$ by definition.

Now for $u \in \mathbb{C} \setminus [1, \infty)$ we have:

$$\log(1-u) = \log|1-u| + i\arg(1-u)$$
$$\frac{du}{u} = d\log|u| + i d\arg u$$

We get

$$\Im(\mathrm{Li}_2(t)) = -\int_0^t d\log|u| \arg(1-u) - \int_0^t \log|1-u| d\arg u$$

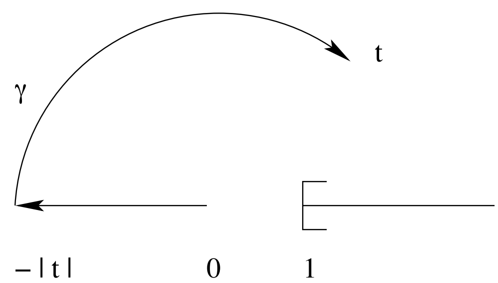


Figure 11: Integration path.

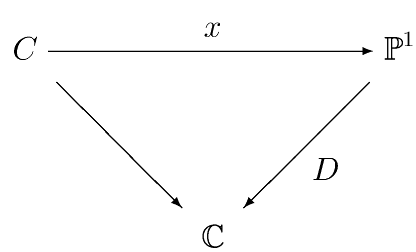
Now choose the integration path as follows: start from zero and go straight to $-|t|$, then go from $-|t|$ to t with constant radio and without crossing the real line (Figure 11). In the first leg, $\arg u$ is constant, then $d\arg u = 0$, also $\arg(1-u) = 0$. In the second leg, $|u|$ is constant and $d\log|u|$ vanishes. Then the first integral vanishes and we get

$$f(t) = \int_0^t \log|u| d\arg(1-u) - \int_0^t \log|1-u| d\arg u = \arg(1-t)\log|t| + \Im(\mathrm{Li}_2(t)) = D(t)$$

$$\Longrightarrow \int_0^t \eta(u, 1-u) = D(t)$$

But the integral of η should be path independent on all of $\mathbb{C}$, so we conclude that the jump over the branch cut of $\mathrm{Li}_2(t)$ must be balanced by the jump of $\arg(1-t)$. Then we get that $D(t)$ is analytic on $\mathbb{C}\setminus\{0,1\}$ and continuous on $0, 1$.

The general case $\eta(x, 1-x) = dD(x)$ is solved via the diagram



□

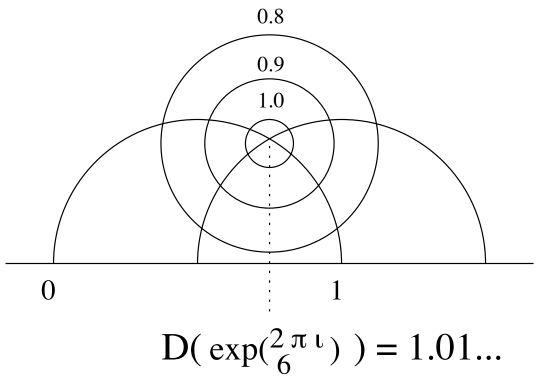


Figure 12: Distribution of values of $D(z)$.

3 Properties of the Bloch – Wigner dilogarithm

We will investigate $D(z)$. It has the following properties:

$$D(e^{i\theta}) = \Im(\mathrm{Li}_2(e^{i\theta})) = \Im\left(\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^2}\right) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$$

$$D(\bar{z}) = -D(z)$$

Then $D \equiv 0$ in $\mathbb{R}$.

Also, D assumes its maximum at $e^{\frac{2\pi i}{6}}$. It can be viewed as a function $\mathbb{P}^1(\mathbb{C}) \longmapsto \mathbb{R}$. Figure 12 sketches how the values of $D(z)$ are distributed.

3.1 Maillot's example

Consider this example due by Maillot:

Example 18 *Let* $a_1x_1 + a_2x_2 + a_3x_3 \in \mathbb{C}[x_1, x_2, x_3]$. *Then*

$$\pi m(a_1x_1+a_2x_2+a_3x_3) = \begin{cases} D\left(\left|\frac{a_1}{a_3}\right| e^{i\alpha_2}\right) + \alpha_1 \log|a_1| + \alpha_2 \log|a_2| + \alpha_3 \log|a_3| & \triangle \\ \pi \log \max\{|a_1|, |a_2|, |a_3|\} & \textit{not } \triangle \end{cases}$$

Here the $\triangle$ stands for the fact of whether $|a_1|, |a_2|$ and $|a_3|$ are the sides of a triangle. In other words, whether we have the inequalities:

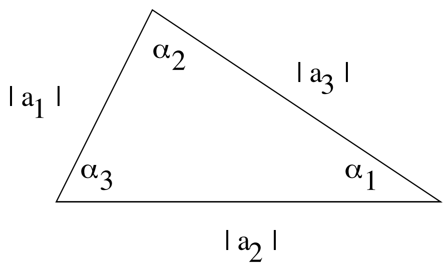


Figure 13: Triangle figure for Maillot's example.

$$|a_1| + |a_2| > |a_3| \qquad |a_2| + |a_3| > |a_1| \qquad |a_3| + |a_1| > |a_2|$$

The α_i is the angle in the triangle which is opposite to the side of measure $|a_i|$. See Figure 13.

Proof. First, let's observe that we can assume $a_i \in \mathbb{R}_{>0}$, because we can change each variable by multiplying it by a complex number of absolute value one and this will not change the Mahler measure of the polynomial.

Consider the second case, assume that we have $a_3 \geq a_1 + a_2$. We apply Jensen's formula and obtain:

$$\begin{aligned} m(a_1x_1 + a_2x_2 + a_3x_3) &= m\left(a_3\left(x_3 - \left(\frac{-a_1x_1 - a_2x_2}{a_3}\right)\right)\right) \\ &= \log a_3 + \frac{1}{(2\pi i)^2}\int_{T^2} \log^+\left|\frac{a_1x_1 + a_2x_2}{a_3}\right| \frac{dx_1}{x_1}\frac{dx_2}{x_2} \end{aligned}$$

Since $a_3 \geq a_1 + a_2$, then $a_3 \geq |a_1x_1 + a_2x_2|$ as long as $x_1, x_2 \in T$. Then $\left|\frac{a_1x_1+a_2x_2}{a_3}\right| \leq 1$ and so the integral vanishes and we have proved this case.

Now for the first case, we will write our polynomial in the form

$$P(x, y) = y + ax - b, \qquad a, b \in \mathbb{R}_{>0}$$

for simplicity of notation. By applying Jensen's formula,

$$\begin{aligned} m(P) &= \frac{1}{2\pi i}\int_{|x|=1} \log^+|b - ax|\frac{dx}{x} = \frac{1}{2\pi}\int_{|x|=1} \log^+|y| d\arg x \\ &= \frac{1}{2\pi}\int_{\gamma} \log|y| d\arg x = \frac{-1}{2\pi}\int_{\gamma} \eta(x, y) \end{aligned}$$

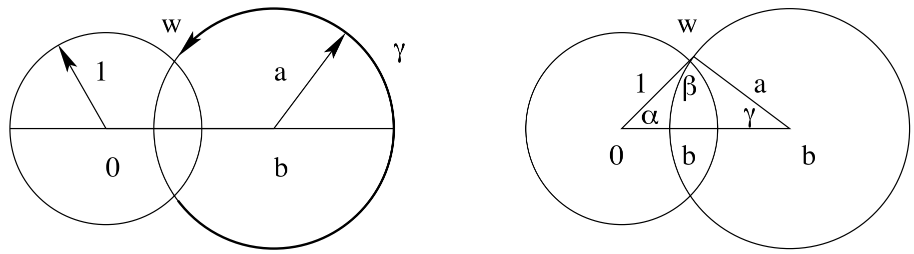


Figure 14: Integration path. Triangle of sides a, b and 1.

Because of the formula $\eta(x, y) = \log |x| d \arg y - \log |y| d \arg x$ and the fact that $|x| = 1$. Here γ is the path where $|b - ax| > 1$, which is part of a circle of center b and radius a. See Figure 14.

It will be convenient to work in $\Lambda^2(\mathbb{C}(\mathbb{P}^1)^\times)_0$ where the subindex 0 stands for the tensorial product over $\mathbb{Q}$.

We claim that:

$$x \wedge y = \frac{ax}{b} \wedge \left(1 - \frac{ax}{b}\right) + \frac{b}{a} \wedge \left(1 - \frac{ax}{b}\right) + x \wedge b \qquad (1)$$

Proof.

$$\frac{ax}{b} \wedge \left(1 - \frac{ax}{b}\right) = \frac{a}{b} \wedge \left(1 - \frac{ax}{b}\right) + x \wedge \left(1 - \frac{ax}{b}\right) = -\frac{b}{a} \wedge \left(1 - \frac{ax}{b}\right) + x \wedge (b - ax) - x \wedge b$$

We conclude the result from the fact that $b - ax = y$. $\square$

Applying η to the equation (1):

$$\eta(x, y) = dD\left(\frac{ax}{b}\right) + \log \frac{b}{a} \, d \arg \left(1 - \frac{ax}{b}\right) - \log b \, d \arg x$$

Now we integrate. The first term can be integrated using Stokes Theorem. Let α be the angle opposite to the side of length a, β opposite to the side of length b and γ to the side of length 1 (Figure 14).

The $d \arg \left(1 - \frac{ax}{b}\right)$ term goes between $\arg \left(1 - \frac{a\bar{x}_0}{b}\right)$ and $\arg \left(1 - \frac{ax_0}{b}\right)$, where $w = b - ax_0$. So, $\frac{\bar{w}}{b} = 1 - \frac{a\bar{x}_0}{b}$ and hence $\arg \left(1 - \frac{a\bar{x}_0}{b}\right) = \arg \bar{w} = -\alpha$. Also $\arg \left(1 - \frac{ax_0}{b}\right) = \arg w = \alpha$. The total difference of arguments is 2α.

The $d \arg x$ term goes between $\arg \bar{x}_0$ and $\arg x_0$. By the use of geometry, it is possible to compute $\arg x_0 = -\gamma$. Then, $\arg \bar{x}_0 = \gamma$, and the difference of arguments is $2\pi - 2\gamma$.

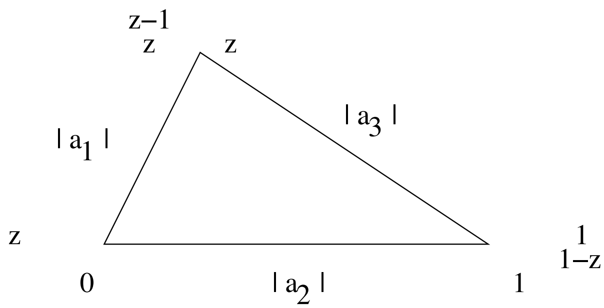


Figure 15: A triangle is described, up to similarity, by a parameter in $\mathcal{H}$.

Putting all of this together:

$$-2\pi m(P) = D\left(\frac{ax_0}{b}\right) - D\left(\frac{a\bar{x}_0}{b}\right) + 2\alpha \log \frac{b}{a} - 2(\pi - \gamma)\log b$$

Then use the fact that $D(z) = -D(\bar{z})$.

$$\pi m(P) = D\left(\frac{a\bar{x}_0}{b}\right) - \alpha \log \frac{b}{a} + (\pi - \gamma)\log b$$

Using the fact that $\alpha + \beta + \gamma = \pi$:

$$\pi m(P) = D\left(\frac{a}{b}e^{i\gamma}\right) + \alpha \log a + \beta \log b$$

And we are done. $\square$

Observe that the formula should be symmetric. In fact, we can describe the triangle (up to similarity) by using only one parameter: any triangle is similar to one that has vertices equal to $0, 1, z$ where $z \in \mathcal{H}$. See Figure 15.

Interchanging sides and angles, the first term should not change. This translates into the equality of this six numbers:

$$D(z) = D\left(\frac{z-1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{z}{z-1}\right)$$

(Using $D(z) = -D(\bar{z})$). The number we have got corresponds to an invariant of the triangle. We associate a number to each vertex by doing the quotient of the two sides that converge to the vertex, always in counterclockwise order. We get: z, $\frac{1}{1-z}$ and $\frac{z-1}{z}$, see Figure 15. Now, if we apply the dilogarithm to each of those numbers, we always get the same value, this is the invariant for the class of similarity of the triangle.

In this frame of work, we get the following result:

Corollary 19

$$D(z)+\log|z|\arg(1-z)+\log|1-z|\arg(z)\geq\pi\max\{\log|z|,\log|1-z|\}$$

Proof. It is enough to apply the fact that the Mahler measure of the Newton Polygon is greater or equal to the Mahler measure of the sides to the polynomial $1+zx+(1-z)y\in\mathbb{C}[x,y]$. □

The simplest particular case of this result by Maillot was proved by Smyth:

$$\pi m(x+y+z)=D\left(e^{\frac{2\pi i}{6}}\right)=1.01494\ldots$$

In fact,

$$\pi m(P)=D(\xi_6)=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{2\pi n}{6}\right)}{n^2}=\frac{\sqrt{3}}{2}\sum_{n=1}^{\infty}\frac{\delta(n)}{n^2}$$

Where δ is a character modulo six with the following values: $0,1,1,0,-1,-1$. If χ is the primitive character modulo three, then $\delta+\chi$ is a character modulo six with the values: $0,2,0,0,0,-2$. This can be expressed in this way:

$$\delta(n)+\chi(n)=\begin{cases}2\chi(n) & n\,\text{odd}\\ 0 & n\,\text{even}\end{cases}$$

Then

$$\sum_{n=1}^{\infty}\frac{\delta(n)}{n^2}=-\sum_{n=1}^{\infty}\frac{\chi(n)}{n^2}+2\sum_{n=1,\,2\nmid n}^{\infty}\frac{\chi(n)}{n^2}=\left(-1+2\left(1+\frac{1}{2^2}\right)\right)L(\chi,2)=\frac{3}{2}L(\chi,2)=L'(\chi,-1)$$

$$m(P)=\frac{3\sqrt{3}}{4\pi}L(\chi,2)=\frac{\sqrt{3}}{2\pi}L'(\chi,-1)$$

3.2 Five term relation

Let $\{z_j\}_{j\geq0}\in\mathbb{C}\cup\{\infty\}$ be a sequence defined by

$$\begin{cases}z_0=a,\ z_1=b\\ z_{j+1}z_{j-1}=1-z_j\end{cases}$$

Where $(a,b) \in (\mathbb{C} \cup \{\infty\})^2 \neq (0,1),(1,0)$. It is easy to see that this sequence repeats itself cyclically modulo 5 reaching the values: $a, b, \frac{1-b}{a}, \frac{a+b-1}{ab}, \frac{1-a}{b}$ in this order.

Incidentally, observe that the first five equations define a surface in $\mathbb{P}^5$ that is called Del Pezzo surface (and it was first studied by Elkies):

$$S = \begin{cases} 1 - z_1 & = & z_2 z_0 \\ 1 - z_2 & = & z_3 z_1 \\ 1 - z_3 & = & z_4 z_2 \\ 1 - z_4 & = & z_0 z_3 \\ 1 - z_0 & = & z_1 z_4 \end{cases}$$

It is easy to prove that:

$$\sum_{j=0}^{4} z_j \wedge (1 - z_j) = 0$$

by developing $z_j \wedge (1 - z_j) = z_j \wedge z_{j+1} + z_j \wedge z_{j-1}$.

On S, we get

$$\sum_{j=0}^{4} \eta(z_j, 1 - z_j) = 0 \quad \Rightarrow \quad d\left(\sum_{j=0}^{4} D(z_j)\right) = 0 \quad \Rightarrow \quad \sum_{j=0}^{4} D(z_j) = \text{constant}$$

In order to compute the constant, take, for instance: $a = i, b = -i$, then the other three values are: $1 - i, -1, 1 + i$. We ge

$$D(i)+D(-i)+D(1-i)+D(-1)+D(1+i) = D(i)-D(i)+D(1-i)-D(1-i) = 0$$

Then the constant is zero and we get the five term relation:

$$\sum_{j=0}^{4} D(z_j) = 0$$

□

Observe that this method, which will be used again in the future, is also good to prove the equalities of the six symmetrical terms of D. Take, for instance, $D\left(\frac{1}{1-z}\right) = D(z)$:

$$\frac{1}{1-z} \wedge \left(1 - \frac{1}{1-z}\right) = -(1-z) \wedge \frac{-z}{1-z} = -(1-z) \wedge (-z) = z \wedge (1-z)$$

(here we are strongly using the fact that we are tensorizing with $\mathbb{Q}$, so there is no torsion on the wedge product). We can see the equality taking, for instance, $z = 2$.

4 Relation with Hyperbolic Geometry

Some of the proofs of this part can be found in [M].

We will be working on $\mathbb{H}^3$, the hyperbolic 3-space. It can be represented as $\mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$ with a special metric that has constant curvature -1. The metric is given by

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

with volume element

$$\frac{dx\, dy\, dz}{z^3}$$

This is analogous to $\mathbb{H}^2$, the upper half plane with the hyperbolic metric. While the group of isometries of $\mathbb{H}^2$ is $PSL_2(\mathbb{R})$, the group of isometries (preserving orientation) of $\mathbb{H}^3$ is $PSL_2(\mathbb{C})$. Keep in mind that the restriction of this action on $\mathbb{C} \cup \{\infty\}$ is as usually:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

4.1 Dilogarithm and Volume of Ideal Tetrahedra

We will be considering the ideal tetrahedron (v_1, v_2, v_3, ∞). See Figure 16. Ideal stands for the fact that its vertices belong to $\mathbb{C} \cup \infty$ and one of them is actually ∞. By the action of $PSL_2(\mathbb{C})$ we can send (v_1, v_2, v_3, ∞) to $(0, 1, \infty, z)$ and this will not change the volume of the tetrahedron. Hence, if we want to consider the volume of those kinds of tetrahedra, it will depend only on the parameter z. Actually, depending on the election of the action, we could end up with any of z, $\frac{1}{1-z}$, $\frac{z-1}{z}$, we will see that they are equivalent in some sense to us. Let's denote this tetrahedron Δ_z.

Theorem 20

$$Vol(\Delta_z) = D(z)$$

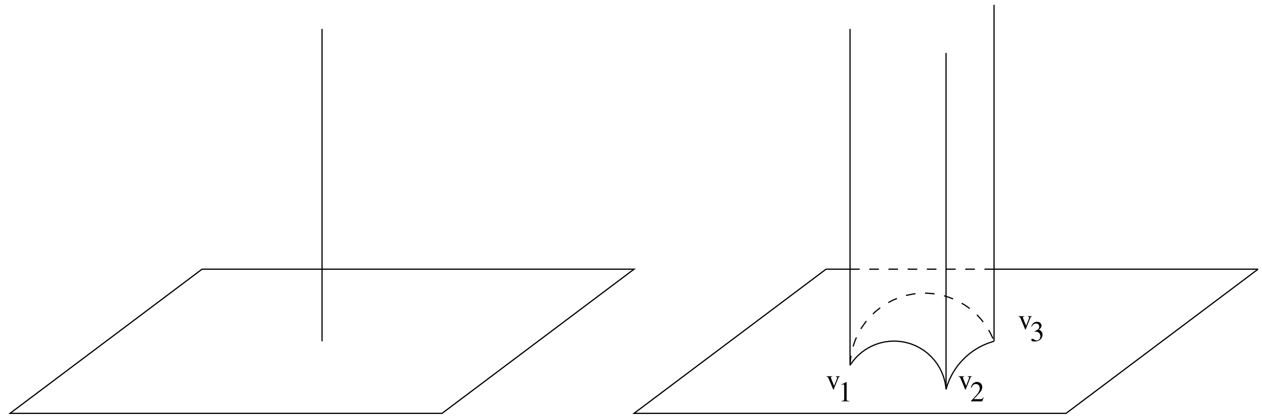


Figure 16: The hyperbolic space and an ideal tetrahedron.

The sign on the volume stands for orientation. This Theorem gives an interpretation to Maillot's formula: the dilogarithm term is the volume of a hyperbolic tetrahedron such that if we cut the tetrahedron by a plane which is parallel to the plane $\mathbb{C}$, we get an Euclidean triangle whose sides are $|a_1|$, $|a_2|$ and $|a_3|$. (Figure 17).

We will need the following:

Lemma 21

$$Vol(\Delta_z) = \mathfrak{L}(\alpha) + \mathfrak{L}(\beta) + \mathfrak{L}(\gamma)$$

Where the α, β and γ are the angles of the Euclidean triangle whose sides are $|a_1|$, $|a_2|$ and $|a_3|$ that we mentioned above. (Observe that this triangle is invariant by similarities). And

$$\mathfrak{L}(\sigma) := -\int_0^\sigma \log|2\sin t|dt$$

is the Lobachevski function.

Proof of Lemma. Suppose that the triangle has all the angles acute (the other case is similar). We can normalize the tetrahedron such that the triangle with vertices in $\mathbb{C}$ is inscribed in a circle of radio 1. Consider the circumcenter, and consider the radios from the center to the vertices and the perpendicular lines from the center to the sides. We get the triangle divided in six small triangles. Consider the triangle $\overset{\triangle}{OCA'}$. See Figure 18.

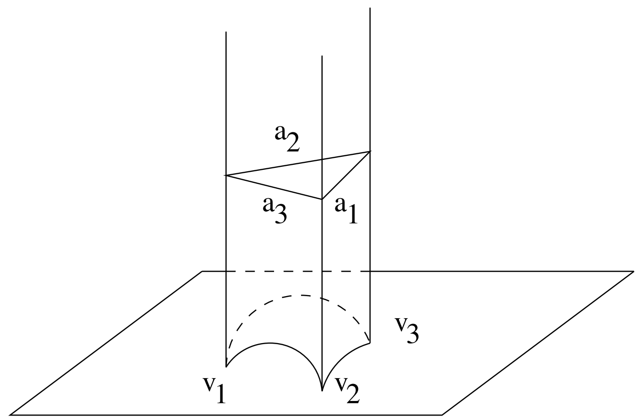


Figure 17: The triangle determined by cutting the tetrahedron with a plane parallel to $\mathbb{C}$ is invariant up to similarity.

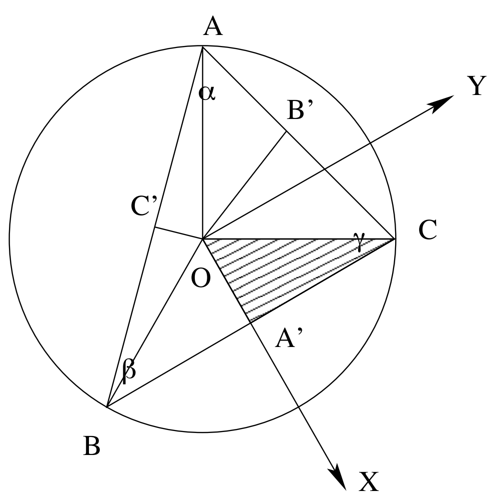


Figure 18: We integrate over the shadowy domain.

Then, the volume of this part of the tetrahedron is

$$\int_0^{\cos\alpha}\int_0^{x\tan\alpha}\int_{\sqrt{1-x^2-y^2}}^{\infty}\frac{dz}{z^3}\,dy\,dx=\int_0^{\cos\alpha}\int_0^{x\tan\alpha}\frac{1}{2}\frac{dy\,dx}{1-x^2-y^2}=$$

$$\int_0^{\cos\alpha}\int_0^{x\tan\alpha}\frac{1}{4\sqrt{1-x^2}}\left(\frac{1}{\sqrt{1-x^2}-y}+\frac{1}{\sqrt{1-x^2}+y}\right)dydx=$$

$$\int_0^{\cos\alpha}\left(\log(\sqrt{1-x^2}+x\tan\alpha)-\log(\sqrt{1-x^2}-x\tan\alpha)\right)\frac{dx}{4\sqrt{1-x^2}}=$$

$$\int_0^{\cos\alpha}\log\left(\frac{\sqrt{1-x^2}\cos\alpha+x\sin\alpha}{\sqrt{1-x^2}\cos\alpha-x\sin\alpha}\right)\frac{dx}{4\sqrt{1-x^2}}=$$

Substituting $x=\cos\theta$, then $\sqrt{1-x^2}=\sin\theta$, $dx=-\sqrt{1-x^2}d\theta$.

$$\frac{1}{4}\int_\alpha^{\frac{\pi}{2}}\log\left(\frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}\right)d\theta=\frac{1}{4}\left(\mathfrak{L}(2\alpha)-\mathfrak{L}\left(\frac{\pi}{2}+\alpha\right)-\mathfrak{L}(0)+\mathfrak{L}\left(\frac{\pi}{2}-\alpha\right)\right)=$$

$$=\frac{1}{4}\left(\mathfrak{L}(2\alpha)-2\mathfrak{L}\left(\frac{\pi}{2}+\alpha\right)\right)$$

We claim that this expression is equal to $\frac{\mathfrak{L}(\alpha)}{2}$ Indeed, starting from the well known identity

$$\sin 2\alpha=2\,\sin\alpha\,\cos\alpha=2\,\sin\alpha\,\sin\left(\frac{\pi}{2}-\alpha\right)=2\,\sin\alpha\,\sin\left(\frac{\pi}{2}+\alpha\right)$$

Multiplying by 2, applying $\log|\cdot|$ and integrating we get the equality we need:

$$\mathfrak{L}(2\alpha)=2\,\mathfrak{L}(\alpha)+2\,\mathfrak{L}\left(\frac{\pi}{2}+\alpha\right)$$

Summing over the six triangles, we prove the Lemma. □

Proof of Theorem. There are two ways of proving this Theorem. One uses the Lemma:

$$-\int_0^{\sigma}\log|2\sin t|dt=-\int_0^{\sigma}\log|\sqrt{2(1-\cos 2t)}|dt=-\int_0^{\sigma}\log|1-e^{2it}|dt$$

$$=-\frac{1}{2}\int_0^{2\sigma}\log|1-e^{is}|ds=\frac{1}{2}D(e^{2i\sigma})$$

We will need:

$$2D(z) = D\left(\frac{z}{\bar z}\right) + D\left(\frac{1-\frac{1}{z}}{1-\frac{1}{\bar z}}\right) + D\left(\frac{\frac{1}{1-z}}{\frac{1}{1-\bar z}}\right)$$

Proof of this identity. We will use a similar idea as the one we used in order to prove the Five-term relation. We claim that

$$2\,\eta(z,1-z) = \eta\left(\frac{z}{\bar z},1-\frac{z}{\bar z}\right)+\eta\left(\frac{\bar z(1-z)}{z(1-\bar z)},1-\frac{\bar z(1-z)}{z(1-\bar z)}\right)+\eta\left(\frac{1-\bar z}{1-z},1-\frac{1-\bar z}{1-z}\right)$$

We will prove, as before, the identity in $\Lambda^2(\mathbb{C}(\mathbb{P}^1)^\times)_0$. The argument in the second term is equal to

$$\frac{z}{\bar z}\wedge\frac{\bar z-z}{\bar z}+\frac{\bar z(1-z)}{z(1-\bar z)}\wedge\frac{z-\bar z}{z(1-\bar z)}+\frac{1-\bar z}{1-z}\wedge\frac{\bar z-z}{1-z}=\frac{z}{\bar z}\wedge(\bar z-z)-\frac{z}{\bar z}\wedge\bar z+$$

$$+\frac{\bar z}{z}\wedge(z-\bar z)+\frac{1-z}{1-\bar z}\wedge(z-\bar z)-\frac{\bar z}{z}\wedge z(1-\bar z)-\frac{1-z}{1-\bar z}\wedge z(1-\bar z)+\frac{1-\bar z}{1-z}\wedge(\bar z-z)-\frac{1-\bar z}{1-z}\wedge(1-z)=$$

$$-\frac{z}{\bar z}\wedge\bar z-\frac{\bar z}{z}\wedge z-\frac{\bar z}{z}\wedge(1-\bar z)-\frac{1-z}{1-\bar z}\wedge z-\frac{1-z}{1-\bar z}\wedge(1-\bar z)-\frac{1-\bar z}{1-z}\wedge(1-z)=$$

$$-\bar z\wedge(1-\bar z)+z\wedge(1-\bar z)-(1-z)\wedge z+(1-\bar z)\wedge z=-\bar z\wedge(1-\bar z)+z\wedge(1-z)=2\,z\wedge(1-z)$$

Applying η, we prove the claim. Then, since η is dD, we get the identity up to a constant. Then, it suffices to see what happens for a specific value.

Take $z=i$. Then $\frac{z}{\bar z}=-1$, $\frac{\bar z(1-z)}{z(1-\bar z)}=\frac{1-\bar z}{1-z}=i$. We get $2D(i)=D(-1)+D(i)+D(i)$ and the identity is proved. □

Then $\mathfrak{L}(\alpha)+\mathfrak{L}(\beta)+\mathfrak{L}(\gamma)=\frac{1}{2}(D(e^{2i\alpha})+D(e^{2i\beta})+D(e^{2i\gamma}))$ and the Theorem follows from the identity. □

Observe that $(0,1,\infty,z)$, $\left(0,1,\infty,\frac{1}{1-z}\right)$, $\left(0,1,\infty,\frac{z-1}{z}\right)$ are isometric tetrahedra and then they have the same volume. This is consistent with our identities of dilogarithm and with the invariant we have defined for the Euclidean triangles.

Another way of proving this Theorem, due by Bloch, is to observe that both sides lie in certain cohomology group which is one dimensional over $\mathbb{R}$ and then we only need to check that the constant is one on a particular case. The particular case could be, for instance, the regular tetrahedron ($z = \frac{1+\sqrt{-3}}{2}$). This was computed by Coxeter.

Where are we heading? We want to investigate the equality: $\int_\gamma \eta(x,y) = 2\pi m(P)$. In order to compute the integral easily, we will concentrate in the case where η is exact. When can we guarantee that? If we have

$$x \wedge y = \sum_{j=1}^{N} r_j \, z_j \wedge (1 - z_j), \;\; r_j \in \mathbb{Q}, \; z_j \in \mathbb{C}(C)^{\times} \text{ in } \Lambda^2(\mathbb{C}(C)^{\times}) \otimes \mathbb{Q} \qquad (2)$$

then

$$\eta(x,y) = d\left(\sum_{j=1}^{N} r_j \, D \circ z_j\right)$$

The advantage in this case is that we can use Stokes Theorem to compute the integral. If we pass to $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$, then the equation (2) implies $\{x, y\} = 0$.

Our idea is to see that the equation (2) comes from hyperbolic geometry.

For instance, let's revisit the Five Term Relation. We have given an algebraic proof of it. It turns out that this algebraic proof has an equivalent proof in the point of view of hyperbolic geometry. Consider five points $\in \mathbb{P}^1(\mathbb{C}) \subset \mathbb{H}^3$. (Here, $\mathbb{P}^1(\mathbb{C})$ is simply the subset of $\mathbb{H}^3$ which consists of $\mathbb{C} \cup \{\infty\}$).

Now, we can consider the five tetrahedra that are formed by choosing all the subsets of four points. We claim that $\sum_{j=0}^{4} \mathrm{Vol}(\Delta_{z_j}) = 0$. Recall that our volumes have a sign according to orientation. This equality can be seen if we sent one of the points to infinity. We will relate this fact with the Five Term Relation.

Suppose we are in the situation that one point is ∞, and the other points are like in Figure 19. Observe that if three triangles converge building a fourth triangle (Figure 19), then the product of the invariants must be 1, i.e., $w_1 w_2 w_3 = 1$. This is clear if we choose wisely the pairs of sides that we use to compute the invariants. Calling the sides which are common to two of the triangles as u_1, u_2, u_3, then $w_1 w_2 w_3 = \frac{u_2}{u_3}\frac{u_3}{u_1}\frac{u_1}{u_2} = 1$.

Now let's look at the four points which are not ∞. We can certainly use this. Suppose that the invariants are z_1, z_4 and $\frac{1}{1-z_0}$. Writing the equation,

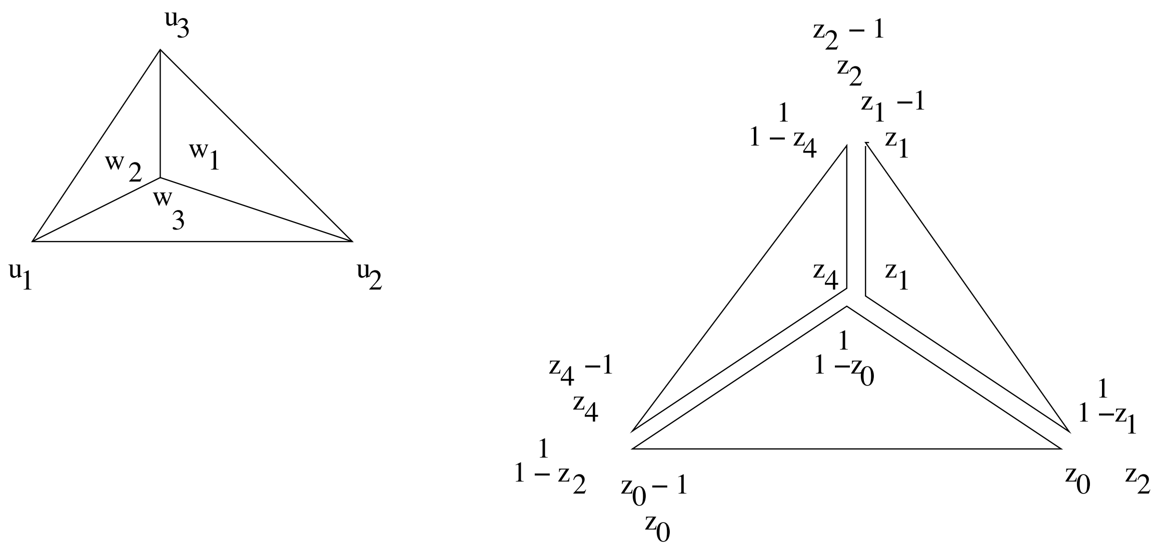


Figure 19: In the first picture, $w_1 w_2 w_3 = 1$. The second picture shows us how we compare the invariants of the different triangles.

we get $z_4 z_1 = 1 - z_0$, which is one of the five equations that define the Del Pezzo surface. We get three other equations by comparing the invariants in the other three vertices (here we will have to use the invariant z_2 of the "big" tetrahedron formed by the vertices which are in the boundary of our figure and ∞. In this way, we get: $z_0 z_2 = 1 - z_1$, then calling $z_1 z_3 = 1 - z_2$, we get $z_3 z_0 = 1 - z_4$, and $z_1 z_3 = 1 - z_2$, (we already had this last condition).

We get equations that are equivalent to the fact that $\sum_{j=0}^{4} D(z_j) =$ constant. But we know that this is equivalent to the sum of volumes being constant. Then it is easy to verify that the sum is actually zero.

4.2 Hyperbolic Structures on 3-manifolds

Definition 22 *A hyperbolic manifold M is a space with a Riemannian metric such that each point has a neighborhood which is isometric to an open subset of $\mathbb{H}^3$.*

In this case it is true that $M \cong \mathbb{H}^3/\Gamma$ where Γ is a discrete torsion free subgroup of $\text{Isom}_+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$. This is analogous to the case of the upper half plane $\mathbb{H}^2$, say for instance, $\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{mod}(2) \right\} \subset$

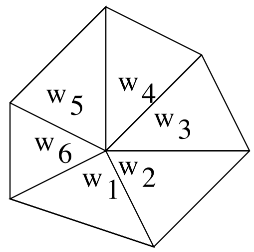


Figure 20: Gluing equation. In this case, $w_1 w_2 w_3 w_4 w_5 w_6 = 1$.

$SL_2(\mathbb{Z})$ acting on $\mathbb{H}^2$.

We have:

Theorem 23 *(Mostow's rigidity). Let $M_1 = \mathbb{H}^3/\Gamma_1$ and $M_2 = \mathbb{H}^3/\Gamma_2$, where Γ_1, Γ_2 are discrete, torsion free, and M_1, M_2 have finite volume. If $\Gamma_1 \cong \Gamma_2$ (isomorphic as groups), then*

$$M_1 \cong M_2 \text{ (isometric)}$$

We will try to glue ideal tetrahedra $z_1, \ldots, z_N \in \mathbb{H}^2$ in order to get manifolds. We will come up with a list of tetrahedra and two sets of equations. The first set corresponds to the gluing equations. For instance, the equation stating that the product of the invariants of the triangles that converge in one vertex and form a polygon must be 1, is a gluing equation (Figure 20). Actually, looking from ∞, these equations are always of the same form, i.e., $w_1 \ldots w_n = 1$ with $w_j \in \left\{z_j, \frac{1}{1-z_j}, \frac{z_j-1}{z_j}\right\}$. The second set of equations is the completeness equations. These are equations of the same nature that will guarantee that the manifold is complete. Summarizing, all the equations are of the form

$$\prod_j w_j^{e_j} = 1, \quad w_j \in \left\{z_j, \frac{1}{1-z_j}, \frac{z_j-1}{z_j}\right\}$$

The options for w_j correspond to the choice in each triangle of what vertex is going to be the common vertex with all the other triangles.

Take the example of the Figure 8 Knot. See Figure 21. The complement of the knot admits a complete hyperbolic structure. We glue two tetrahedra. The equations are:

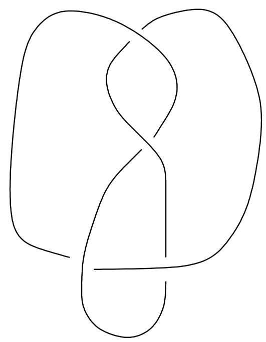


Figure 21: Figure 8 Knot.

$$G: \quad z(1-z)w(1-w) = 1$$

$$C: \quad \begin{array}{l} x = z(1-z) = 1 \\ y = w(1-z) = 1 \end{array}$$

If we take only G, we get an affine curve C (which will be an elliptic curve over $\mathbb{Q}$ of conductor 15), where x, y are rational functions on C.

$$x \wedge y = z(1-z) \wedge w(1-z) = z(1-z) \wedge w + z \wedge (1-z) =$$
$$= -\left(w(1-w) \wedge w\right) + z \wedge (1-z) = w \wedge (1-w) + z \wedge (1-z)$$

$\Rightarrow \eta(x,y) = d(D(z) + D(w))$.
Eliminating z, w, we get

$$\begin{array}{ccc} & 1 & \\ & -1 & \\ 1 & -2 & 1 \\ & -1 & \\ & 1 & \end{array}, \text{ in other words, } P(x,y) = xy^4 - xy^3 + x^2y^2 - 2xy^2 + y^2 - xy + x = 0$$

which is called the A-polynomial of the knot and is an invariant for the knot. We have,

$$2\pi m(P(x,y)) = \mathrm{Vol}(M) = 2\,D\left(\frac{1+\sqrt{-3}}{2}\right)$$

Let's do the other way, start with the elliptic curve C, whose minimal Weierstrass equation is:

$$v^2 + uv + v = u^3 + u^2$$

We have $E(\mathbb{Q}) = \langle Q \rangle, Q = (0,0)$, which has order 4. Let L be the (multiplicative) lattice of functions modulo constants whose divisors are supported by $E(Q) = \langle Q \rangle$. L has dimension 3, because it is $\cong \mathbb{Z}^4$ with the additional conditions $\sum_{j=0}^{3} n_j = 0$ and $\sum_{j=0}^{3} n_j j \equiv 0 \mod 4$. But this last condition does not influence the dimension. It is generated by u, $u+1$ and v:

n	nQ	u	$u+1$	v
0	$\mathbf{O}$	-2	-2	-3
1	$(0,0)$	1	0	2
2	$(-1,0)$	0	2	1
3	$(0,-1)$	1	0	0

There is an involution on C, the curve defined by the gluing equation, which takes x to x^{-1} and y to y^{-1}. In other words, the polynomial relating x and y is reciprocal: $P(x^{-1}, y^{-1}) = x^a y^b P(x,y)$.

Now we are after this involution. We do not know anything about it, except that it is defined over $\mathbb{Q}$. Recall that an involution does not need to preserve the origin of the curve, it is defined on the algebraic curve, but not necessarily defined over the group. There are two kinds of involutions: the ones that have fixed points, which are of the form $\sigma : P \to -P + R$; and the ones that do not have fixed points, of the form $\sigma : P \to P + R$ with $2R = \mathbf{O}$. The involution $x \to x^{-1}$, $y \to y^{-1}$ has a fixed point $(x,y) = (1,1)$.

We will pick:

$$\sigma : P \to -P + Q$$

This involution preserves L. We want $x, y \in L$ such that $x^\sigma = x^{-1}, y^\sigma = y^{-1}$. Their divisors δ must satisfy $\delta^\sigma = -\delta$. If

$$\delta = \sum_{j=0}^{3} n_j(jQ) \quad n_j \in \mathbb{Z}$$

then, δ must verify the following conditions:

1. $\sum_{j=0}^{3} n_j = 0$

2. $\sum_{j=0}^{3} n_j j \equiv 0 \bmod 4$

3. $n_{1-j} = -n_j$

The first two conditions are equivalent to the fact that δ is a divisor and the last one expresses that $\delta^\sigma = -\delta$, in fact, $\sigma : \mathbf{O} \leftrightarrow Q$, and $2Q \leftrightarrow 3Q$.

Putting the conditions together: $\delta = n_0((\mathbf{O}) - (Q)) + n_2((2Q) - (3Q))$ with $n_0 + n_2 \equiv 0 \bmod 4$. Hence the space of the δ is spanned by $(2, -2, 2, -2)$ and $(1, -1, -1, 1)$. Define

$$x := -\frac{u+1}{u^2}, \; y := -\frac{u}{v}$$

Then $(x) = (2, -2, 2, -2)$ and $(y) = (1, -1, -1, 1)$. (The signs are included in order that the final equation is exactly our original polynomial P). We have $x^\sigma = x^{-1}$ up to a constant and the same with y. We want to verify that this constant is 1. In order to do that, we compute the involution according to (u, v).

$$(u, v)^\sigma = (0, 0) - (u, v) = (0, 0) + (u, -v - u - 1) = \left(\frac{u+1}{v}, \frac{u(u+1)}{v^2} \right)$$

Then

$$x^\sigma = -\frac{\left(\frac{u+1}{v} + 1\right)}{\left(\frac{u+1}{v}\right)^2} = -\frac{uv + v + v^2}{(u+1)^2} = -\frac{u^2}{u+1} = x^{-1}$$

$$y^\sigma = -\frac{\frac{u+1}{v}}{\frac{u(u+1)}{v^2}} = y^{-1}$$

Next, we want to get a minimal polynomial relating x and y. We have:

$$\begin{cases} w = v^2 + uv + v - u^3 - u^2 = 0 \\ A = u^2 x + u + 1 = 0 \\ B = yv + u = 0 \end{cases}$$

Computing resultants, it is possible to get the polynomial:

$$\mathrm{Res}_v(w, A) = A_1(x, y, u)$$

$$\mathrm{Res}_v(w, B) = B_1(x, y, u)$$

We may need to factorize A_1 and B_1 and get rid of common factors. After that, we compute

$$R(x, y) = \text{Res}_u(A_1, B_1)$$

Again, we may need to factorize R in order to get the minimal polynomial. In this particular case, we get

$$R(x, y) = P(x, y)^2$$

where $P(x, y)$ is as before:

$$\begin{matrix} & 1 & \\ & -1 & \\ 1 & -2 & 1 \\ & -1 & \\ & 1 & \end{matrix}, \quad P(x, y) = xy^4 - xy^3 + x^2y^2 - 2xy^2 + y^2 - xy + x = 0$$

the A-polynomial of the figure 8 knot.

Observe that the Newton polytope has three interior points, while we started with an elliptic curve, which has genus one. The fact is that $P(x, y)$ is singular, and hence, the number of interior points does not need to be a equal to the genus.

From the manifold point of view, we had

$$x \wedge y = z \wedge (1 - z) + w \wedge (1 - w) \Rightarrow \{x, y\} = 0 \quad \text{in} \quad K_2(\mathbb{Q}(E))$$

Note $\{x, y\}^\sigma = \{x^\sigma, y^\sigma\} = \{x^{-1}, y^{-1}\} = \{x, y\}$.

Theorem 24 *Bass – Tate.* $\{x, y\}$ *arises from* $K_2(\mathbb{Q}(E)^\sigma) = K_2(\mathbb{Q}(\mathbb{P}^1))$

We want to solve $x \wedge y = \sum_{j=1}^N r_j \, z_j \wedge (1 - z_j) \quad r_j \in \mathbb{Q}$. This is almost the same as decomposing the manifold into tetrahedra.

We want to find the z_j. We are looking for functions such that $z,\ 1-z \in L$. Two of them are $z_1 = -u$ and $z_2 = \frac{u^2}{v}$. Now $\{u \wedge (u+1), u \wedge v, (u+1) \wedge v\}$ is a basis for $\Lambda^2(L)$. We will express $z_i \wedge (1 - z_i)$ as a linear combination of elements in this basis.

	$-u \wedge (u+1)$	$\frac{u^2}{v} \wedge \left(1 - \frac{u^2}{v}\right)$	$x \wedge y$
$u \wedge (u+1)$	1	-2	-1
$u \wedge v$	0	2	2
$(u+1) \wedge v$	0	-1	-1

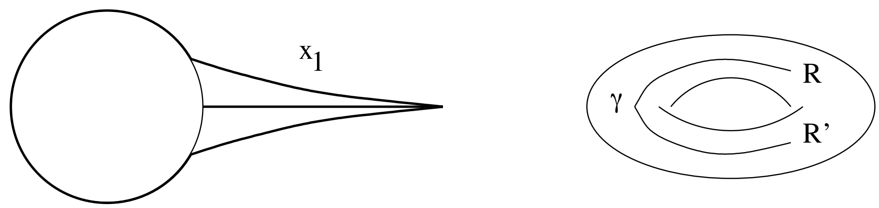


Figure 22: Integration path. First picture: $|x_1(y)| \geq 1$, while $|y| = 1$. Second picture: the corresponding path γ in the elliptic curve.

(Here we do not care about constants). Then

$$x \wedge y = z_1 \wedge (1 - z_1) + z_2 \wedge (1 - z_2)$$

modulo torsion. The decomposition is not unique.

Our goal is to show

$$\pi m(P) = 2L'(\chi_{-3}, -1) = \mathrm{Vol}(M)$$

We have $2D\left(\frac{1+\sqrt{-3}}{2}\right) = L'(\chi_{-3}, -1)$.
We know $\eta(x, y) = d\left(D(-u) + D\left(\frac{u^2}{v}\right)\right)$
Claim:

$$2\pi m(P) = \int_\gamma \eta(x, y) \qquad \gamma \text{ is a 1-cycle in } E$$

By Jensen's formula,

$$m(P) = \sum_{j=1}^{2} \frac{1}{2\pi i} \int_{|y|=1} \log^+ |x_j(y)| \frac{dy}{y}$$

where $x_j(y)$ are the roots of P for a given y. Since we integrate on $|y| = 1$, and $\eta(x, y)|_{|y|=1} = \log|x|\, d\arg y$, the claim will be proved if we observe that $|x_1(y)| \geq 1$, and $|x_2(y)| \leq 1$ in the whole domain. This is true and the approximate path that $x_1(y)$ does as long as y goes through the unit circle, is described by Figure 22.

At the same time, in the elliptic curve, the corresponding path γ connects two conjugate points R and $R' = \bar{R}$. Observe that there is no contradiction in the fact that γ is not a loop. The fact is that R and R' are two different points on the elliptic curve that lie over the same singular point in the polynomial P (both lie over $(1, 1)$).

Now we are able to apply Stokes Theorem. Define $V((u,v)) := D(-u) + D\left(\frac{u^2}{v}\right)$

$$2\pi m(P) = \int_\gamma \eta(x,y) = V(R) - V(R') = 2V(R)$$

Here, $R = (\xi_3, -\xi_3)$, $R' = (\xi_3^{-1}, -\xi_3^{-1})$. This can be computed directly from the formulas that relate x, y and u, v. Hence,

$$\pi m(P) = 2D(\xi_6) = \mathrm{Vol}(M) = 2L'(\chi_{-3}, -1)$$

Our last step was to show that η was exact. In order to do that, we computed the primitive. There is an algorithm to find the primitive in the general case. We suppose that the Tame symbols are trivial. We want to write

$$x \wedge y = \sum_{j=1}^{N} r_j \, z_j \wedge (1 - z_j) + c_1 \wedge c_2 \qquad r_j \in \mathbb{Q},\ z_j \in \bar{\mathbb{Q}}(t),\ c_i \in \bar{\mathbb{Q}}$$

We will use the following identity due to Tate:

$$(t-a)\wedge(t-b) = -\left(\frac{t-a}{b-a} \wedge \left(1 - \frac{t-a}{b-a}\right)\right) + (t-a)\wedge(a-b) - (t-b)\wedge(b-a)$$

This is a formal identity which can be easily proved by observing that $\frac{t-a}{b-a} \wedge \left(1 - \frac{t-a}{b-a}\right) = \frac{t-a}{b-a} \wedge \frac{b-t}{b-a}$. The idea is to reduce $x \wedge y$ to a sum of terms of the form $* \wedge (1 - *)$ and constants.

$$x = A \prod_j (t - a_j)^{n_j} \qquad y = B \prod_k (t - b_k)^{m_k} \qquad a_j \neq b_k$$

$$x \wedge y = -\sum_{j,k} n_j m_k \frac{t-a_j}{b_k - a_j} \wedge \left(1 - \frac{t-a_j}{b_k - a_j}\right) + \sum_j n_j (t - a_j) \wedge \frac{y(a_j)}{B} - \sum_k m_k (t - b_k) \wedge \frac{x(b_k)}{A}$$

$$+ A \wedge \frac{y}{B} + \frac{x}{A} \wedge B + A \wedge B =$$

$$= -\sum_{j,k} n_j m_k \frac{t-a_j}{b_k - a_j} \wedge \left(1 - \frac{t-a_j}{b_k - a_j}\right) + \sum_j \frac{x}{A} \wedge \frac{y(a_j)}{B} - \sum_k \frac{y}{B} \wedge \frac{x(b_k)}{A} + A \wedge \frac{y}{B} + \frac{x}{A} \wedge B + A \wedge B =$$

$$= -\sum_{j,k} n_j m_k \frac{t-a_j}{b_k-a_j} \wedge \left(1-\frac{t-a_j}{b_k-a_j}\right) + \sum_j \frac{x}{A} \wedge y(a_j) - \sum_k \frac{y}{B} \wedge x(b_k) + A \wedge B$$

Now when we tensor with $\mathbb{Q}$ and apply η the last terms are zero because A, B are constants and $y(a_j)$, $x(b_j)$ are roots of unity. This is because the Tame symbols are trivial and those numbers are essentially the Tame symbols (up to a sign). Here we use that $a_j \neq b_k$) and η is zero on constants. We get,

$$\eta(x,y) = d\left(-\sum_{j,k} n_j m_k D\left(\frac{t-a_j}{b_k-a_j}\right)\right)$$

Let's do another example,

$$x = \frac{t^2+t+1}{(t-1)^2} \quad y = \frac{3t^2}{(t-1)^2}$$

(this example has not been related to hyperbolic manifolds yet).

The minimal polynomial relation is:

$$\begin{matrix} 1 & & \\ -1 & -2 & \\ 1 & -2 & 1 \end{matrix} \quad , \quad P(x,y) = x^2 - 2xy + y^2 - 2x - y + 1$$

By looking at the roots of the side polynomials and observing that they have to be roots of the unity, we conclude that the tame symbols are trivial. We can also compute some of them

$$(x,y)_{t=1} = (-1)^{2\cdot 2}\left(\frac{3t^2}{t^2+t+1}\right)^2\Bigg|_{t=1} = 1$$

$$(x,y)_{t=\xi_3} = (-1)^{1\cdot 0}\frac{1}{y(\xi_3)} = \frac{(\xi_3-1)^2}{3\xi_3^2} = -\xi_3^{-1}$$

We want $\eta(x,y) = d(D(\xi(t)))$. $D : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ can be extended by linearity to $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})]$. Applying the above procedure:

$$\xi(t) = 2\left[\frac{t-\xi_3}{-\xi_3}\right] + 2\left[\frac{t-\xi_3^{-1}}{-\xi_3^{-1}}\right] - 2\left[\frac{t-\xi_3}{1-\xi_3}\right] - 2\left[\frac{t-\xi_3^{-1}}{1-\xi_3^{-1}}\right] - 4[1-t]$$

Observe that this expression is Galois invariant.

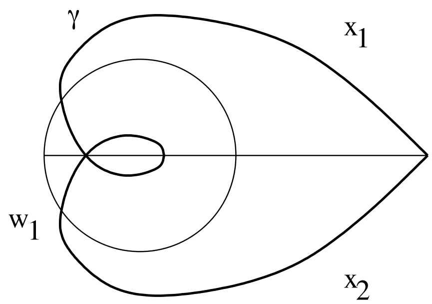


Figure 23: Integration path. $|x_1(y)|$ and $|x_2(y)|$ go through symmetrical paths.

Now the Mahler measure is

$$m(P) = \frac{1}{2\pi i}\sum_{j=1}^{2}\int_{|y|=1}\log^+|x_j(y)|\frac{dy}{y} = \int_\gamma \eta(x,y)$$

For y fixed there are two possible solutions to x. The path that x_1 and x_2 do while $|y| = 1$ is described by Figure 23.

In order to compute the integral, since η is exact and we want to apply Stokes Theorem, we need to find w_1 and $\bar{w}_1$. They are the intersection of $P(x,y) = 0$ and $|x| = |y| = 1$. To find this intersection, we consider the reciprocal polynomial:

$$P^*(x,y) = x^2y^2P(x^{-1},y^{-1}) = x^2y^2-2xy^2-x^2y+y^2-2xy+x^2 = \begin{array}{ccc} 1 & -2 & 1 \\ & -2 & -1 \\ & & 1 \end{array}$$

$P, P^* \in \mathbb{R}[x,y]$. We want $|x| = |y| = 1$, then $\bar{x} = x^{-1}$, $\bar{y} = y^{-1}$.

$$0 = \overline{P(x,y)} = P(\bar{x},\bar{y}) = P(x^{-1},y^{-1})$$

Then

$$P(x,y) = P^*(x,y) = 0 \quad \text{for} \quad |x| = |y| = 1$$

Let's compute the minimal polynomial of $w_1 = (x_1,y_1) = (x_1(t_1), y_1(t_1))$.

$$\mathrm{Res}_y(P,P^*) = (x^2-4x+1)(x^4-2x^3-2x+1)$$

The first factor has two real roots whose module is $\neq 1$. The second factor has two roots of module 1, which must correspond to x_1 and $\bar{x}_1$. Now $x_1, y_1, t_1 \in F$, a number field of degree 4, $r_2 = 1$, and $\Delta_F = -1728$. Claim

$$m(P) = s \cdot \frac{3 \cdot 1728^{\frac{3}{2}} \zeta_F(2)}{2^5 \pi^7}$$

for some $s \in \mathbb{Q}^\times$. (s "looks a lot like" $\frac{1}{6}$). [1]

5 Bloch Group

Let F be a field, $\xi = \sum_j n_j [a_j] \in \mathbb{Z}[\mathbb{P}^1(F)]$. We have a function:

$$\mathbb{Z}[\mathbb{P}^1(F)] \xrightarrow{\partial} \Lambda^2(F^\times)$$
$$\xi \to \sum_j n_j \, a_j \wedge (1 - a_j)$$

(we omit the sum if $a_j = 0, 1$ or ∞).

In $K_2(F)$, $\{a, a\} = \{a, -1\}$ and this one is torsion of order 2. Then, up to 2-torsion we have an exact sequence:

$$\begin{array}{ccccc} \mathbb{Z}[\mathbb{P}^1(F)] & \xrightarrow{\partial} & \Lambda^2(F^\times) & \longrightarrow & K_2(F) \\ & & a \wedge b & \longrightarrow & \{a, b\} \end{array}$$

Define

$$\mathcal{A} = \ker \partial$$

Note that if $(z_0, \ldots z_4) \in S(F)$ (Del Pezzo surface over the field F), then

$$\sum_{j=0}^{4} [z_j] \in \mathcal{A}(F) \qquad (\text{i.e. } \partial \left(\sum_{j=0}^{4} [z_j] \right) = 0)$$

Define

$$\mathcal{G}(F) = \text{group generated by } \left\{ \sum_{j=0}^{4} [z_j] \,|\, (z_0, \ldots, z_4) \in S(F) \right\} \subset \mathcal{A}(F)$$

[1] How do we compute s? In GP:
`zv = zetakinit(x^4 - 2x^3 - 2x + 1)`
`zetak(zv, 2) * 1728^1.5/(Pi^6 * 2^5)`
$= 3.60526644...$
$\stackrel{?}{=} \pi \, m(P)$

Finally, define the Bloch group to be

$$\mathcal{B} = \mathcal{A}/\mathcal{G}$$

5.1 Relation with K-theory

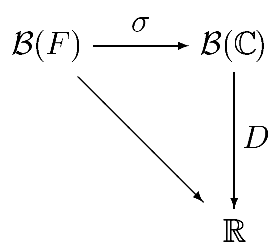
Define $K_3^{ind}(F) = K_3(F)/K_3^{dec}$. Where *dec* and *ind* stand for decomposable and indecomposable respectively. Then

$$K_3^{ind}(F) \otimes \mathbb{Q} \cong \mathcal{B} \otimes \mathbb{Q}$$

Given $\sigma : F \longrightarrow L$, it induces $\mathcal{B}(F) \longrightarrow \mathcal{B}(L)$. Then for

$$\sigma : F \hookrightarrow \mathbb{C}$$

We have the following diagram:



We get one different map for each complex embedding.

We have the following facts:

- D is, up to a factor in $\mathbb{Q}^\times$, the Borel's regulator.
- The rank of $\mathcal{B}(F)$ is the order of vanishing of ζ_F at $s = -1$, namely $n_- = r_2$.

Theorem 25 *Let $\xi_1, \ldots, \xi_{r_2}$ be a $\mathbb{Q}$ - basis for $\mathcal{B} \otimes \mathbb{Q}$. Let $\sigma_1, \ldots, \sigma_{r_2}$ pairwise non-conjugate complex embeddings of F. Then*

$$det\left(D\left(\sigma_j(\xi_k)\right)\right) \sim_{\mathbb{Q}^\times} \frac{|\Delta_F|^{\frac{3}{2}}\zeta_F(2)}{\pi^{2n_+}}$$

We do not know how small the left term can be. If we knew this, we might be able to estimate the rational constant.

In particular, if $r_2 = 1$,

$$D\left(\sigma(\xi)\right) \sim_{\mathbb{Q}^\times} \frac{|\Delta_F|^{\frac{3}{2}}\zeta_F(2)}{\pi^{2(n-1)}}$$

Some examples:

$$\mathbb{Q}(\sqrt{-7}): \qquad 2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right]$$

is likely to be a generator of the Bloch group, which has rank 1.

$$\mathbb{Q}(\sqrt{-23}): \qquad 21\left[\frac{1+\sqrt{-23}}{2}\right] + 7\left[2+\sqrt{-23}\right] + \left[\frac{3+\sqrt{-23}}{2}\right] - 3\left[\frac{5+\sqrt{-23}}{2}\right] + [3+\sqrt{-23}]$$

If we knew how small $D(\xi)$ can be, we would get some clue about whether those numbers are generators of their Bloch groups or not.

Consider our former example

$$x = \frac{t^2+t+1}{(t-1)^2}, \quad y = \frac{3t^2}{(t-1)^2} \qquad \begin{matrix} 1 & & \\ -1 & -2 & \\ 1 & -2 & 1 \end{matrix}$$

Then $2\pi m(P) = V(w_1) - V(\bar{w}_1)$, $w_1 = (x_1, y_1)$, $|x_1| = |y_1| = 1$.

In order to find V, we need a decomposition of the form $x \wedge y = \sum_j r_j\, z_j \wedge (1 - z_j)$, $r_j \in \mathbb{Q}$. We have found this decomposition in the example above. Note that the decomposition is not unique, since we can always add elements that come from the five term relation. We claim that:

$$2\pi m(P) \sim_{\mathbb{Q}^\times} \frac{1728^{\frac{3}{2}}\zeta_F(2)}{2^8\pi^6}$$

It is a consequence of Borel's Theorem.

$$V(w_1) = D(\xi(t_1)) = \sum_j r_j D(z_j(t_1)) \quad \text{where} \quad \xi(t) = \sum_j r_j[z_j(t)]$$

Borel's Theorem, for $r_2 = 1$ and $\xi \in \mathcal{B}(F)$ not in the torsion, says that

$$D(\xi) \sim_{\mathbb{Q}^\times} \frac{|\Delta_F|^{\frac{3}{2}}\zeta_F(2)}{\pi^{2n_+}} \qquad n_+ = r_1 + r_2$$

$$\partial(\xi(t_1)) = \sum_j r_j \; z_j(t_1) \wedge (1 - z_j(t_1)) = x(t_1) \wedge y(t_1) \stackrel{?}{=} 0$$

Where $\xi(t_1) = \sum_j r_j[z_j(t_1)]$. The last statement is true (always modulo torsion) if and only if $\exists\, a, b \in \mathbb{Z}$ such that

$$x(t_1)^a = y(t_1)^b$$

there is no reason a priori for this to hold. However, it does.

Claim: $y_1 = x_1^2$. In this particular case, this fact is not surprising since $x_1, y_1 \in \mathcal{O}_F^\times$. We have

$$\mathcal{O}_F^\times \longrightarrow \mathbb{R}$$

$$\epsilon \to \log|\sigma(\epsilon)|$$

x_1, y_1 are in the kernel of this function. The rank of $\mathcal{O}_F^\times = r_1 + r_2 - 1 = 2$, hence ker is of rank 1. Then, there has to be a relation (initially, up to roots of unity, but then you can raise to a higher power).

Therefore, $\xi(t_1) \in \mathcal{B}$, and we can use Borel's Theorem:

$$2\pi m(P) = D(\xi(t_1)) \sim_{\mathbb{Q}^\times} \frac{1728^{\frac{3}{2}} \zeta_F(2)}{\pi^6}$$

If

$$x \wedge y = \sum_j r_j \, z_j \wedge (1 - z_j) \quad r_j \in \mathbb{Q} \tag{3}$$

then $\eta(x, y) = dD(\sum_j r_j[z_j])$, and we use Stokes Theorem to compute $2\pi m(P)$ in terms of D.

$$2\pi m(P) = \int_\gamma \eta(x, y) \quad \text{where } \partial\gamma = \sum_k \epsilon_k[w_k] \qquad \epsilon_k = \pm 1$$

γ is a path on the surface $P(x, y) = 0$ (see Figure 24) and can be chosen in such a way that the end points w_k correspond to (x_k, y_k) with $|x_k| = |y_k| = 1$. Then by Stokes Theorem: $2\pi m(P) = D(\xi)$ where $\xi = \sum_k \xi_k, \quad \xi_k = \epsilon_k \sum_j r_j \, [z_j(w_k)]$.

The question now is when $\xi_k \in \mathcal{B}(\bar{\mathbb{Q}})$.

$$\partial(\xi_k) = \epsilon_k \sum_j r_j \; z_j(w_k) \wedge (1 - z_j(w_k)) = \epsilon_k \, x(w_k) \wedge y(w_k)$$

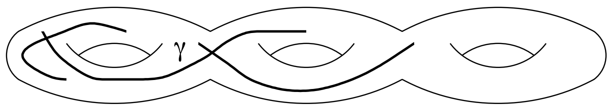


Figure 24: γ may have several components over C.

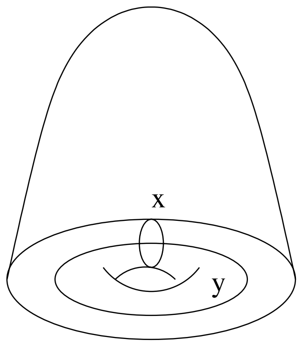


Figure 25: M is a hyperbolic 3-manifold such that $[\partial M] = 0$ in $H_2(M)$. For instance, in this picture $\partial M \simeq T^2$.

$\xi_k \in \mathcal{B}(\bar{\mathbb{Q}}) \Longleftrightarrow x(w_k) \wedge y(w_k) = 0$, (e.g. $x(w_k) \in \mu_\infty$). If $\xi_k \in \mathcal{B}(\bar{\mathbb{Q}})$, then Borel's theorem will give us some information about $2\pi m(P)$.

If we start with a 1-cusped hyperbolic 3-manifold, then we get the equation (3). See Figure 25.

We want $x(w) \wedge y(w) = 0$, $w \in C(\bar{\mathbb{Q}})$. The condition corresponds to Hyperbolic Dehn surgery:

$$x(w)^a = y(w)^b$$

We have the following situation:

$$\begin{array}{ccc} \text{Mahler measure} & \longrightarrow & D(\xi) \\ & & \Big\downarrow \text{Borel's theorem relates to} \\ \mathrm{Vol}(M) & \longrightarrow & \zeta_F(2) \end{array}$$

We will see the connection between the volume of the manifold M and $\zeta_F(2)$.

G. Humbert: Let K be an imaginary quadratic field, $\Gamma = SL_2(\mathcal{O}_K) \hookrightarrow SL_2(\mathbb{C}) \subset \mathbb{H}^3$. ($\Gamma$ is called the Bianchi group). Then $M = \mathbb{H}^3/\Gamma$ is an orbifold.

Theorem 26 *(Humbert Formula)*

$$Vol(M) = \frac{|\Delta_K|^{\frac{3}{2}}\zeta_K(2)}{4\pi}$$

This is a true equality, without hidden factors.

One proof of this uses Tamagawa numbers and another proof uses Einsestein series.

Before proving it, observe that

$$\#SL_2(\mathbb{F}_q) = \frac{\overbrace{(q^2-1)(q^2-q)}^{GL_2(\mathbb{F}_q)}}{q-1} = (q+1)(q^2-q) = q^3 - q = q^3(1-q^{-2})$$

Note that $1 - q^{-2} = 1 - \mathbb{N}(\mathcal{P})^{-2}$ is related to 2 in $\zeta_K(2)$.

Idea of the Proof (Humbert Formula). This proof can be found in [GK]. Look at the binary Hermitian positive definite forms

$$\lambda : (a,b,c) \longrightarrow \begin{pmatrix} 2a & b \\ \bar{b} & 2c \end{pmatrix} \qquad a, c \in \mathbb{R}_{>0},\ b \in \mathbb{C},\ \Delta(\lambda) = b\bar{b} - 4ac$$

Note that $SL_2(\mathbb{C})$ acts on these forms preserving Δ.

Let

$$\begin{array}{ccc} V & = & \mathbb{R} \times \mathbb{C} \times \mathbb{R} \\ \cup & & (a,b,c) \\ V^+ & = & a, c > 0,\ b\bar{b} - 4ac < 0 \end{array}$$

The map

$$\lambda \longrightarrow \left(\frac{b}{2c}, \frac{\sqrt{|\Delta|}}{2c} \right) \in \mathbb{C} \times \mathbb{R}_{>0} = \mathbb{H}^3$$

is compatible with the action of $SL_2(\mathbb{C})$. The pull back by this map of a fundamental domain for $SL_2(\mathcal{O}_k)$ in $\mathbb{H}^3$ is a cone $X \subset V^+$. Figure 26.
The function $b\bar{b} - 4ac$ is a quadratic form of signature $(3,1)$ for V, which has dimension 4. We have $\Lambda \subset V^+$ lattice: $(a,b,c),\ a,b,c \in \mathcal{O}_K$. Now,

$$Z(s) = \sum_{\lambda \in \Lambda \cap X} |\Delta(\lambda)|^{-2s} \qquad \text{for } \Re(s) > 1$$

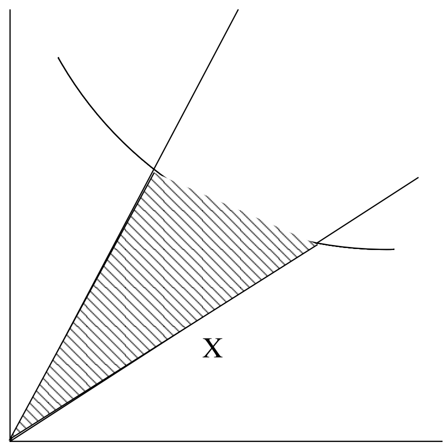


Figure 26: The area of the cone X is proportional to the hyperbolic measure of the graph.

Dirichlet principle states that Z has a pole at $s = 1$ and $\mathrm{Res}_{s=1} Z(s) = \frac{\mathrm{Vol}\{x \in X \mid |\Delta(x)| \leq 1\}}{|\Lambda|}$

Humbert identified and related $Z(s)$ to $\zeta_F(2s)$ by using quaternion algebras. What is the relation between the Euclidean volume of the shadowed region and the hyperbolic measure of the graph above it? There is a constant relating both numbers. It can be proved in dimension 1 by observing the invariance by the action of SL_2, but it is true for several dimensions. □

General case: Let B/F be a quaternion algebra. Let $\mathcal{O}$ be a maximal order of B. Let $\Gamma \subset \mathcal{O}$ be the units of norm 1. We have,

$$B \overset{\sigma}{\hookrightarrow} B \otimes \mathbb{R} \text{ or } \mathbb{C} \cong \begin{cases} \mathbb{H} & \mathbb{R} \quad \sigma \text{ ramified} \\ M_2(\mathbb{R}) & \mathbb{R} \quad \sigma \text{ unramified} \\ M_2(\mathbb{C}) & \mathbb{C} \end{cases}$$

If $k = \#$ of unramified real places, Γ acts on $\mathbb{H} = (\mathbb{H}^2)^k \times (\mathbb{H}^3)^{r_2}$ discretely. e.g. if $F = \mathbb{Q}$, $B = M_2(\mathbb{Q})$, then $\mathcal{O} = M_2(\mathbb{Z})$, $\Gamma = SL_2(\mathbb{Z})$. We have in this case that $SL_2(\mathbb{Z})$ acts on $\mathbb{H}^2$.

We have

$$\mathrm{Vol}(\mathbb{H}/\Gamma) = \left(\prod_{v \in S} (\mathbb{N}(v) - 1) \right) \frac{2|\Delta_F|^{\frac{3}{2}} \zeta_F(2)}{2^{2r_1+3r_2-2k} \pi^{2r_1+2r_2-k}}$$

where S are the finite places where B ramifies.

In the example above, $\mathrm{Vol}(\mathbb{H}^2/SL_2(\mathbb{Z})) = \frac{2\frac{\pi^2}{6}}{\pi} = \frac{\pi}{3}$.

The following development can be found in [CCGLS].

We had the following situation: M is a 3-manifold, $\partial M = T^2$ (torus), (see Figure 25) $M = \bigcup \Delta_{z_j}$ (disjoint union of tetrahedra). The gluing equations give rise to a curve. From the work by Neumann – Zagier in the 80's:

$$x \wedge y = \sum_{j=1}^{N} z_j \wedge (1 - z_j) \qquad \text{up to torsion in} \quad \Lambda^2(\mathbb{C}(C)^\times) \otimes \mathbb{Q}$$

Note that $\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q} \Rightarrow \eta(x, y)$ is exact. (And therefore, $m(P)$ can be computed using Stokes Theorem). We are going to prove that $\{x, y\} = 0$.

Recall: if A is a commutative ring with 1, then

$$1 \longrightarrow K_2(A) \longrightarrow St(A) \xrightarrow{\phi} E(A) \longrightarrow 1$$

is the universal central extension.

If we have a presentation of G,

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

define

$$H_2(G) := [F, F]/[F, R]$$

We have

$$K_2(A) \cong H_2(E(A))$$

We want to give a curve C and two functions on the curve, x, y such that $\{x, y\} = 0$ in $K_2(\mathbb{C}(C))$.

Let

$$u, v \in A^\times, \qquad \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} \in E(A)$$

such that u, v commute. Then

$$\{u, v\} := [\phi^{-1}(u), \phi^{-1}(v)] \in K_2(A)$$

Take $x, y \in G$ group, such that $[x, y] = 1$. For instance, $G = \pi_1(M)$, M hyperbolic 3-manifold, $\partial M = T^2$ and x, y generators of $\pi_1(\partial M) \cong \mathbb{Z}^2$.

Consider

$$\rho : G \longrightarrow E(A)$$

a representation of G. Then $[\rho(x), \rho(y)] = 1$. If $\rho(x), \rho(y)$ are diagonal, then we have a symbol $\{\rho(x), \rho(y)\}$. Define $H := \langle x, y \rangle$. Then

$$\begin{array}{ccc} H_2(H) & \longrightarrow & H_2(G) \\ & \searrow^{\rho|_H} & \downarrow \rho \\ & & H_2(E(A)) \cong \quad K_2(A) \end{array}$$

And the diagram commutes.

Now $\{\rho(x), \rho(y)\} = 0$ if $[x, y]$ is trivial in $H_2(G)$. For example, if M is a manifold as before, $G = \pi_1(M)$, and $H = \pi_1(T^2) \cong \mathbb{Z}^2$.

Hopf's formula says that

$$H_2(\pi_1(X)) \cong H_2(X)/\operatorname{Im} \pi_2(X)$$

x, y generators of T^2, $[x, y] = 1 \Rightarrow$

$$\begin{array}{ccc} H_2(H) & \xrightarrow{\quad f \quad} & H_2(G) \\ \downarrow \cong & & \downarrow \cong \\ H_2(\partial M)/\operatorname{Im} \pi_2(\partial M) & \longrightarrow & H_2(M)/\operatorname{Im} \pi_2(M) \end{array}$$

But $T^2 = \partial M \Rightarrow T^2$ is trivial in $H_2 \Rightarrow f \equiv 0 \Rightarrow \rho|_H(H_2(H)) = 0$, i.e. $\{x, y\} = 0$.

$\pi_1(\partial M) = \langle u, v \rangle$.

$$\begin{array}{rcl} \rho : \pi_1(M) & \longrightarrow & SL_2(\mathbb{C}) \\ u & \to & \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \\ v & \to & \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \end{array}$$

then $\{\rho(x),\rho(y)\} = \{x,y\}\{x^{-1},y^{-1}\} = \{x,y\}^2 \Rightarrow \{x,y\}^2 = 1$ in $K_2(\mathbb{C})$. We want to do this in a universal way. i.e., $u,v \in E(A), [u,v] = 1, \rightsquigarrow \{u,v\} \in K_2(A)$ this process is stable by conjugation, in this case, universal means up to conjugation.

We want to describe such representations. If G is a finitely generated group, $g_1,\dots g_n$ generators, $r_1,\dots r_m$ relations,

$$\begin{array}{ccc} \rho : G & \longrightarrow & SL_2(\mathbb{C}) \\ g_j & \to & A_j \\ r_i & \to & \mathrm{Id} \end{array}$$

then all the representations are described by an algebraic set.

Universal construction: look at all

$$\begin{array}{ccl} \rho : \pi_1(M) & \longrightarrow & SL_2(\mathbb{C}) \\ u & \to & \text{diagonalizable matrices} \\ v & \to & \end{array}$$

this is parameterized by some algebraic set. If C is a component of this algebraic set, we get that $\{x,y\} = 1$ in $K_2(\mathbb{C}(C))$.

In the case of the Figure 8 Knot, C is an elliptic curve of conductor 15. What is the significance of the primes 3, 5? We will see:

$$m\begin{pmatrix} 1 & 1 & 1 & \\ & 3 & 3 & \\ & 1 & 1 & 1 \end{pmatrix} = \frac{1}{6}L'(\chi,-1)$$

where χ is the conductor of $\mathbb{Q}(\sqrt{-15})/\mathbb{Q}$.

We will find the decomposition of $x \wedge y$ by hand.

The polynomial has to do with the same elliptic curve of conductor 15:

$$E : v^2 + uv + v = u^3 + u^2$$

$$P(x,y) = 0 \Longleftrightarrow w^2 = \mathrm{disc}_y(P) = -(x-1)^2 x(4x^2+7x+4).$$

$$\begin{cases} x &= -(u+1) \\ y &= \frac{v(v+1)}{u(u+v)} \end{cases}$$

x, y have divisors supported on $H = \langle Q \rangle \subset E(\mathbb{Q})$ where $Q = (\xi_3, -\xi_3)$ and $|H| = 8$.

$$\sum_{j=0}^{7} n_j(jQ) \quad \text{is a divisor iff} \quad \begin{cases} \sum_j n_j = 0 \\ \sum_j j n_j \equiv 0 \pmod 8 \end{cases}$$

L is the lattice of functions whose divisors are supported on H modulo constants (and has dimension 7). We want $z,\ 1-z \in L$.

$L/\mathbb{Q}$ has rank 5, spanned by $\{u, v, u+1, v+1, u+v\}$.

Now, in wedge 2,

$$\left.\begin{array}{rl} -u, & u+1 \\ -v, & v+1 \\ -(u+v), & 1+u+v \\ -\frac{v+1}{u}, & \frac{u+v+1}{u} \\ \frac{u+v}{v}, & -\frac{u}{v} \end{array}\right\} \in L$$

Note: $R \longrightarrow R+2Q$ acts on $L/\mathbb{Q}$. ($2Q = (0,0)$ which has order 4). Applying to these facts, we get a lattice of rank 8 in $\Lambda^2(L)$.

$$3x \wedge y = 3z_1 \wedge (1-z_1) + z_2 \wedge (1-z_2) + z_3 \wedge (1-z_3) + z_4 \wedge (1-z_4) + z_5 \wedge (1-z_5) + 2z_6 \wedge (1-z_6) + 2z_7 \wedge (1-z_7)$$

$$z_1 = \frac{-v^2}{u^2(u+1)}, \qquad z_2 = \frac{-v}{(u+1)^2}, \qquad z_3 = -v, \qquad z_4 = \frac{v(v+1)}{u(u+1)^2}$$

$$z_5 = -(u+v), \qquad z_6 = \frac{-(v+1)}{u}, \qquad z_7 = \frac{u+v}{v}$$

$$\eta(x,y) = d(D(\xi(x,y)), \qquad \xi = [z_1] + \frac{1}{3}([z_2]+[z_3]+[z_4]+[z_5]) + \frac{2}{3}([z_6]+[z_7])$$

In the integration only one root matters, since the other has module ≤ 1. See Figure 27.

$w_0 = \left(-2, \frac{-1+\sqrt{-15}}{2}\right)$, w_0 corresponds to the point $(x,y) = (1,-1)$, which is a fixed point of $\begin{array}{rcl} \sigma : x & \to & x^{-1} \\ y & \to & y^{-1} \end{array}$.

The fixed points of $\sigma : R \to -R + 4Q$ are: $2R = 4Q$, then $R = 2Q + U$ with $2U = \mathbf{O}$. Then the primes involved cannot be different from 2, 3, 5.

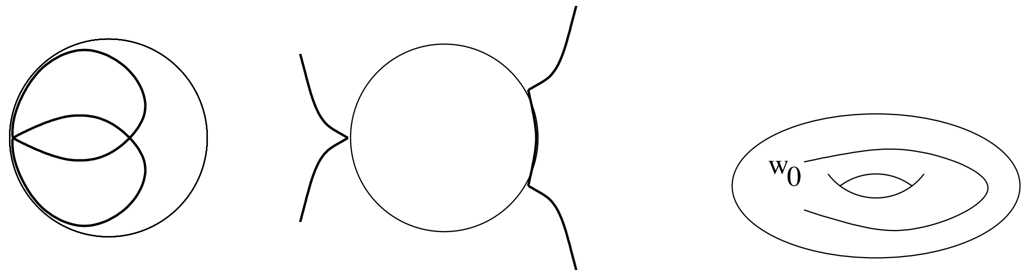


Figure 27: One of the roots has always module ≤ 1.

$$\pi m(P) = D(\xi(w_0))$$

Does Borel Theorem apply? i.e. Is $\xi(w_0) \in \mathcal{B}(\mathbb{Q}(\sqrt{-15}))$? $x \wedge y = \sum r_j z_j \wedge (1 - z_j)$, plug in w_0, we get

$$0 = 1 \wedge -1 = x(w_0) \wedge y(w_0) = \partial(\xi(w_0))$$

Then $\xi(w_0) \in \mathcal{B}(\mathbb{Q}(\sqrt{-15}))$, therefore, by Borel,

$$D(\xi(w_0)) \sim_{\mathbb{Q}^\times} \frac{15^{\frac{3}{2}}\, \zeta_{\mathbb{Q}(\sqrt{-15})}(2)}{\pi^2}$$

This implies that

$$m(P) \sim_{\mathbb{Q}^\times} L'(\chi, -1)$$

By Humbert's formula for $K = \mathbb{Q}(\sqrt{-15})$,

$$M = \mathbb{H}^3 / PSL_2(\mathcal{O}_K), \qquad \mathrm{Vol}(M) = \frac{15^{\frac{3}{2}} \zeta_K(2)}{4\pi^2}$$

We want to relate $\pi m(P)$ and $\mathrm{Vol}(M)$, this will be done through $D(\xi(w))$. We know $\zeta_K(s) = \zeta(s) L(\chi, s)$, and $\zeta(2) = \frac{\pi^2}{6}$.

Recall that if

$$M = \bigcup_j A_{w_j}, \quad \text{then,} \quad \mathrm{Vol}(M) = \sum_j D(w_j)$$

There are two possible approaches:

1. Gangl, H. M is decomposed into tetrahedra by finding a fundamental domain for $\Gamma = PSL_2(\mathcal{O}_K)$ on $\mathbb{H}^3$. We get that $\mathrm{Vol}(M) = D(\delta), \quad \delta \in \mathbb{Z}[K]$. The decomposition we found for $x \wedge y$ matches this decomposition, $\xi(w_0) = \delta$ This proves:

$$m(P) = \frac{1}{6} L'(\chi, -1) = \text{const.} \times L(\chi, 2) \quad \chi \leftrightarrow \mathbb{Q}(\sqrt{-15})/\mathbb{Q}$$

If we are not lucky: $\frac{15^{\frac{3}{2}} \zeta_K(2)}{4\pi^2} = \mathrm{Vol}(M) = D(\delta)$, and $\pi m(P) = D(\xi(w_0))$, so we only need

$$\xi(w_0) = \delta \text{ in } \mathcal{B}(K)$$

2. Nathan. Observe that

$$P = \begin{matrix} 1 & 1 & 1 & \\ & 3 & 3 & \\ & 1 & 1 & 1 \end{matrix}$$

is the A-polynomial of a 3-manifold and this is actually more information than the decomposition into tetrahedra.

- Get a Bianchi orbifold $M = \mathbb{H}^3/PSL_2(\mathcal{O}_K)$, with $K = \mathbb{Q}(\sqrt{-15})$. Baker, M.D: when is M or some other $\mathbb{H}^3/\Gamma'$, with $\Gamma' \subset \Gamma$ (finite index), the complement if some link? (e.g. the Fig 8 knot, $M \cong \mathbb{H}^3/\Gamma$, $\Gamma \subset PSL_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right)$). [0]

$$\mathbb{Q}(\sqrt{-15}) \xrightarrow{\text{produces}} \text{link}$$

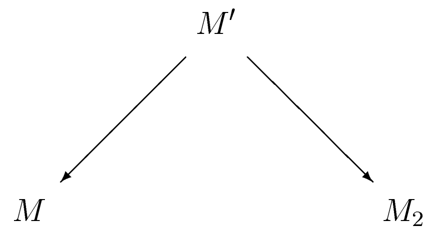
(the question of how to glue the tetrahedra is combinatorial).

In this particular case, we have $(2,0)\,(3,0)\,(0,0)\,(0,0)$ and apply Dehn's surgery to fill out one cusp.

- Now produce two other manifolds $M_1 = \mathfrak{m}129\,(0,3)\,(0,0)$, and $M_2 = \mathfrak{m}412\,(0,0)\,(3,0)$, by surgery which have a decomposition into tetrahedra over the field $\mathbb{Q}(\sqrt{-15})$ (this is done numerically).
- M and M_1 have a common 6-fold covering.

[0] It is proved that there are only finitely many discriminants and finitely many orbifolds that are the complement of some link for $|\text{disc}| \leq 71$.

- M_1 and M_2 have a common 3-fold covering.
- The A-polynomial of M_2 is P.
- The point $(1,1)$ (the point of intersection with the circle)$\leftrightarrow$ completes the structure on M_2, i.e., this gives $2\pi m(P) = \mathrm{Vol}(M_2)$.
- Then, M and M_2 have a common covering M':



  So the volume of M and M', and of M_2 and M' are related according to the degree of the correspondent coverings. In this way we can relate the volume of M_2 and M, but we have $\mathrm{Vol}(M_2) \sim m(P)$, and $\mathrm{Vol}(M) \sim \zeta_K(2)$.

Given M a 3-hyperbolic manifold, with a certain triangulation, we get

$$G : \prod_j z_j^{n_j}(1 - z_j^{n_j}) = \pm 1$$

$$C : \prod_k z_k^{m_k}(1 - z_k^{m_k}) = \pm 1$$

relaxing C, G produces a variety, if the manifold has one cusp, the component G is an algebraic curve.

So, projecting to x, y, (rational functions on this curve), we get an affine curve in $\mathbb{C}^\times \times \mathbb{C}^\times$, whose equation $A(x,y) = 0$ is the equation of the A-polynomial [1].

Construction of A-type polynomials. We want

- $\eta(x,y)$ exact: i.e., C curve, x, y rational functions and $\eta(x,y)$ exact.
- $P(x,y) = 0$, P reciprocal.

[1] The A-polynomial is an invariant which has been discovered recently.

$x \wedge y = \sum_j r_j\, z_j \wedge (1 - z_j) \Rightarrow \{x, y\} \in K_2(C)$

Universal construction of the trivial case:

$$a = \frac{1 + \sqrt{-7}}{2}, \qquad K = \mathbb{Q}(\sqrt{-7}), \qquad 2[a] + [a+1] \in \mathcal{B}(K)$$

(this data corresponds to the manifold $\mathfrak{m}009$).

We want

$$2\, a \wedge (1 - a) + (a + 1) \wedge [1 - (a + 1)] = 0 \quad \text{(up to torsion)}$$

This is true since $a + 1 = -(1 - a)^2$ and $1 - (a + 1) = -a$. Then, a, $1 - a$, generate (up to torsion, i.e., ± 1), the subgroup of $\mathbb{C}^\times$ containing the four quantities: $a, 1 - a, a + 1, -a$. In fact, we can write those numbers as (multiplicative) linear combinations of the two elements:

	a	$1-a$	$a+1$	$-a$
a	1	0	0	1
$1-a$	0	1	2	0

Lemma 27 *U of rank r, $u_1, \ldots, u_r$ generators. $z_1, w_1, \ldots z_N, w_N \in U$. Then*

$$\sum_{j=1}^{N} z_j \wedge w_j = \sum_{j<k} a_{jk}\, u_j \wedge u_k$$

Let $A = (a_{jk})$ with $a_{kj} = -a_{jk}$ then

$$A = MJM^t$$

where

$$M = (m^1, n^1, \ldots, m^N, n^N), \quad J = \left.\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & -1 & 0 & & & \\ & & & & \ddots & 0 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & 1 \\ 0 & 0 & \cdots & & 0 & -1 & 0 \end{pmatrix}\right\} 2N$$

and $z_j = u^{m^j}$ *and* $w_j = u^{n^j}$ *(using the multi index notation).*

In our case,

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

taking $u_1 = a$, $u_2 = 1 - a$. Then, $z_1 = z_2 = u_1 = a$, $z_3 = u_2^2 = a + 1$.

We get $2z_1 \wedge (1 - z_1) + z_3 \wedge (1 - z_3) = 0$, the sum of three tetrahedra.

In order to see that it is in the Bloch group, (it is easy to) verify that $MJM^t = 0$.

We want to extend M, by adding two rows so that $MJM^t = 0 \perp 0 \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Take,

$$M = \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right)$$

Now, each column of M gives a monomial in u_1, u_2, x, y.

$$\left\{\begin{array}{llllll} u_1x^{-1}+u_2 & =1 & \rightsquigarrow & z_1+w_1=1 & \rightsquigarrow & z_1+(1-z_1)=1 \\ u_1x+u_2y & =1 & \rightsquigarrow & z_2+w_2=1 & \rightsquigarrow & z_2+(1-z_2)=1 \\ u_2^2y+u_1 & =1 & \rightsquigarrow & z_3+w_3=1 & \rightsquigarrow & z_3+(1-z_3)=1 \end{array}\right. \tag{4}$$

By the Lemma, MJM^t corresponds to $z_1 \wedge w_1 + z_2 \wedge w_2 + z_3 \wedge w_3 = x \wedge y$. Now, the equation (4), leads to $z_1 \wedge (1-z_1) + z_2 \wedge (1-z_2) + z_3 \wedge (1-z_3) = x \wedge y$. It is much easier working with matrices than working with this expression.

The curve C that we get is given by (4). Projecting onto x, y, we get the polynomial

$$\begin{array}{ccc} 1 & 1 & \\ & -2 & \\ & -2 & \\ & 1 & 1 \end{array} \qquad P(x,y) = xy^3 + y^3 - 2xy^2 - 2xy + x^2 + x$$

Then we need to solve $P(x,y) = 0$, $|x| = |y| = 1$. We get $(1,1) \leftrightarrow 2[a] + [a+1]$.

Therefore,

$$\pi m(P) = \mathrm{Vol}(\mathfrak{m009}) = \frac{c}{\pi^*}\zeta_{\mathbb{Q}(\sqrt{-7})}(2) \quad c \in \mathbb{Q} \text{ known}$$

This last equality comes from the Bianchi manifold theory.

Now, in a general setting, we start with $M \in \mathbb{Z}^{4\times 6}$, $MJM^t = 0 \perp 0 \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We can modify the example above. Take $U \in Sp_6(\mathbb{Z})$ (or, more precisely, $UJU^t = J$). Then can take $M' = MU$ and $M'JM'^t = 0 \perp 0 \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The polynomial P associated to M' may not be reciprocal. How do we guarantee that P is reciprocal?

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \longleftrightarrow \mathfrak{m009}$$

We have an involution $x \to x^{-1}$, $y \to y^{-1}$.

$$\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_4 \to R_2 + R_4} \left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right)$$

We can pick $U \in Sp_6(\mathbb{Z})$ such that it preserves this involution. By the shape of the involution, U must be of the form:

$$U = \begin{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & & \\ & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \\ & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \quad \text{with } det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$$

In general, $M = (m^1, n^1, m^2, n^2, m^3, n^3)$

$$\begin{cases} \pm u^{m_1} \pm u^{n_1} = 1 \\ \pm u^{m_2} \pm u^{n_2} = 1 \\ \pm u^{m_3} \pm u^{n_3} = 1 \end{cases} \quad \text{where } u = (u_1, u_2, x, y)$$

(here signs can change without affecting the torsion part).

For instance, take $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ in the example, then we get

$$M' = \left(\begin{array}{cc|cc|cc} -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{array}\right) \qquad M'JM'^t = 0 \perp 0 \perp \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the equations:

$$M' : \begin{cases} u_1^{-1}x + u_2^{-1} & = & 1 \\ u_1^{-1}x^{-1} + u_2^{-1}y^{-1} & = & 1 \\ u_2^2 y + u_1 & = & 1 \end{cases} \qquad M : \begin{cases} u_1^{-1}x + u_2 & = & 1 \\ u_1 x + u_2 y & = & 1 \\ u_2^2 y + u_1 & = & 1 \end{cases}$$

And we get the polynomial

$$P_{M'}: \quad \begin{array}{ccccc} & -1 & & \\ -1 & 0 & 1 & \\ & 2 & 1 & \\ & 4 & -4 & \\ & -1 & -2 & \\ & -1 & 0 & 1 \\ & & 1 & \end{array} \qquad -xy^6+x^2y^5+x^2y^4-y^5+2xy^4-4x^2y^3+4xy^3-2x^2y^2+x^3y-xy^2-xy+x^2$$

which is not exactly reciprocal, but it is just a question of changing one sign.

We have $2\pi m(P) = \int_\gamma \eta(x,y)$, where,

$$\partial\gamma \subset \{P(x,y)=0\} \cap \{|x|=|y|=1\} = \{(\pm 1, \pm 1)\}$$

On this example, only two points contribute ti the Mahler measure: $(1,1)$ produces a cubic field F_1 [2] of discriminant $\Delta = -23$ and $(1,-1)$ produces a cubic field F_2 of discriminant $\Delta = -31$. Finally,

$$m(P) \stackrel{?}{=} 3\zeta_{F_1}(2) + 3\zeta_{F_2}(2)$$

The equality has to do with the theory of Bianchi manifolds. (This polynomial is actually the A-polynomial for $\mathfrak{m}367$).

5.2 Maillot's calculation revisited

Let C be a curve, x,y rational functions such that $\eta(x,y) = dV$. Then $2\pi m(P) = \int_\gamma \eta(x,y) = V(\partial\gamma)$.

E.g. $C = \mathbb{P}^1$, $x = t$, $y = 1-t$. $P(x,y) = x+y-1$, $V = D(t)$. If we scale x,y by $a,b \in \mathbb{C}^\times$,

$$x_1 = \frac{x}{a} \qquad y_1 = \frac{y}{b}$$

[2] In order to determine which field resolves the point $(1,1)$, we use the equations

$$\begin{cases} u_1^\alpha u_2^\beta x^\alpha + u_1^\gamma u_2^\delta x^\gamma & = 1 \\ u_1^\alpha x^{-\alpha} u_2^\beta y^{-\beta} + u_1^\gamma x^{-\gamma} u_2^\delta y^{-\delta} & = 1 \\ u_2^2 y + u_1 & = 1 \end{cases}$$

Then set $x = y = 1$ and we get

$$(1-u_2^2)^\alpha u_2^\beta + (1-u_2^2)^\gamma u_2^\delta = 1$$

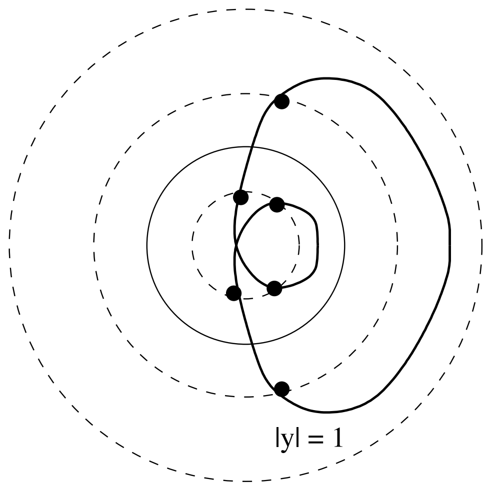


Figure 28: There are three different cases when scaling by b

$$P_1(x_1, y_1) = P(ax, by)$$

e.g. $ax + by - 1 = 0$. Then,

$$2\pi m(P_1) = \int_{\gamma_1} \eta(x_1, y_1)$$

but

$$\eta(x_1, y_1) = \eta(x, y) - \eta(a, x) - \eta(x, b) + \eta(a, b) = dV - \log|a| d\arg y + \log|b| d\arg x$$

E.g.

$$\begin{matrix} 1 & & \\ -1 & -2 & \\ 1 & -2 & 1 \end{matrix}$$

We get a formula for the Mahler measure of the general polynomial P_1 starting from the formula for the Mahler measure of the particular polynomial P:

$$2\pi m(P_1) = \sum_w V(w) + \log|a| \sum \alpha' s + \log|b| \sum \beta' s$$

Observe that the number of summands may depend on the parameters a, b, for instance, varying b we may have four, two, or zero points w. See Figure 28.

5.3 Ladders

We will say a few words about relations among polylogarithms of powers of a fixed number α, known as ladders.

Question: How do we write down elements in $\mathcal{B}(\bar{\mathbb{Q}})$? Suppose $\alpha \in \bar{\mathbb{Q}}$ satisfies a cyclotomic equation, i.e.

$$\prod_{n=1}^{\infty} (\alpha^n - 1)^{c_n} = \zeta \alpha^N, \quad c_n \in \mathbb{Z},\ c_n = 0 \text{ for almost all } n,\ \zeta \in \mu_\infty$$

Claim:

$$\xi = \sum_{n=1}^{\infty} \frac{c_n}{n} [\,\alpha^n] \in \mathcal{B}(\mathbb{Q}(\alpha)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Proof. It is enough to see that $\partial(\xi) = 0$.

$$\sum_{n=1}^{\infty} \frac{c_n}{n} \alpha^n \wedge (1 - \alpha^n) = \sum_{n=1}^{\infty} \alpha \wedge (1 - \alpha^n)^{c_n} = \alpha \wedge \prod_{n=1}^{\infty} (1 - \alpha^n)^{c_n} = \alpha \wedge \pm \zeta \alpha^N = 0$$

□

If α satisfies many linearly independent cyclotomic equations, we will eventually find one with $D(\xi) = 0$.

In what follows, [CLZ] is a good reference.

If α is the real root of the Lehmer polynomial of degree 10 which is also > 1, then $\prod_{n=1}^{\infty} (\alpha^n - 1)^{c_n} = \zeta \alpha^N$ implies that ζ must be real and positive, then $\zeta = 1$. Also, $\frac{1}{\alpha}$ is conjugate to α. Conjugating, we get

$$\prod_{n=1}^{\infty} \left(\frac{1}{\alpha^n} - 1 \right)^{c_n} = \alpha^{-N}$$

Comparing the powers of α, we conclude:

$$\sum_{n=1}^{\infty} n\, c_n = 2N$$

If we define

$$P_1(z) := -\log |1 - z\,| + \frac{1}{2} \log |z\,|$$

More generally, the Polylogarithm function P_n is a modified version of

$$Li_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{n^m}$$

Then

$$\sum_{n=1}^{\infty} c_n \, P_1(\alpha^n) = 0$$

For the same α, [CLZ] found a numerical relation of the form:

$$\sum_{n=1}^{360} c_n \, P_{16}(\alpha^n) = c\pi^{16}$$

Now, finding multiplicative relations among α and $\alpha^n - 1$ is the same as finding the relations among α and cyclotomic polynomials $\Phi_k(\alpha)$. The equation $\prod_{n=1}^{\infty} (1-\alpha^n)^{c_n} = \zeta\alpha^N$ translates into multiplicative relations for the $\mathbb{N}\Phi_k(\alpha)$. It is known that $\mathbb{N}\Phi_k(\alpha) = \pm 1$ for 66 values of $k \leq 1000$ (for α the Lehmer root).

$$\mathbb{N}\Phi_k(\alpha) = \pm \prod_{\alpha,\xi} (\alpha - \xi)$$

$$\frac{1}{\phi(k)} \log |\mathbb{N}\Phi_k(\alpha)| = \frac{1}{\phi(k)} \sum_{\alpha,\xi} \log |\alpha - \xi| = \frac{1}{\phi(k)} \sum_{\xi} \log |P(\xi)|$$

$$\text{(Riemann sum for)} \sim \frac{1}{2\pi i} \int_{|x|=1} \log |P(x)| \frac{dx}{x} \sim m(P)$$

They have found 71 linearly independent cyclotomic relations for α. In fact, they found

$$\sum c_n \, P_k(\alpha^n) \quad \text{for } k = 1, 2, \ldots, 16$$

Consider also the example in [B-RV]. The Smyth's calculation:

$$P(x,y) = p(x)y - q(x) \quad \in \mathbb{Z}[x,y]$$

where p, q have roots in $\mu_\infty \cup \{0\}$. Then $P(x,y) = 0$ is $\mathbb{P}^1$, so $\eta(x,y) = dV$.

Proposition 28

$$\frac{q(x)}{p(x)} = \pm \, x^k \prod_{n=1}^{\infty} (1 - x^n)^{c_n}$$

then

$$2\pi m(P) = \pm \sum_{n=1}^{N} (-1)^n \sum_{m=1}^{\infty} \frac{c_m}{m} D(\alpha_n^m)$$

where α_n are the roots of $p(x)p(x^{-1}) - q(x)q(x^{-1}) = 0$ which have odd multiplicity (this is a cyclotomic equation).

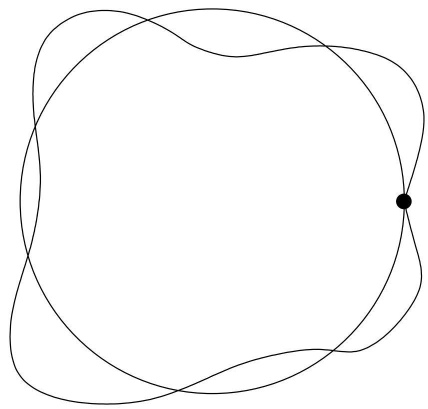


Figure 29: Case of α_n of even multiplicity.

The hypothesis of odd multiplicity is needed so that as long as x moves on the unit circle, $\frac{q(x)q(x^{-1})}{p(x)p(x^{-1})}$ enters or leaves the unit circle in each α_n. If α_n had even multiplicity we could have a case as the one in Figure 29.

Question: Can you find A, B cyclotomic (i.e., rational functions such that the roots and poles $\in \mu_\infty \cup \{0\}$) such that $A+B=1$? A good example due to Mossinghoff, Pinner, Vaaler:

$$\left(\frac{x^{n+1}-1}{x-1}\right)^2 - x^n = \frac{(x^{n+2}-1)(x^n-1)}{(x-1)^2}$$

Dividing by x^n we get two cyclotomic equations whose sum is 1.

In [B-RV]:

$$m((x+1)^2y+x^2+x+1) = \frac{9}{4}m((x+1)^4y+x^4+1)$$

$$\xi = 7[\alpha]+[\alpha^2]-3[\alpha^3]+[-\alpha^4] \qquad \alpha = \frac{-3+\sqrt{-7}}{4}$$

It is quite hard to prove that actually$D(\xi)=0$.

6 The elliptic curve case

Let $C/\mathbb{Q}$ be a smooth curve. The regulator map:

$$\text{reg} : K_2(C) \longrightarrow H^1(C,\mathbb{R})^-$$

$$\{f,g\} \to \eta(f,g)$$

Observe that $H^1(C,\mathbb{R})^-$ is a $\mathbb{R}$- vector space of dimension equal to the genus g of the curve. We have

$$\eta(\bar{f},\bar{g}) = -\eta(f,g)$$

This should be an analogous of what happens with number fields (Borel). The simplest case is with $g=1$, E an elliptic curve. We should have:

$$\text{Covolume of image of reg} \overset{?}{\sim}_{\mathbb{Q}^\times} L'(E,0)$$

This is not a Theorem but a Conjecture. Let $E/\mathbb{Q}$ be an elliptic curve. Supposed that it is given in Weierstrass form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad a_i \in \mathbb{Z}$$

Then consider

$$\#E(\mathbb{F}_p) = p + 1 - a_p$$

If E mod p is not singular, we get

$$Z(E \bmod p, T) = \frac{1 - a_pT + pT^2}{(1-T)(1-pT)} = \frac{H^1}{H^0\ H^2}$$

If E mod p is singular, E^{ns} mod p (nonsingular points) carries a group structure and has only one singular point.

E^{ns} mod $p \cong$

	# points	$\#E$ mod p	a_p	kind of reduction
$(\mathbb{F}_p,+)$	p	$p+1$	0	additive
$(\mathbb{F}_p^\times,\times)$	$p-1$	p	1	split multiplicative
$(\mathbb{F}_{p^2}^\times/\mathbb{F}_p^\times,\times)$	$p+1$	$p+2$	-1	non split multiplicative

and

$$Z(E \bmod p, T) = \frac{1 - a_pT}{(1-T)(1-pT)}$$

Observe that in both cases we have the same denominator and it does not depend on the elliptic curve. So, it makes sense to codify the information of the numerator. Define:

$$L(E,s) = \prod_{p\notin S}(1 - a_pp^{-s} + p^{1-2s})^{-1}\prod_{p\in S}(1 - a_pp^{-s})^{-1} \qquad \Re(s) > \frac{3}{2}$$

where S is the set of primes of bad reduction.

Note: the Euler factor vanishes at $s = 0$ iff $a_p = 1$, i.e. if E has split multiplicative reduction.

Modularity means

$$L(E, s) = L(f, s) \qquad f \text{ newform of level } N \text{ weight } 2$$

Define

$$L^*(E, s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(E, s)$$

Then

$$L^*(E, s) = \pm L^*(E, 2 - s)$$

The product converges for $\Re(s) > \frac{3}{2}$, then $L^*(E, 2) \neq 0$. Therefore, $L^*(E, 0) \neq 0$. Since $\Gamma(s)$ has a simple pole at $s = 0$, we conclude that $L(E, s)$ has a simple zero at $s = 0$.

Conjecture 29 *(Bloch – Beilinson)*

$$r : K_2(\mathcal{E}) \longrightarrow \mathbb{R}$$

$K_2(\mathcal{E})$ is of rank 1 and

$$L'(E, 0) \sim_{\mathbb{Q}^\times} r(\xi)$$

where $\xi \in K_2(\mathcal{E})$ is a non-torsion element.

What is $K_2(\mathcal{E})$? $\mathcal{E}$ is the Néron model of E and it corresponds to a condition of integrality. To fix ideas, we would say that $\mathcal{E}$ is to E the same as $\mathcal{O}_K$ is to K.

We had

$$K_2(E) \longrightarrow K_2(\mathbb{Q}(E)) \xrightarrow{\text{tame symbols}} \bigoplus_w \mathbb{C}^\times$$

Now we have

$$K_2(\mathcal{E}) \longrightarrow K_2(E) \longrightarrow \bigoplus_p K_1'(\mathcal{E} \bmod p)$$

We have defined the regulator function:

$$\text{reg} : K_2(C) \longrightarrow H^1(C, \mathbb{R})^-$$

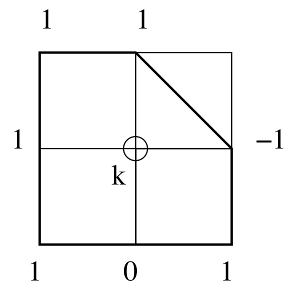


Figure 30: E_k is generically a curve of genus 1.

Now $K_2(\mathcal{E}) \subset K_2(E)$ and we can restrict reg to $K_2(\mathcal{E})$, then we get a number by integrating:

$$r(\{f,g\}) = \int_{\gamma^-} \eta(f,g) \quad \in \mathbb{R}$$

where $\mathbb{Z}\gamma^- = H_1(E,\mathbb{Z})^-$.

$K_1'(E \bmod p)$ is a group such that $K_1'(E \mathrm{mod}\, p) \otimes \mathbb{Q}$ is trivial unless p is a prime of split multiplicative reduction.

Next, we would like to write elements of $K_2(E)$ explicitly.

1. **Proposition 30** *(Bloch) Let $f, g \in \mathbb{Q}(E)$, such that their divisors are supported on torsion points, then there exists $f_i \in \mathbb{Q}(E)^\times$, $c_i \in \mathbb{Q}^\times$ such that:*

$$\{f,g\} + \sum_i \{f_i, g_i\} \in K_2(E)$$

2. Newton polygons.

   - Pick a polygon Δ with only one interior point (Figure 30). There are 16 possible choices up to $GL_2(\mathbb{Z})$. [3]
   - Pick integer coefficients for the sides such that the tame symbols are torsion. We have a finite number of choices for this, but the tame symbols do not impose any condition on the center coefficient.

   This produces a family of curves E_k that generically have genus 1, with $\{x,y\} \in K_2(E_k)$.

[3]Scott proved that given $r > 0$, there are finitely many such polygons with $\#\Delta^\circ = r$. For $r = 0$ this is clearly false, take for instance any rectangle with one side equal to one and the other side any length.

Other versions of the regulator:

The Tate curve is

$$E(\mathbb{C}) \cong \mathbb{C}^\times / q^{\mathbb{Z}}$$

for

$$E(\mathbb{C}) \cong \mathbb{C}/L \qquad \text{where} \quad L = \mathbb{Z} + \mathbb{Z}\tau \quad \tau \in \mathcal{H}, \quad q = e^{2\pi i \tau}$$

the correspondence is given by

$$\mathbb{C} \longrightarrow \mathbb{C}^\times$$

$$z \in \mathbb{C} \to u = e^{2\pi i z}$$

Well defined since clearly

$$z + 1 \to u$$

$$z + \tau \to uq$$

If $u \in \mathbb{C}^\times$, define

$$D(u,q) := \sum_{n \in \mathbb{Z}} D(uq^n)$$

It can be proved that this series converges absolutely and uniformly over compact sets. Note that summing in this way we assure that the function is defined on the quotient $\mathbb{C}^\times / q^{\mathbb{Z}}$, i.e,

$$D(uq,q) = D(u,q)$$

So we get the elliptic dilogarithm:

$$\mathcal{D} : E(\mathbb{C}) \longrightarrow \mathbb{R}$$

Proposition 31 *(Bloch) If $\sum_i \{f_i, g_i\} \in K_2(E)$. Then*

$$\sum_i \int_{\gamma^-} \eta(f_i, g_i) = r\left(\sum_i \{f_i, g_i\}\right) = c \sum_i \sum_{P,Q} v_P(f_i)\, v_Q(g_i)\, \mathcal{D}(P - Q)$$

(c is a well-known constant).

$$\mathcal{D} = \Re\left(\frac{(\Im\tau)^2}{\pi} \sum_{\gamma \in \Gamma \setminus \{(0,0)\}} \frac{(w,\gamma)}{\gamma^2 \bar{\gamma}}\right)$$

where

$$(w,\gamma) := \exp\left(\frac{2\pi i (z\bar{\gamma} - \bar{z}\gamma)}{\tau - \bar{\tau}}\right)$$

and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$.

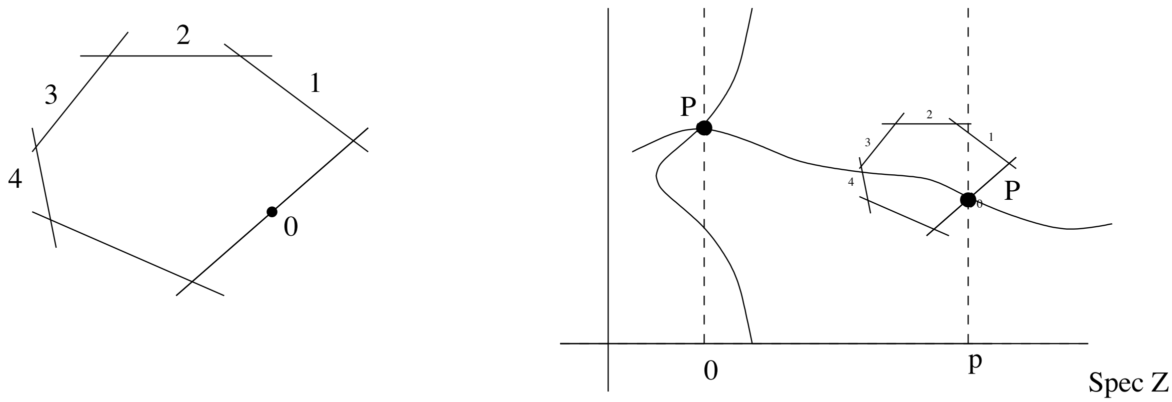


Figure 31: Néron Model.

If E is an elliptic curve with CM by an imaginary quadratic field of class number one, the formula with the elliptic dilogarithm is essentially analogous to the relation of the L-function (see [R]). For example, take

$$E : y^2 = x^3 - 1$$

If $K = \mathbb{Q}(\xi_3)$ and $L(E,s) = L(\varphi, s)$ where φ is a Hecke character, $\varphi((\alpha)) = \epsilon(\alpha)\alpha$.

$$L(E,s) = \frac{1}{6} \sum_{\alpha \in \mathcal{O}_K^\times} \frac{\epsilon(\alpha)\bar{\alpha}}{\mathbb{N}\alpha^s}$$

if $s = 2$ we get $\sum_{\alpha\in\mathcal{O}_K^\times} \frac{\epsilon(\alpha)}{\alpha^2\bar{\alpha}}$ and this can be seen as $\sum_w c_w \mathcal{D}(w)$ with w 6-torsion points.

Note: CM curves have always additive reduction and so we do not have to worry about the integrality condition.

Integrality: if p is a prime with split multiplicative reduction, then the Néron model is like in Figure 31.

$$\partial_P(\{f,g\}) = \sum_{P,Q} v_P(f)\, v_Q(g)\, B_3\left(\frac{d(i_P, i_Q)}{N}\right)$$

where N is the number of sides of the Néron model, d is a distance, the number of sides to pass to go from one point to the other and

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

is the Bernoulli 3-polynomial.

Back to the construction that started from the Newton polygon (Figure 30), we got $\{x, y\} \in K_2(E_k)$. Then, if k is such that E_k is an elliptic curve,

$$\{x, y\} \in K_2(\mathcal{E}_k) \Leftrightarrow k \in \mathbb{Z}$$

Numerically Boyd proved that

$$r(\{x, y\}) = m(P_k) \sim_{\mathbb{Q}^\times} L'(E_k, 0) \quad k \in \mathbb{Z}$$

Idea: if $p|$ denominator of k then we should have that $\partial_P(\{x, y\}) \neq 0$ and that would imply no integrality.

We can relate the Néron polygon for p with the Newton polygon. The dual of the Newton polygon is the Néron polygon.

There is a combinatorial formula for $\partial_P(\{x, y\})$ in terms of the polygon, and it is easy to compute this number and see that it is different from zero.

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