teaching/504...

I

1

Gray Code

Engineer Bell labs '30s

Binary Code

Encode numbers as strings of bits, i.e. digits 0 or 1.

	8	4	2	1	C
0	0	0	0	0	
1	0	0	0	1	1
2	0	0	1	0	2
3	0	0	1	1	1
4	0	1	0	0	3
5	0	1	0	1	1
6	0	1	1	0	2
7	0	1	1	1	1

	8	4	2	1	C
8	1	0	0	0	4
9	1	0	0	1	1
10	1	0	1	0	2
11	1	0	1	1	1
12	1	1	0	0	3
13	1	1	0	1	1
14	1	1	1	0	2
15	1	1	1	1	1

$n = 13$

Algorithm 1

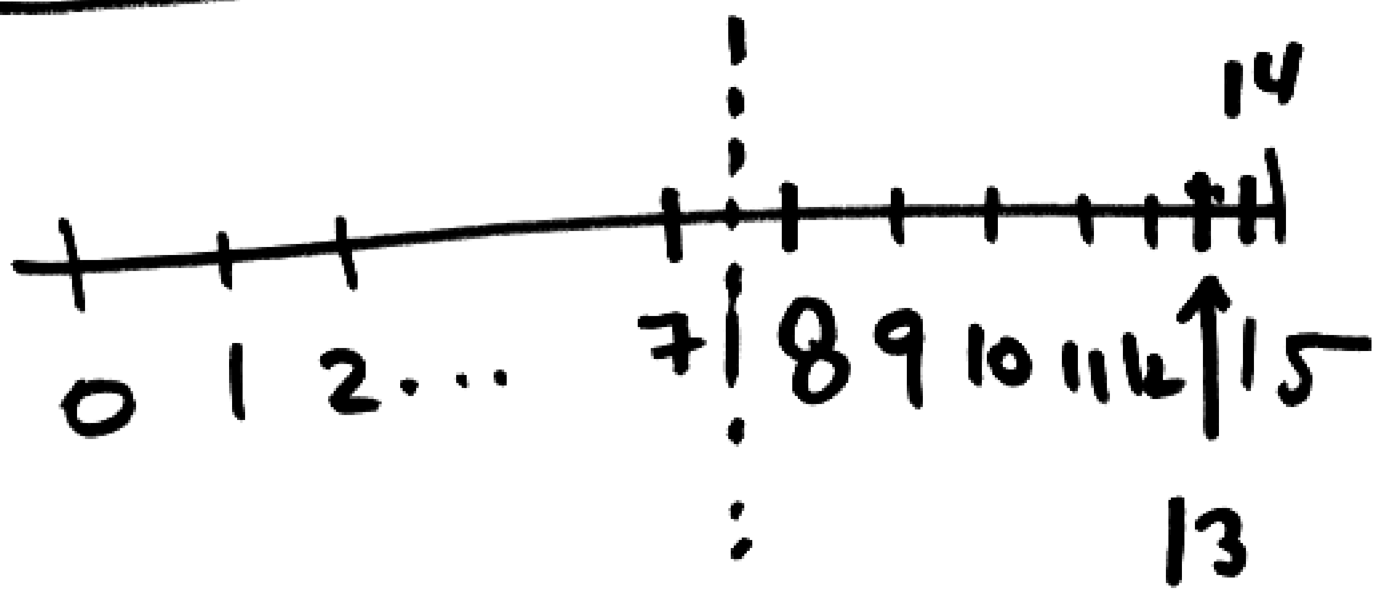
Euclidean algorithm

$$
\begin{aligned}
13 &= \underline{6} \times 2 + 1 \\
6 &= \underline{3} \times 2 + 0 \\
3 &= \underline{1} \times 2 + 1 \\
1 &= 0 \times 2 + 1
\end{aligned}
$$

$13 = (1101)_2$

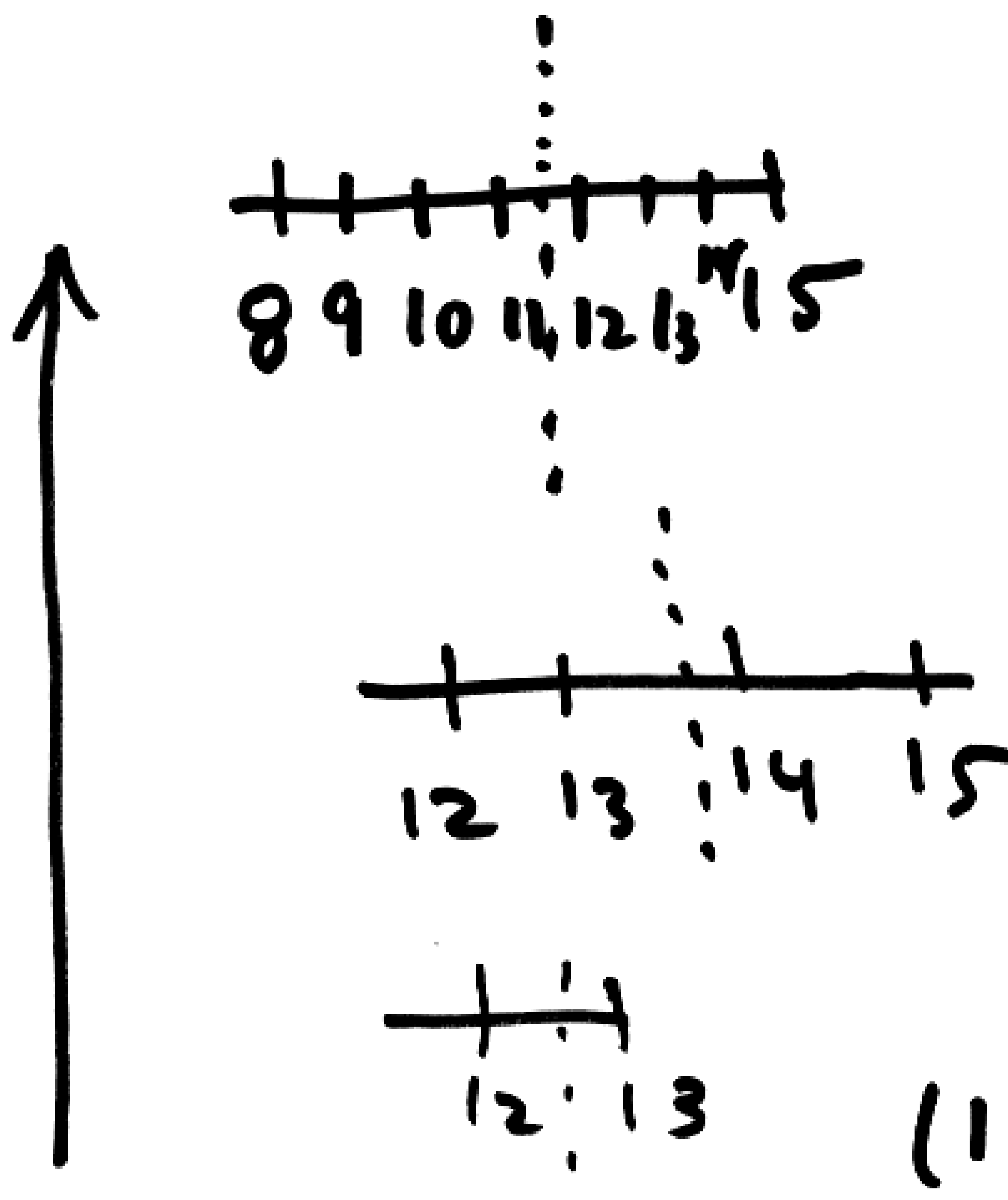
Algorithm 2

Dissection method



Q: Is x in top half? 1

1



Q: 1

Q: 0

Q: 1

$(1101)_2 = 13$

$$m = 2^r \cdot k \qquad r = 0, 1, \ldots$$

$$2 \nmid k \qquad (k \text{ is odd})$$

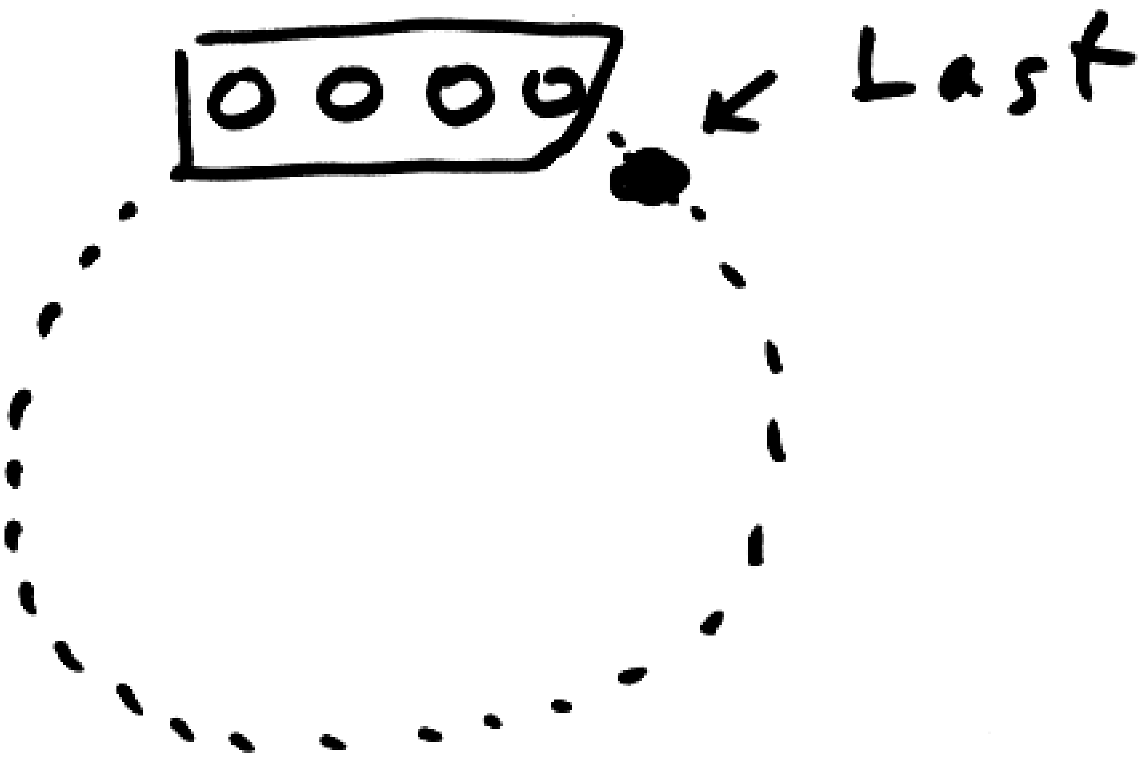
$$V_2(m) := r$$

<u>valuation of m at 2</u>

m	$V_2(m)$
1	0
2	1
3	0
4	2
5	0
6	1
7	0
8	3
9	0

Gray Code

Way to encode numbers such that the code for m and for m+1 differ in exactly <u>one</u> bit.



Reflected Gray Code

Binary Code

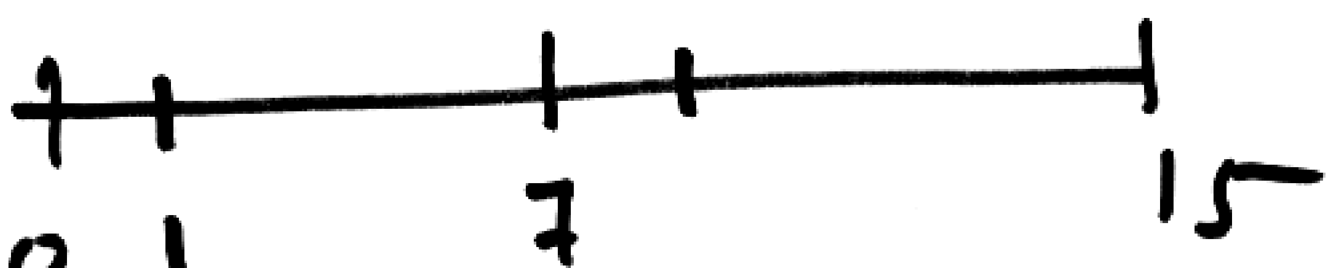
Recursively.

$B_0 = 0$

$B_1 = 0, 1$

$B_{n+1} := 0\,B_n \cup 1\,B_n$

001



$1 \mapsto 0001$

$8 \mapsto 1001$

Reflected Gray Code

$$C_1 = 0, 1$$

$$C_{n+1} := 0\,C_n \cup 1\,C_n'$$

$$C_n' = C_n \text{ backwards}$$

n	
1	0, 1
2	00, 01, 11, 10
3	000, 001, 011, 010, 110, 111, 101, 100

0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9	1101
10	1111
11	1110
12	1010
13	1011
14	1001
15	1000

- Chinese rings
- Slide

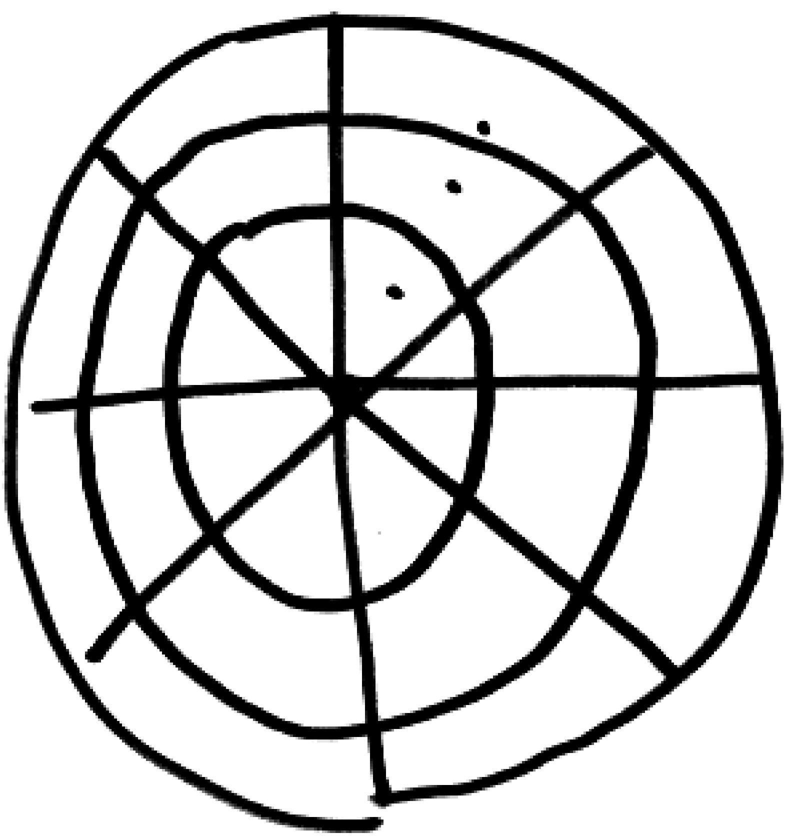
$$\boxed{x}\ \boxed{1}\ 0 \cdots\cdots 0$$

can change.

R Gray code

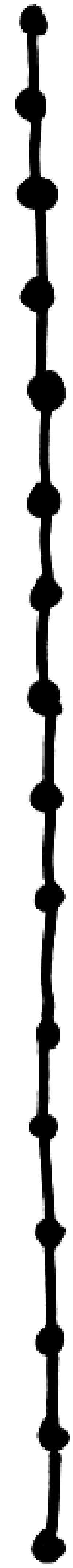
has good ways to encode / decode.

Robot arm



$\boxed{\text{II}}$ 1

k	m	BINARY	GRAY
	0	0000	0000
1	1	0001	0001
2	2	0010	0011
1	3	0011	0010
3	4	0100	0110
1	5	0101	0111
2	6	0110	0101
1	7	0111	0100
4	8	1000	1100
1	9	1001	1101
2	10	1010	1111
1	11	1011	1110
3	12	1100	1010
1	13	1101	1011
2	14	1110	1001
1	15	1111	1000



Bit that changes in Gray code is either the first or the bit number k where $k = \#$ bits that change in the binary code.

$k = v_2(m) + 1$

$v_2(m) = r$

$2^r \cdot l = m, \quad 2 \nmid l$

$$m = (b_{n-1}, \ldots, b_1, b_0)_2$$

Binary code

$$m = b_{n-1} 2^{n-1} + \cdots + b_1 2 + b_0 2^0$$

$$r = V_2(m)$$

$$\begin{array}{cccccccccc} & 7 & 6 & 5 & 4 & \boxed{3} & 2 & 1 & 0 & \\ m = (& 1 & 0 & 1 & 1 & \boxed{1} & 0 & 0 & 0 &)_2 \\ & 128 & 64 & 32 & 16 & 8 & 4 & 2 & 1 & \end{array}$$

$$m = 128 + 32 + 16 + 8 = (\underbrace{16+4+2+1}_{\text{odd}}) \cdot 8$$

$8 \mid 16$

$8 \mid 32$

$8 \mid 128$

$$V_2(m) = 3$$

In Gray code

<u>Either</u>

1) change first bit

OR

2) change this bit

X 1 0 0 0 . . .

Moving the first ring has this effect

$$\boxed{m \longmapsto m + (-1)^{b_0}}$$

b_0 = # rings on

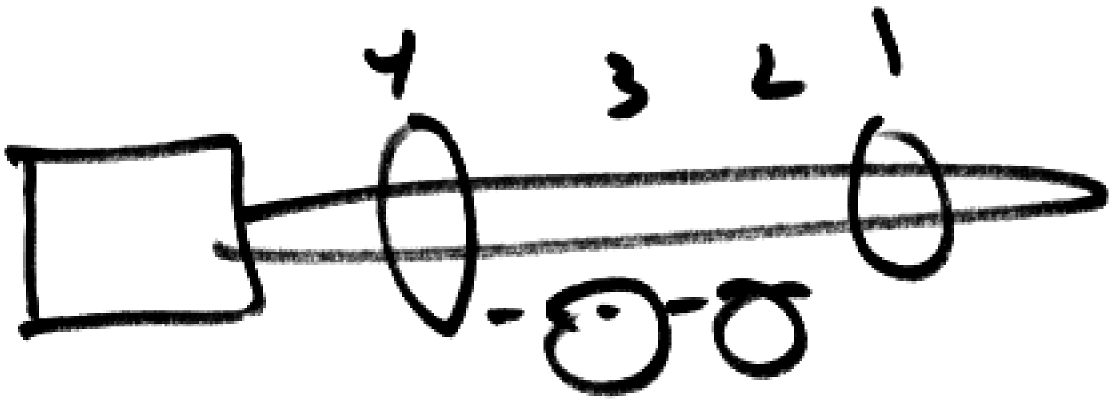
$$m = (b_{n-1} b_{n-2} \dots b_1 b_0)_2$$

$$b_0 = \begin{cases} 0 & m \text{ even} \\ 1 & m \text{ odd} \end{cases}$$

position in chinese ring

(6)

1 0 0 1



we should put ring 2 back on

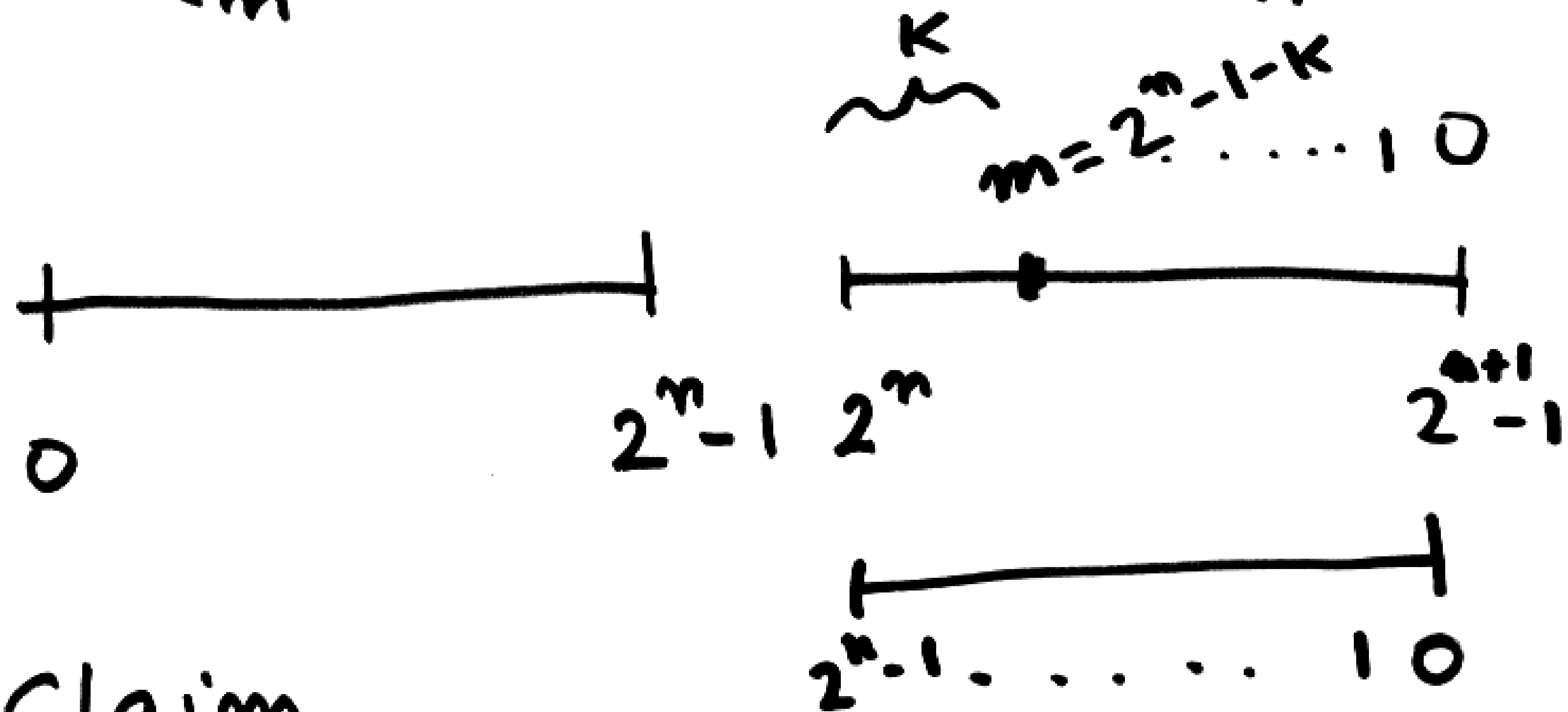
---

given m denote by $g(m)$ the gray code word thought of as binary.

e.g. $g(4) = 6$

$(0100)_2 \mapsto (0110)$

$$0\,\underbrace{\qquad\qquad}_{C_n} \;\cup\; 1\,\underbrace{\qquad\qquad}_{C_n'}$$



Claim

$$\boxed{g(m) = 2^n + g(2^n - 1 - k)}$$

if $\quad m = 2^n + k$

$\qquad 0 \le k < 2^n$

What is the binary code for

$2^n - 1 - k$?

$n = 3$

$K = 2$

$2^3 - 1 - 2 = 5$

0	000
1	001
2	010
3	011
4	100
5	101
6	111
7	111

$$2 \xrightarrow{\text{reversed}} 5$$

010   101

Reversing order just means flip every bit (in binary).

$$m = (b_{n-1}\, b_{n-2} \cdots b_1\, b_0)_2$$

$$g(m) = (c_{n-1}\, c_{n-2} \cdots c_1\, c_0)_2$$

Claim

$$c_j \equiv b_j + b_{j+1} \mod 2$$

$$1 + 1 \equiv 0 \mod 2$$
$$1 + 0 \equiv 1 \mod 2$$
$$0 + 1 \equiv 1 \mod 2$$
$$0 + 0 \equiv 0 \mod 2$$

$$\boxed{\text{Binary} \longrightarrow \text{Gray}}$$

Gray → Binary

$$b_j \equiv c_j + c_{j+1} + c_{j+2} + \cdots \mod 2$$

$$
\begin{array}{rlllll}
 & c_j = & b_j + b_{j+1} & & & \\
 & c_{j+1} = & & b_{j+1} + b_{j+2} & & \\
+ & c_{j+2} = & & & b_{j+2} + b_{j+3} & \\
 & \vdots & & & & \vdots \\
\hline
 & & b_j + 0 & + 0 & + & 0 +
\end{array}
$$

How many steps it takes to solve the Chinese rings? with n rings

n	Gray	Binary	
1	1	1	1
2	11	10	2
3	111	101	5
4	1111	1010	10
5	11111	10101	21
⋮	⋮	⋮	⋮

n odd

$\ldots 10101$

$$1 + 4 + 4^2 + 4^3 + \ldots + 4^{\frac{n-1}{2}}$$

$$= \frac{4^{\frac{n+1}{2}} - 1}{4-1} = \frac{1}{3}\left(2^{n+1} - 1\right)$$

$$1 + a + a^2 + \ldots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

$a \neq 1$

$$\frac{1}{3}\left(2^{n+1} - 1\right) = \frac{2}{3} \cdot 2^n - \frac{1}{3}$$

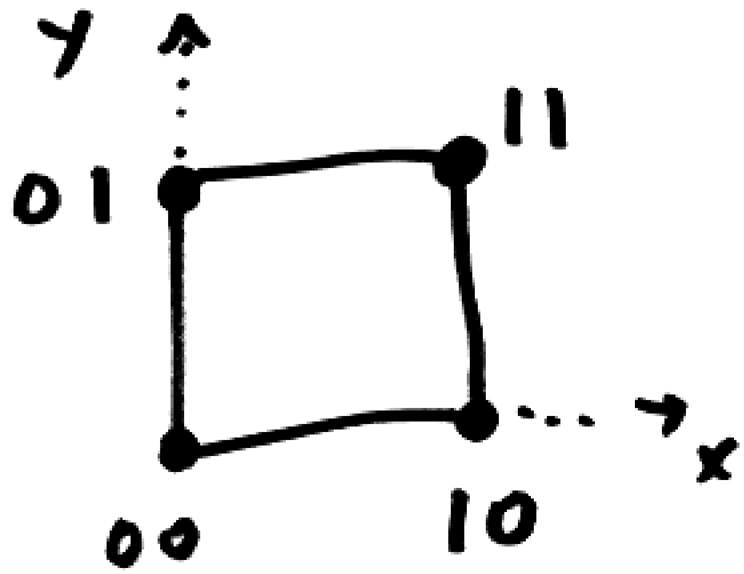
total # of positions

$n = 3$

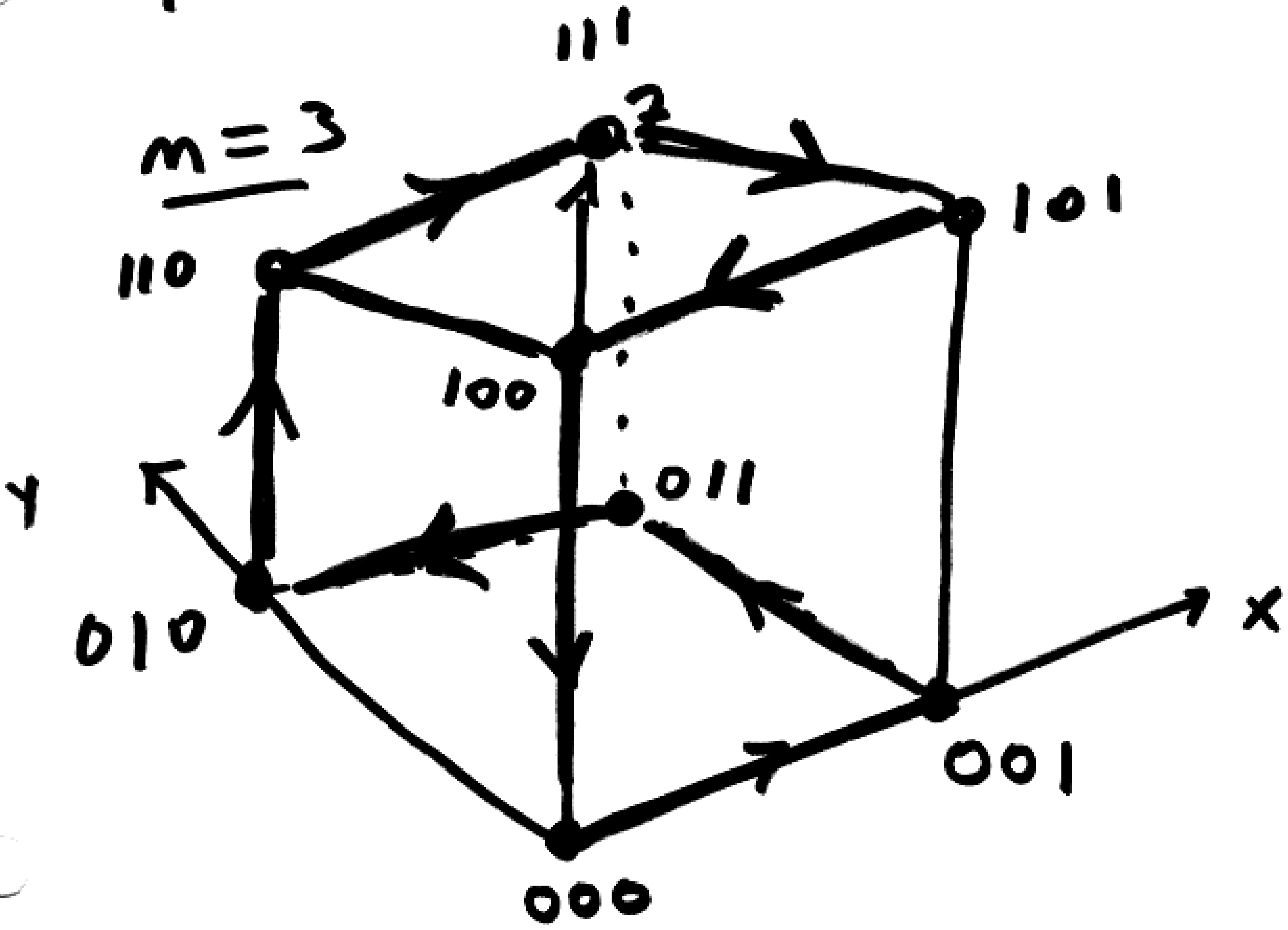
Hamilton $\underline{\text{paths}}$ / $\underline{\text{cycles}}$

Put binary words $\underbrace{\text{on}}_{}$ length n on the vertices of a n-dim'l cube

0	0 0
1	0 1
2	1 0
3	1 1

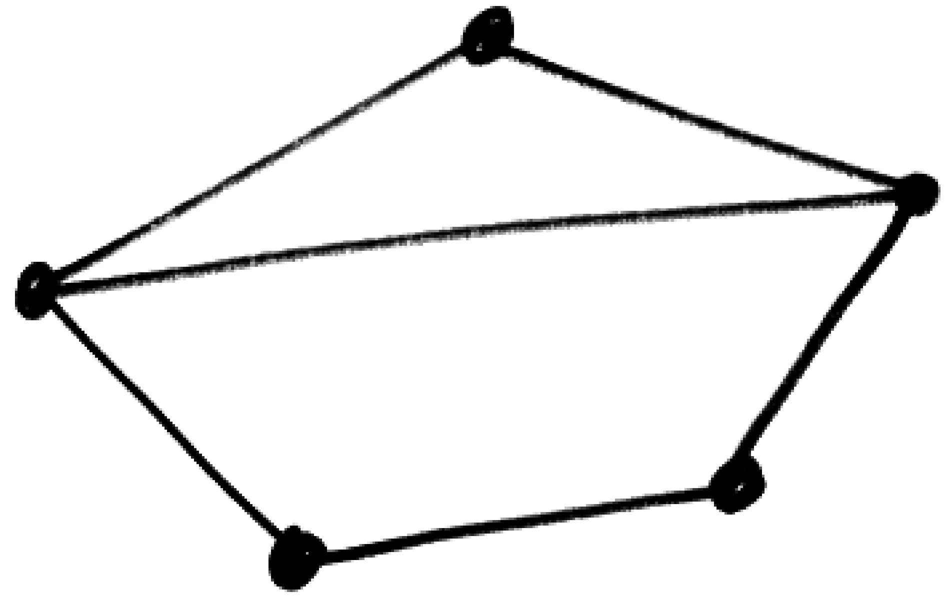


2-dim'l



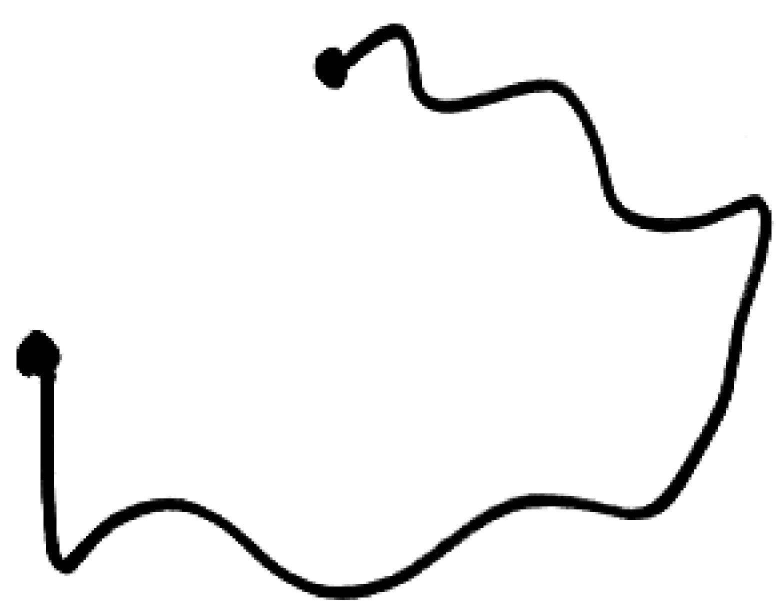
	z y x
0	0 0 0
1	0 0 1
2	0 1 0
3	0 1 1
4	1 0 0
5	1 0 1
6	1 1 0
7	1 1 1

on a graph



Hamiltonian path

moves on edges traversing the graph visiting all vertices without repeating an edge or vertex



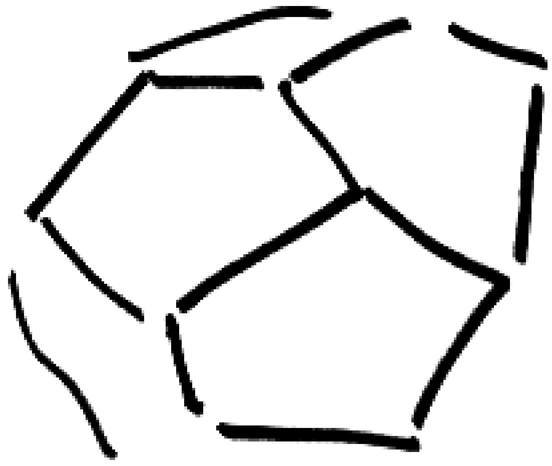
path

circuit or cycle

A gray code on n bits is the same than a Hamiltonian cycle on the n-dimensional cube.

Hamilton

Invented a game called the Icosian. Finding a Hamiltonian path on the vertices of a dodecahedron



Marketed the game. Total flop.

Gray codes $\leftrightarrow$ Hamiltonian cycles on n-dim'l cube

(Reflected gray code is one among many possible ones)

Number of gray codes on n-bits

n	
1	2
2	8
3	96
4	43008
5	58018928640
6	?

Finding Hamiltonian cycles is NP complete

Martin Gardner

(Scientific ~~Am~~ American)

Knotted Doughnuts

Hmwk list all subsets of $\{1,2,\ldots,n\}$ in such a way that two consecutive sets differ by only one element

$\underline{n=3}$ $\quad \phi$

$\{1\}$ $\{2\}$ $\{3\}$

$\{1,2\}$ $\{1,3\}$, $\{2,3\}$

$\{1,2,3\}$

We could do a ternary Gray code

digits: 0, 1, 2

Reflect ternary gray code.

Knuth's vol 4

- in his website

Loony Loop

m:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\rho(m)$:	1	2	1	3	1	2	1	4	1	2	1	3	1	2	1	5

$\rho(m) =$ ruler function

$$\rho(m) = V_2(m) + 1$$

= # digits in binary from right to left to reach the first 1

$\rho(m) =$ # bits that change in binary $m-1 \to m$

$=$ bit that changes in Gray code $m-1 \to m$

$=$ disk # that needs to moved for the optimal solution to the Hanoi towers puzzle.

Recursive way to construct the list of values of ρ's.

1	1
2	121
3	1213121
4	121312141213121

Binary $\rightarrow$ Gray

$(b_{n-1} \cdots b_1 b_0)_2 \longrightarrow (c_{n-1} \cdots c_1 c_0)$

$$c_j \equiv b_j + b_{j+1} \mod 2$$

<u>Basic Idea</u>

$g(m) =$ number whose binary expansion is the Gray code for m.

<u>E.g.</u> $m = 6$ gray code for 6 is 0101

is the binary for 5

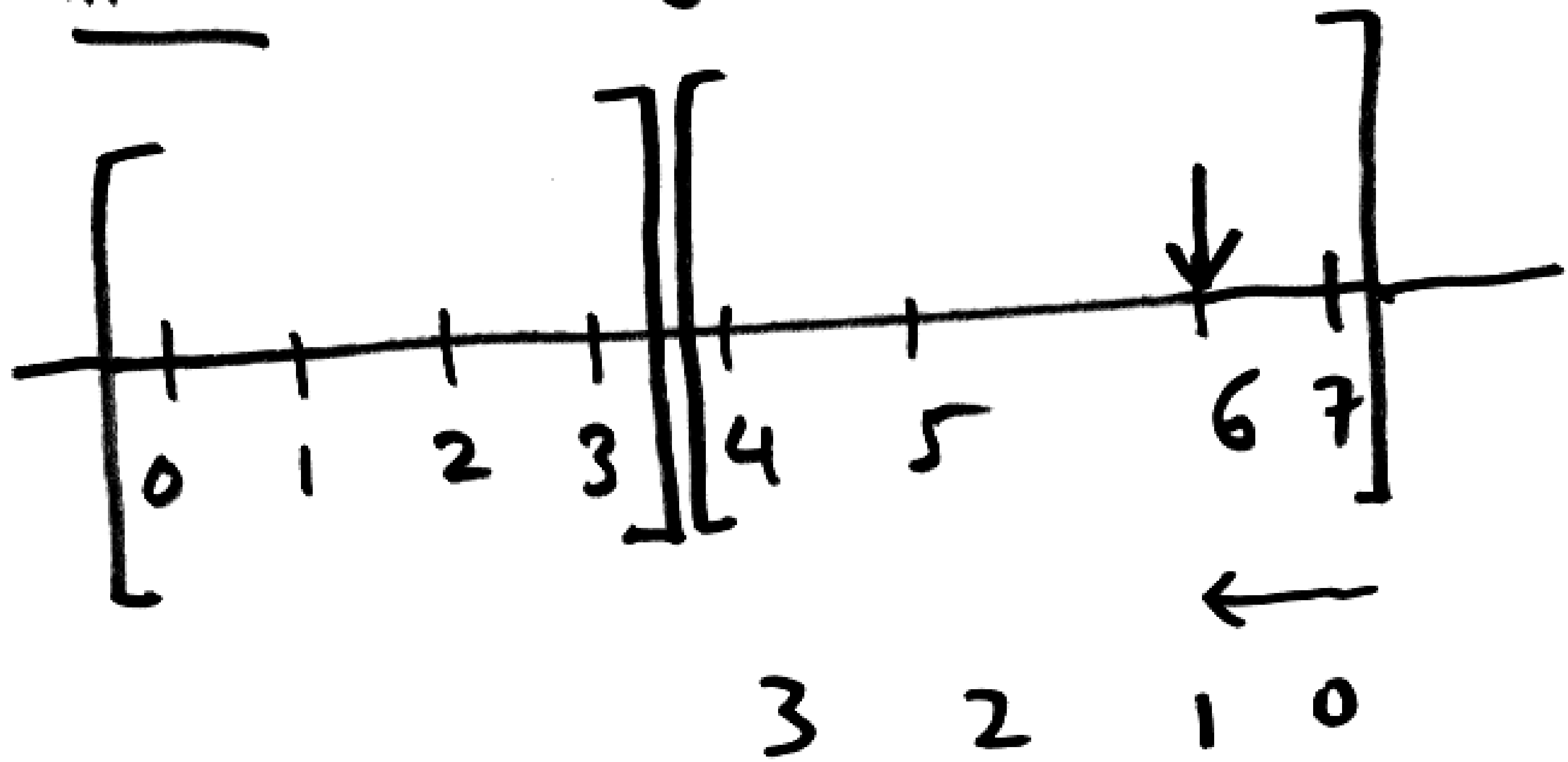
$g(6) = 5$

Claim

$$m = 2^n + k$$

$$0 \le k < 2^n$$

m = 6 $6 = 2^2 + 2$



gray code for 6

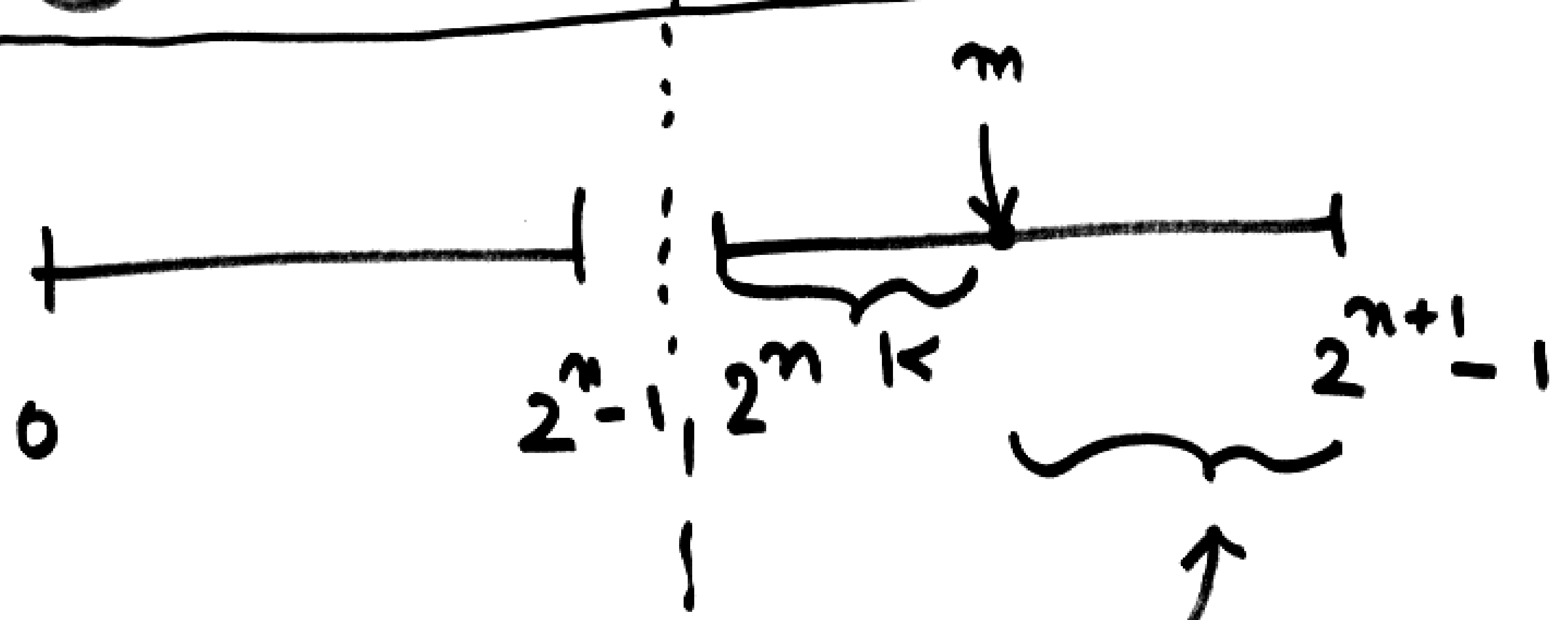
is $1\,\underbrace{01}_{\text{gray code for 1}}$

$$g(6) = 2^2 + g(1)$$

$$m = 2^n + K$$

$$0 \le K < 2^n$$

$$\boxed{g(m) = 2^n + g(2^n - 1 - K)}$$



$$2^n - 1 - K$$

$$K \longleftrightarrow 2^n - 1 - K$$

in binary

$n = 3$

$K = 6$

6 $\qquad$ $2^3 - 1 - 6 = 1$

$(110)_2$ $\qquad$ $(001)_2$

$$\boxed{c_j \equiv b_j + b_{j+1} \mod 2}$$

is a consequence of our boxed formula.

$$\boxed{g(m) = 2^n + g(2^n - 1 - K)}$$

binary bits are flipped from those of K

$m = 2^n + K$

m in binary $(\underset{n-1}{1} \underline{\hspace{3em}})_2$

binary code for K

Proof is by induction on n

Gray $\longrightarrow$ Binary

$$b_j = c_j + c_{j+1} + \cdots \mod 2$$

$$c_j \equiv b_j + b_{j+1} \mod 2$$

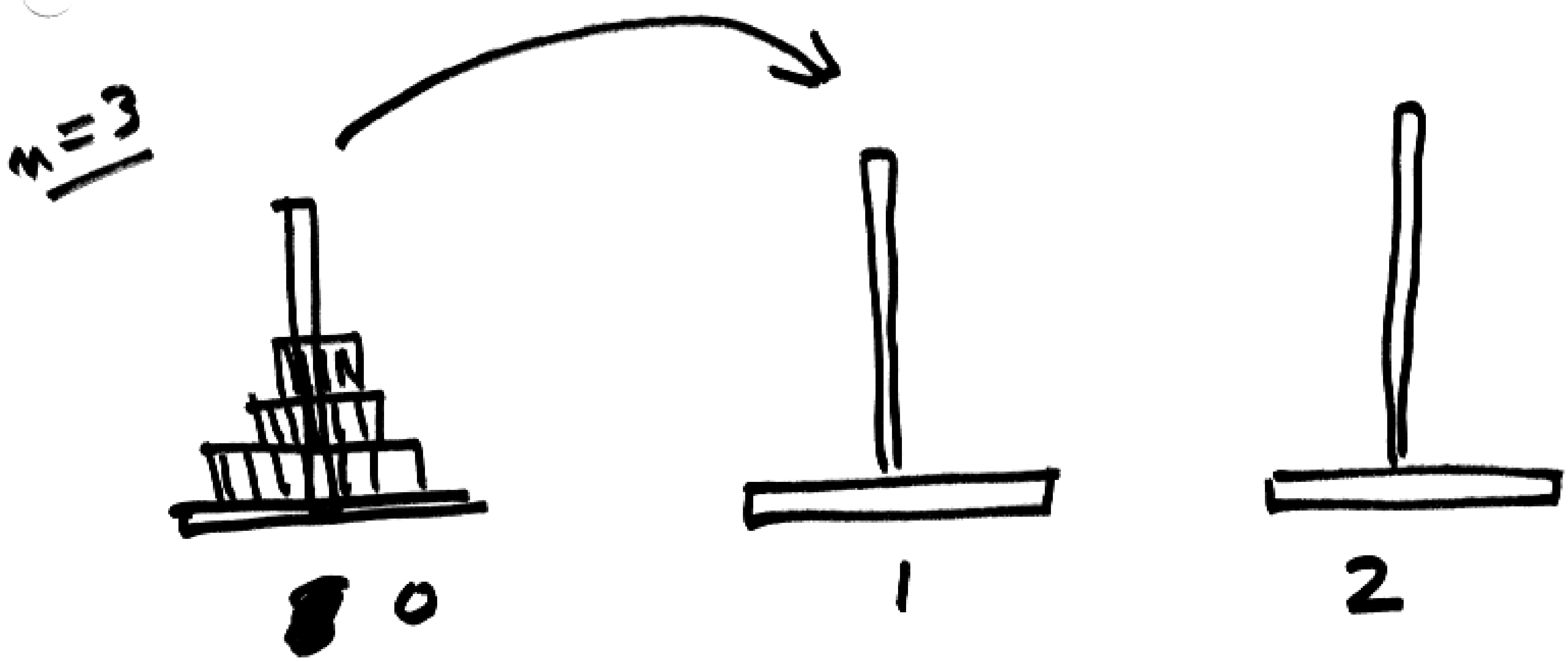
$$c_{n-1} = b_{n-1}$$

$$c_{n-2} = b_{n-1} + b_{n-2}$$

$$b_{n-2} = c_{n-2} + b_{n-1}$$

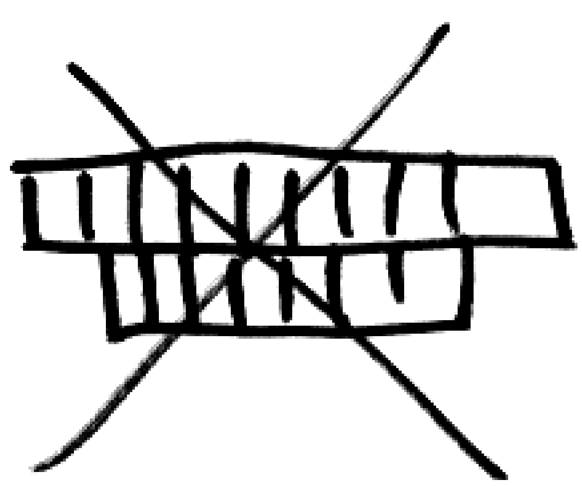
$$b_{n-3} = c_{n-3} + b_{n-2}$$

Hanoi Towers



n-disks

Rule: smaller disks should be always on top of biggerones.



Unique optimal solution

$\text{hanoi}(n, S, D, A)$

source (S), destination (D), auxiliary (A)

$\underline{n>0}$

$\text{hanoi}(n, S, D, A) =$
if $n > 0$
$\quad \text{hanoi}(n-1, S, A, D)$

$\quad$ move disk n from S to D.

$\quad \text{hanoi}(n-1, A, D, S)$

Recursive procedure.

How many steps?

h_n say

$$\begin{cases} h_1 = 1 \\ h_n = 2h_{n-1} + 1 \end{cases}$$

closed formula

$$h_n = 2^n - 1$$

$h_1 = 1 \quad h_2 = 2 \times 1 + 1 = 3$

$h_3 = 2 \times 3 + 1 = 7$

$h_4 = 2 \times 7 + 1 = 15 \ldots$

<u>Proof</u> $\quad n = 1$ ✓

$$h_n = 2 \times (2^{n-1} - 1) + 1$$
$$= 2^n - 2 + 1 = 2^n - 1 \quad \square$$

$n = 4$ pegs 4

.	.	2
1234 .	234 1	34 1

12	12	
34 .	4 3	

2	.	.
14 3	14 23	4 123

4	14	14	
. 123	. 23	2 3	

4	34	34	234	1234
12 3	12 .	2 1	. 1	. .

Disk moved in step m
is the $\rho(m)$ (ruler function)

1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

$$\underbrace{\qquad\qquad}_{n-1} \quad n \quad \underbrace{\qquad\qquad}_{n-1}$$

Exactly the recursive way
to construct the ruler function

1
1 2 1
1 2 1 3 1 2 1
1 2 1 3 1 2 1 4 1 2 1 3 1 2 1

optimal Solution to Hanoi $\leftrightarrow$ Gray
code

position $\longleftrightarrow$ gray code word
1 for i if
disk i was
moved and odd...

In optimal solution disk 1 moves counterclockwise

even disks

---

direction of move of disks depends only on its parity

to solve optimally we know disk n moves only once so this determines the direction of all disks of same parity as n and hence all others as well

disks of same parity as n go

opposite parity go

Puzzles

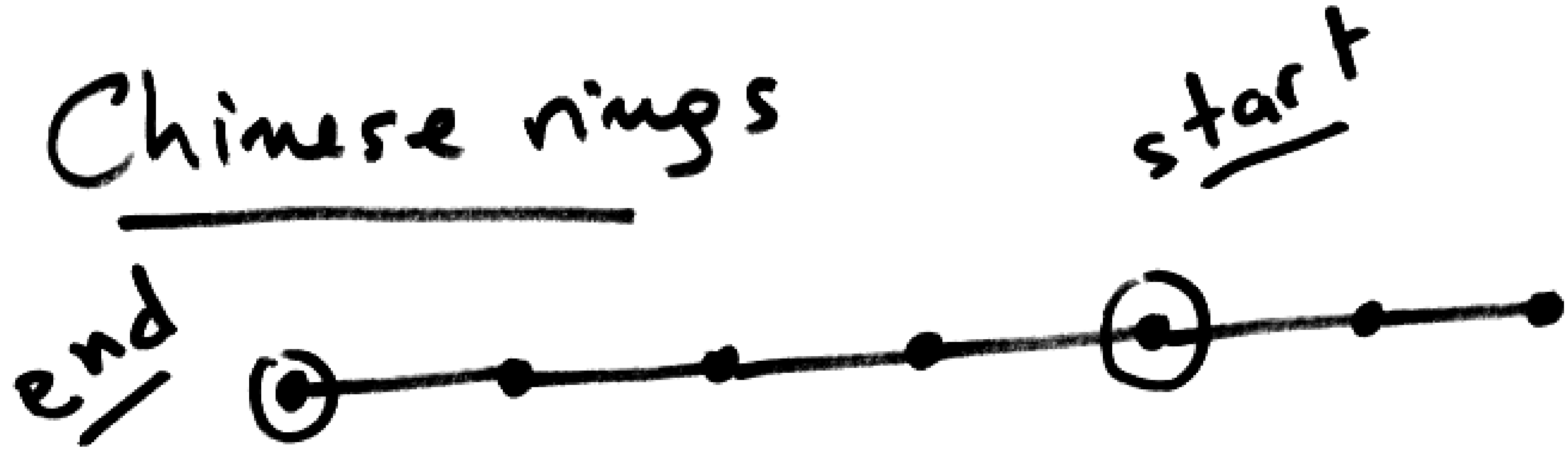
- Positions
- moves

graph

- vertices
- edges.

Goal: start position
↓
end position

Chinese rings



Questions

- How many positions?
- How many steps to solve?
- Orientation?

• Code positions in a useful way. (7)

Hanoi Towers

label disks $1, 2, \ldots, n$

code a position

$(t_1, \ldots, t_n)$

$t_i = 0, 1$ or 2

$=$ label of peg where disk i is

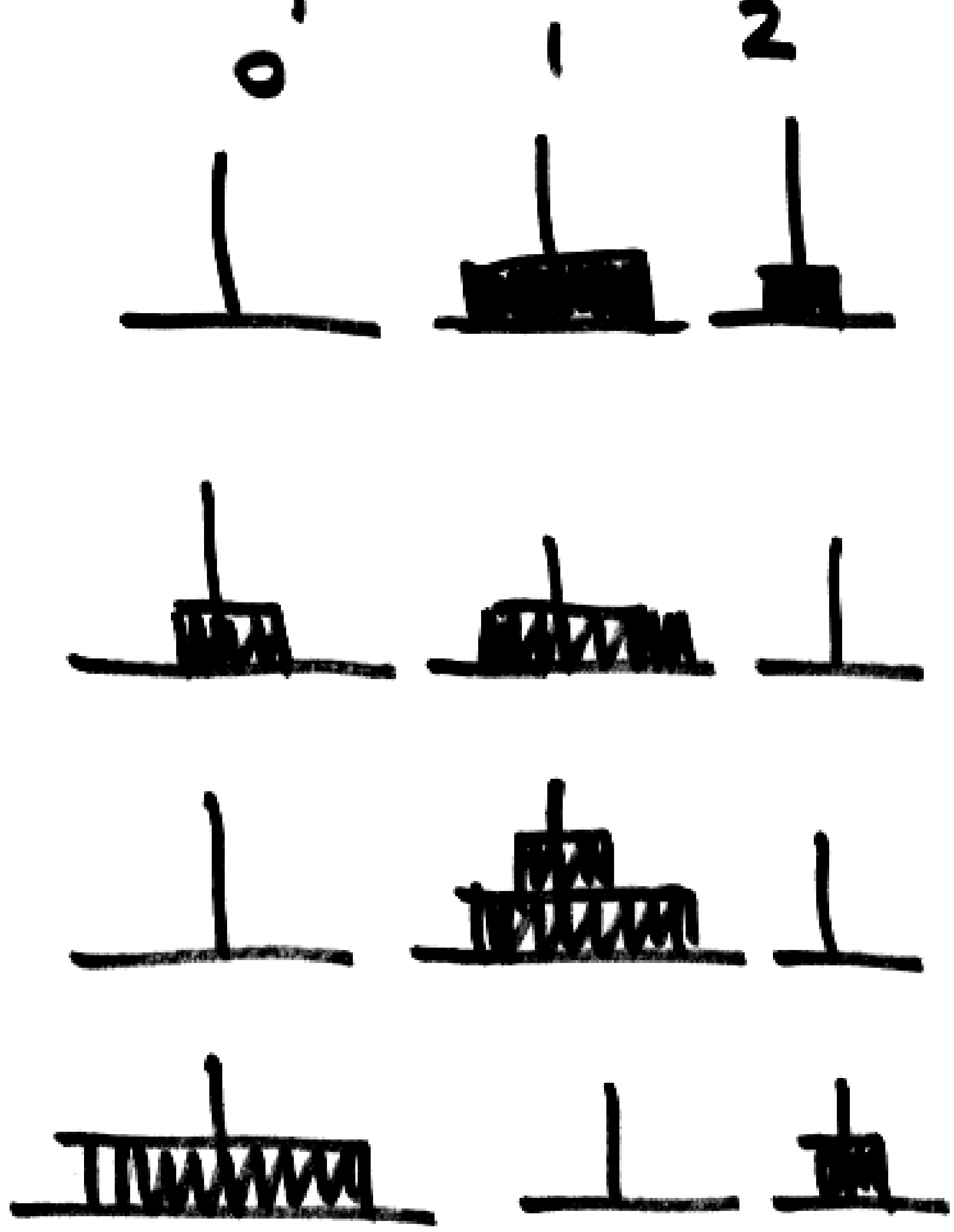
E.g.

(120)

Graph?

How many positions?

3^n total positions.



$n=2$

(21)

(01)

(11)

(20)

Recall: graph of a $\boxed{V}$ puzzle

1

vertices $\longleftrightarrow$ positions

edges $\longleftrightarrow$ moves

graph is <u>connected</u>

e.g.

dis<u>connected</u>

Hanoi Towers has a connected graph.

In the 15-puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	•

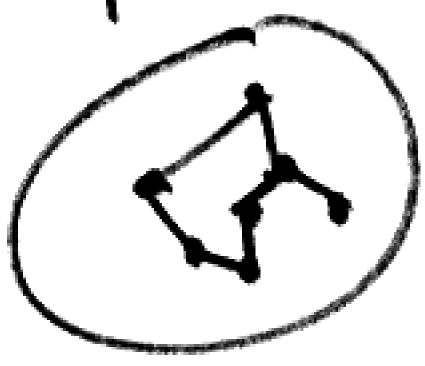
Sam LLoyd.

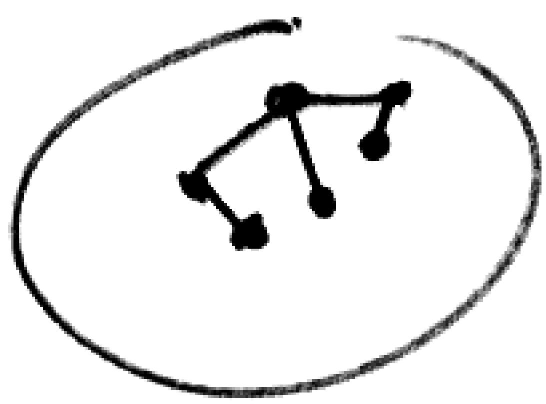
14-15 puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	•

cannot be done!

Graph





Pascal's triangle

Binomial coefficients.

$\binom{n}{k}$ $\quad k = 0, 1, \ldots, n$

$=$ # subsets of n things of size k.

E.g. $\underline{n=3}$ $\quad \{1,2,3\}$

$\underline{k=2}$ $\quad \{1,2\} \quad \{1,3\} \quad \{2,3\}$

$$\binom{3}{2} = 3$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$=$ # permutations of n things

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

combinatorial proof

Permutation Puzzles

X a set

U moves $\qquad X \circlearrowleft^{U}$

a <u>move</u> is a permutation of the set X.

$X = \{1, 2, 3\}$

$$u = \begin{array}{c} 1 \rightarrow 2 \\ \nwarrow \;\; \swarrow \\ 3 \end{array} \qquad \begin{array}{c} 1\;2\;3 \\ \downarrow u \\ 2\;3\;1 \end{array}$$

- Moves are reversible
- Can always be done.

$$\begin{array}{c} 1\ 2\ 3 \\ \downarrow u \\ 2\ 3\ 1 \end{array} \qquad \begin{array}{c} 3\ 1\ 2 \\ \downarrow u \\ 1\ 2\ 3 \end{array}$$

X set

$S(X)$ = all permutations of X

forms a group

- $\sigma, \tau \in S(X)$

$\tau \cdot \sigma$ another permutation

τ: then do this

σ: do this first

$$\sigma : X \longrightarrow X$$
$$x \longmapsto \sigma(x)$$

We can "multiply" permutations

$$\sigma : \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{matrix}$$

$$\tau : \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{matrix}$$

$$\tau \cdot \sigma : \begin{matrix} & 1 & 2 & 3 & 4 \\ \sigma \downarrow & 2 & 1 & 4 & 3 \\ \tau \downarrow & 3 & 2 & 1 & 4 \end{matrix}$$

- $\sigma \in S(X)$ is <u>reversible</u>

$\sigma^{-1} \cdot \sigma = 1$ identity in $S(X)$

$\sigma \cdot \sigma^{-1} = 1$

operation $\cdot$ in $S(X)$

- identity 1
- elements have inverses
- operation is associative.

$$\boxed{\sigma_1 \cdot (\sigma_2 \cdot \sigma_3) = (\sigma_1 \cdot \sigma_2) \sigma_3}$$

Associativity.

- Commutativity

$$\sigma \cdot \tau \neq \tau \cdot \sigma$$

In general

Permutations

Finite set X

$$\sigma : X \longrightarrow X \quad \text{(bijections)}$$

permutations

$$X = \{1, 2, \dots, n\}$$

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

Permutations act on <u>labels</u> not on position.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$$

Can compose permutations
product

$$X \xrightarrow{\tau} X \xrightarrow{\sigma} X$$
$$i \longmapsto \tau(i) \longmapsto \sigma(\tau(i))$$

$\sigma \circ \tau$

↖ first τ then σ

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$$

Permutations with this product form a group.

- Product is associative

$$\sigma \cdot (\tau \cdot \eta) = (\sigma \cdot \tau) \cdot \eta$$

- Permutations have inverses

$$\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1$$

- 1 identity permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

Much better notation

cycle decomposition

$$\sigma = (1\,2)(3\,4\,5)$$

$$\tau = (1)(2)(3)(4\,5)$$

$$= (4\,5)$$

$$\sigma \cdot \tau = (1\,2)(3\,4\,5)(4\,5)$$

$$= (1\,2)(3\,4)$$

Cycle decomposition

$\sigma \in S_n$ (= group of permutations of n things)

can be written as a product of disjoint cycles.

In essentially a unique way.

$$\sigma = (12)(345) = (12)(453) \\ (534)$$

$$= (345)(12)$$

disjoint cycles commute

2 <u>not</u> disjoint cycles typically do <u>not</u> commute.

$$(12)(23) = (123)$$
$$(23)(12) = (132)$$

$$(1\,2\,3\,4\,5)$$

Convention on cycles

$$1 \to 2 \to 3 \to 4 \to 5 \to 1$$

Permutation Puzzles

- X finite set of objects
- U moves (permutations of X)

Moves are reversible and don't depend on the status of X.

Puzzle's goal

is to take scrambled version of X to a desired form by a sequence of moves.

$\{u_1, u_2, \dots, u_m\} = U$

σ "scrambling" permutation

To solve

$\sigma^{-1} = u_{i_1} u_{i_2} \cdots\cdots u_{i_N}$

(Assume U contains the inverses)

Better: assume moves are of finite order (X is finite in fact). Then this is not necessary

Examples

1) Bubbling algorithm for sorting.

$U = \{ \text{transposition} \}$

<u>transposition</u> : $(i\,j)$

(2-cycle) $i \longleftrightarrow j$

swaps i with j

[redacted] [redacted]

$$\sigma^{-1} : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

$$(1\,2\,3\,4\,5) = (1\,5)(1\,4)(1\,3)(1\,2)$$

1 2 3 4 5
2 1 3 4 5
2 3 1 4 5
2 3 4 1 5
2 3 4 5 1

Any cycle is a product of transpositions.

But any permutation is a product of (disjoint) cycles. Hence any permutation is a product of transpositions.

Writing σ as a product of u's in U need not be unique.

$$(1\,2\,3\,4\,5) = (1\,5)(1\,4)(1\,3)(1\,2)$$

$$=$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{array} \qquad\qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 & 1 \\ 3 & 2 & 4 & 5 & 1 \\ 2 & 3 & 4 & 5 & 1 \end{array} \uparrow$$

$$= (2\,3)(3\,4)(4\,5)(5\,1)$$

Permutations obtained as products of moves in U (recall that U contains all inverses of U)

$u_{i_1} \cdots u_{i_m}$

form a subgroup $= H < G = S_n$

In general H need not be the set of all permutations.

$$U = \{(12)\} \subseteq S_5$$

$$\langle U \rangle = H = \{1, (12)\} \subsetneq S_5$$

GAP

For a permutation puzzle $|H| =:$ size of H is the number of possible different positions (states).

---

one generator (one move) only gives a cyclic group,

$$u = (12)(345)$$

$$u^{-1} = (12)(543)$$

$$\{(12)(345), (12)(543)\}$$

12

$H = \langle u \rangle$

Explain by using cyclic permutation as rotation. Then is quite clear what you can possibly do!

$u = (12)(345)$

$u^2 = (12)(345)(12)(345)$

$= (12)^2(345)^2$

$= (354)$

$u^3 = (12)^3(345)^3$

$= (12)$

$u^4 = (12)^4(345)^4$

$= (345)$

$u^{-1} = u^5 = (12)^5(345)^5$

$= (12)(354)$

$u^6 = (12)^6(345)^6 = 1$

$u^7 = u$

Any product

$$u^{n_1} u^{n_2} u^{n_3} \cdots$$

$$= u^{n_1 + n_2 + n_3 \cdots}$$

$$= u^r$$

$$r \equiv n_1 + n_2 + \cdots \mod 6$$

$0 \leq r < 6$

There are many large and complicated groups which are generated with just <u>two</u> moves.

S_n is such an example

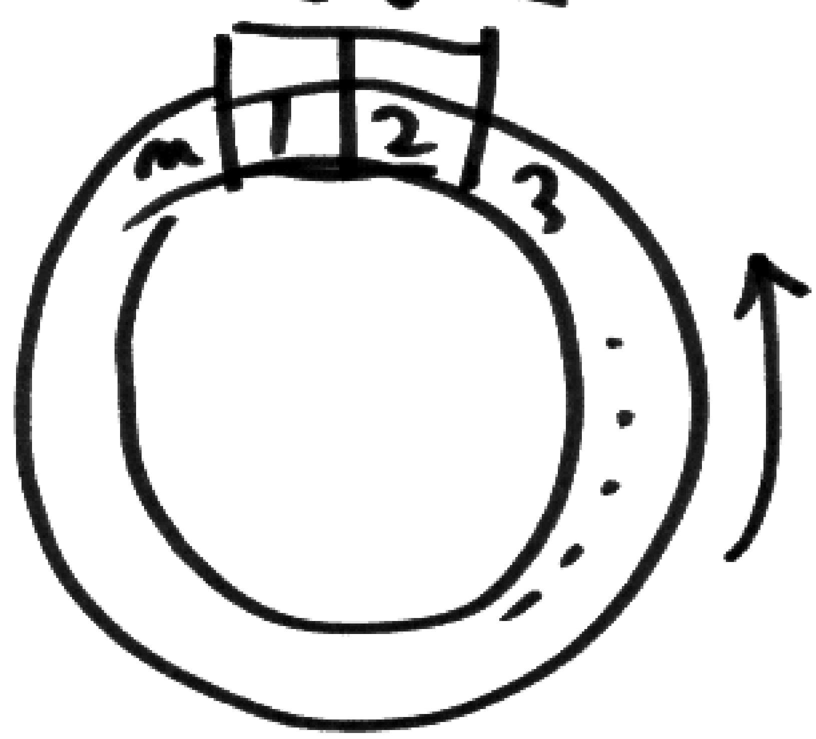
u's $= (1\,2), \; (1\,2\,3 \ldots n)$

[(and their inverses)]

⇝ puzzle

(12)

2 This acts on positions!



$(12\ldots n)$

\# of possible states is $n!$

$$\boxed{(23) = (123\ldots n)(12)(123\ldots n)^{-1}}$$

→

(34) :

(45) :

X = finite set

U = moves
(permutations of X)

Puzzle goal is to write

$\sigma \in H = \langle U \rangle \subseteq S(X)$

as a product

$\sigma^{-1} = u_{i_1} \cdot u_{i_2} \cdot \ldots \cdot u_{i_m}$

with $u_i \in U$

- This is a hard problem.
- No clear algorithm.
- Solution may not be unique.

If we only have one move u

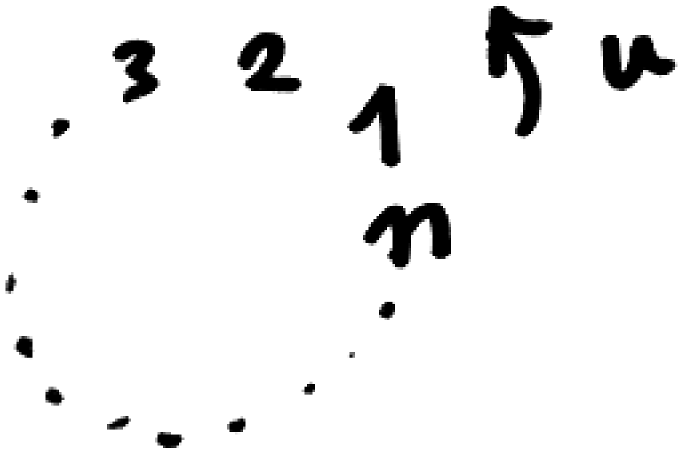
$$1,\ u,\ u^2,\ u^3,\ u^4, \dots$$

For some exponent $r > 0$ we'll have $u^r = 1$

If r the smallest such r then ~~the~~ all the permutations in H are simply

$$1,\ u,\ u^2,\ \dots,\ u^{r-1}$$

H is <u>cyclic</u> group



With 2 moves a lot of things can happen! 3

Homework:

<u>Fact</u>

$(1\,2), \quad (1\,2 \dots n)$

they generate S_n.

Infact if u_1, u_2 with

$u_1^2 = 1 \quad u_2^3 = 1$

Abstractly, the group generated by these two moves can be infinite.

$H \ni u_1^{-1} u_2 u_1 u_2^2 u_1 u_2^{-1} \dots$

$(u_1 u_2)^6 \neq 1 \quad u_1, u_2$ need not commute

$$u_1 u_2 u_1 u_2 = (u_1 u_2)^2$$

$$\neq u_1^2 u_2^2 \text{ possibly}$$

Lack of commutation

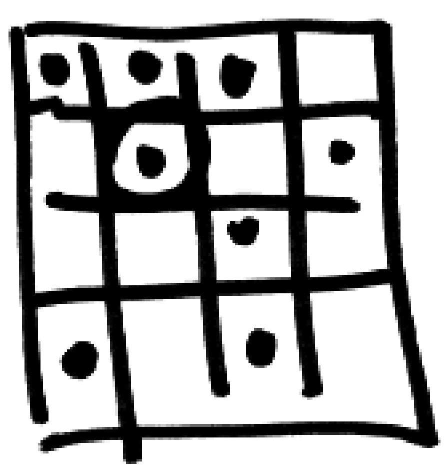
$(\sigma\tau = \tau\sigma)$

can generates great complexity.

Lights out

Merlin squares

⋮



moves: press a button which will switch the neighboring lights

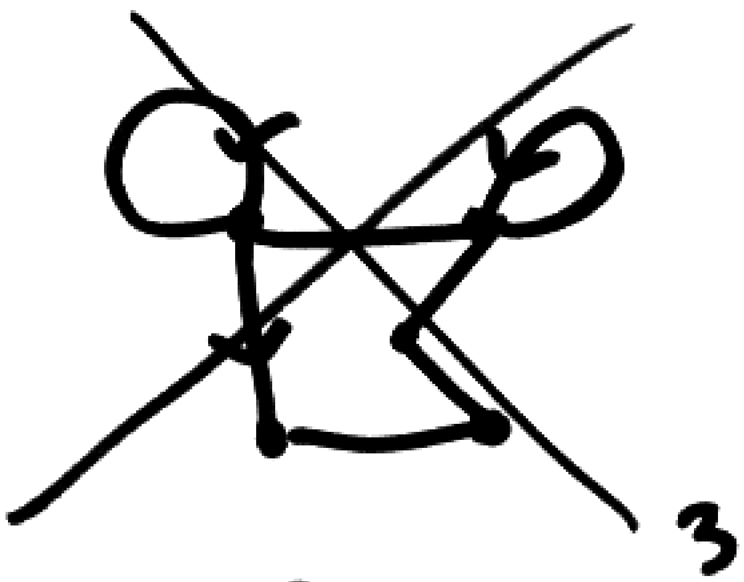
Here the group is abelian (commutative) (Abel)

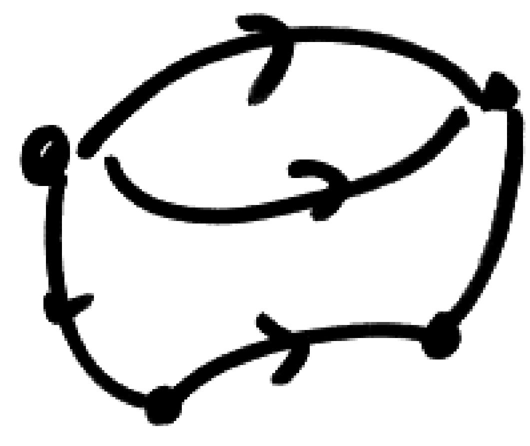
15 - puzzle

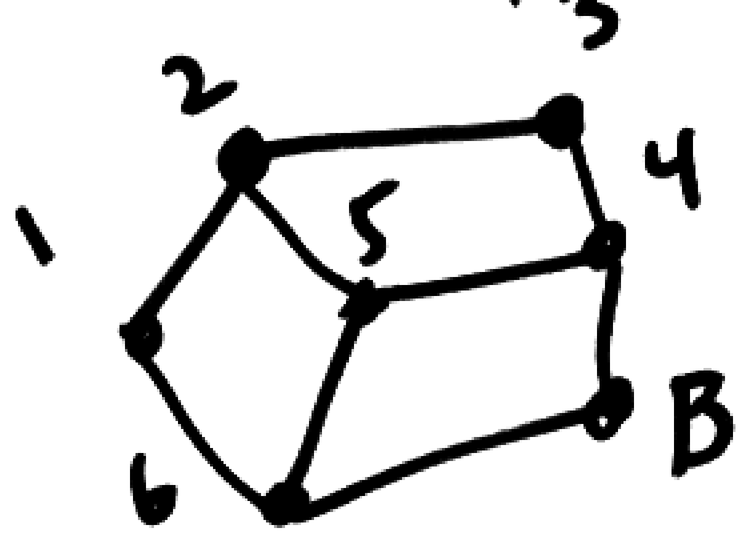
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	•

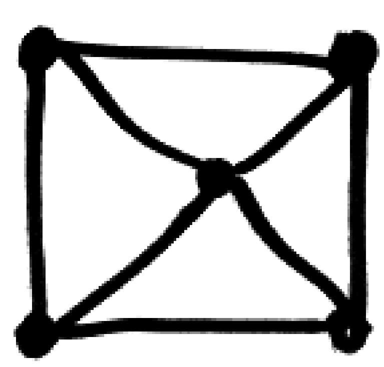
square slide

We can play this puzzle in any simple graph (no looops, no double edges)

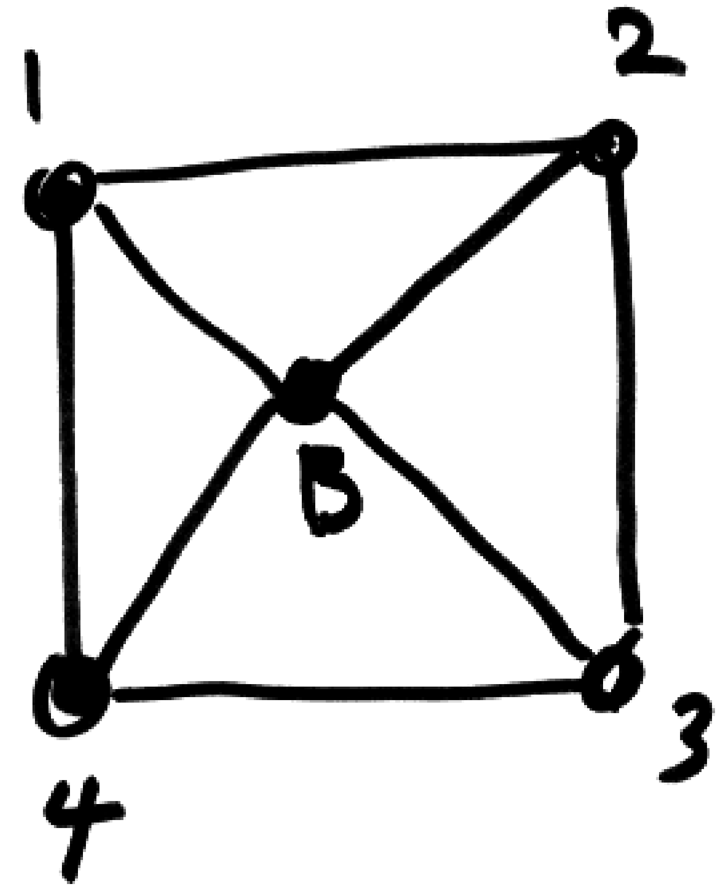






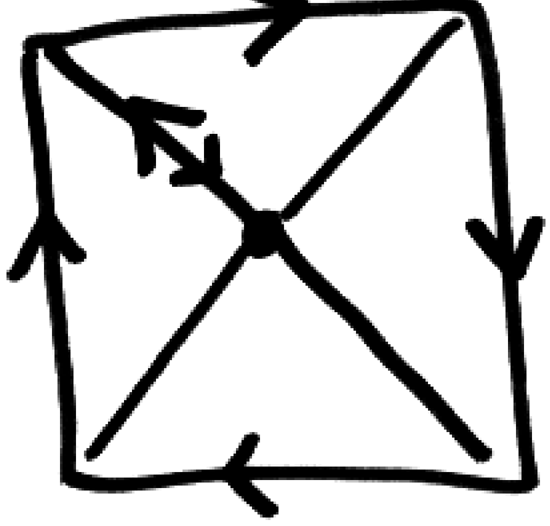


THEOREM. R. Wilson '74 6



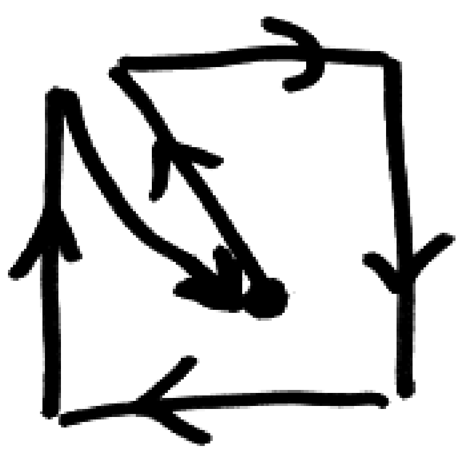
Closed paths

Blank moves on the graph comes to its original position.



Each closed path gives a permutation of the 4 numbers.
We'll encode this permutation as follows.

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \quad \xrightarrow{\gamma_0} \quad \begin{matrix} 1 & 3 \\ 2 & 4 \end{matrix}$$

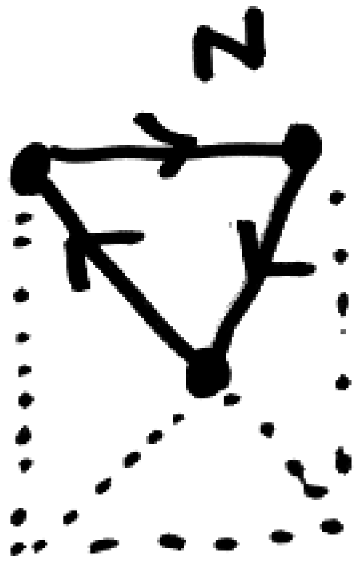


$$\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{matrix}$$

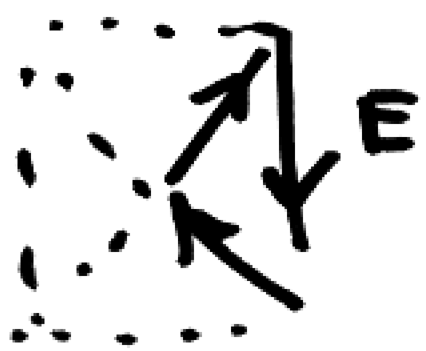
$$\gamma_0 \longmapsto \sigma = (243)$$

All possible closed paths will give us a subgroup H of S_4.

Basic moves



$\longleftrightarrow$ (12)

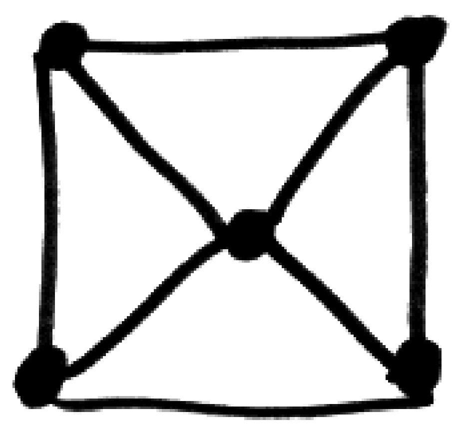


$\longleftrightarrow$ (23)

$\longleftrightarrow$ (34)

$\longleftrightarrow$ (14)

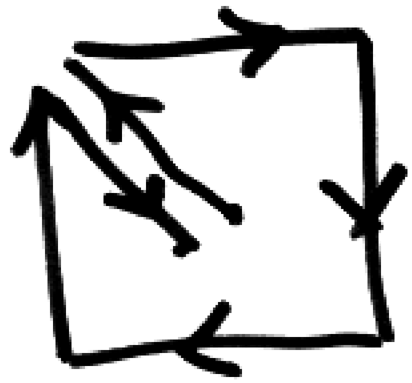
All permutations possible
are generated by these four.



<u>Note</u>: $\gamma : B \ldots\ldots PQP \ldots B$

simplify

$B \ldots\ldots P \ldots\ldots B$



$$B\;1\;\underbrace{2\;B\;2}_{2}\;\underbrace{3\;B\;3}_{3}\;\underbrace{4\;B\;4}_{4}\;1\;B$$

$$\gamma_0 = W \cdot S \cdot E \cdot N$$

We may multiply paths
We get a group

This group is called the <u>fundamental group</u> of the graph. Γ

(Topology).

$$\pi_1(\Gamma) \longrightarrow S_4$$

$$\gamma \longmapsto \sigma$$

$$\gamma_2 \cdot \gamma_1 \longmapsto \sigma_2 \cdot \sigma_1$$

Homomorphism

Image of this map are the permutations H achievable in this puzzle.

$U = \{ (12), (23), (34), (14) \}$

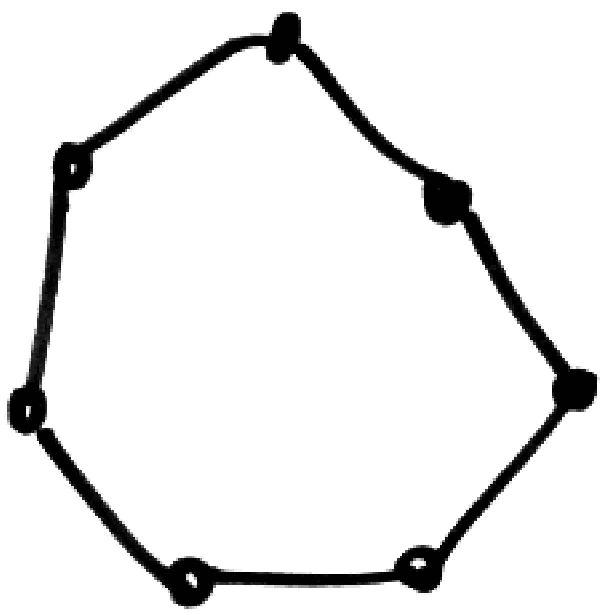
$H = \langle U \rangle$?

In fact $H = S_4$.

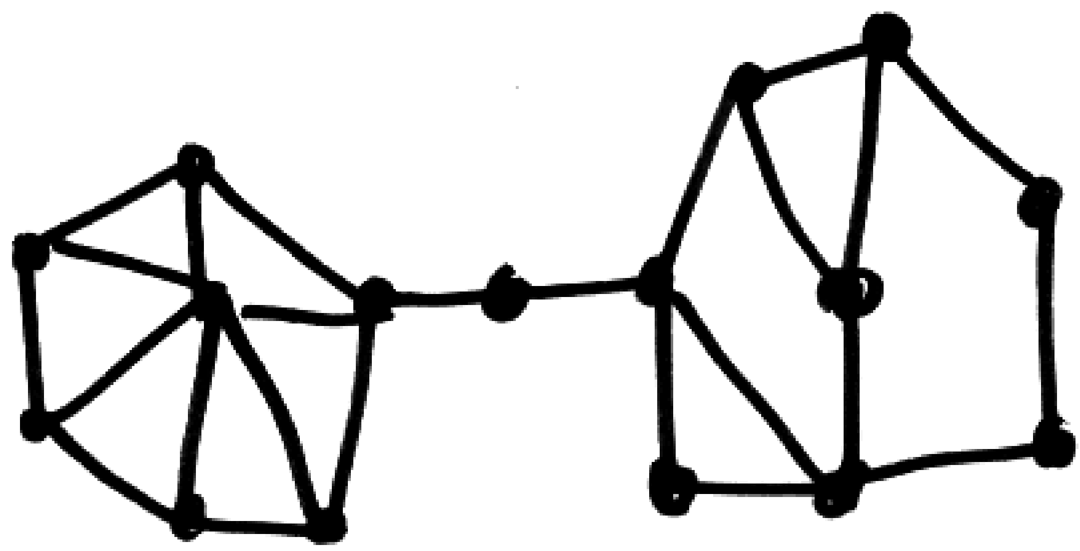
$(1234) = (12)(32)(43)$

Now we know that H contains (12), (1234) and hence all sequences of those which we know generate all of S_4.

THM (Wilson)



cyclic group.



cannot swap from to

Removing one vertex leaves a disconnected graph.
We actually have two separate puzzles.

otherwise:

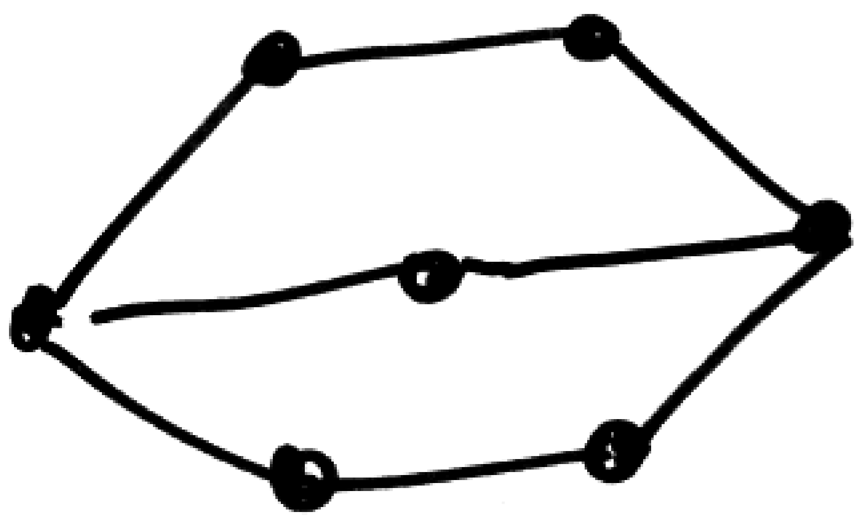
- H is either 1) all of S_n

  or 2) A_n (even permutations)

  $|S_n| = n!$

  $|A_n| = n!/2$

- 3)

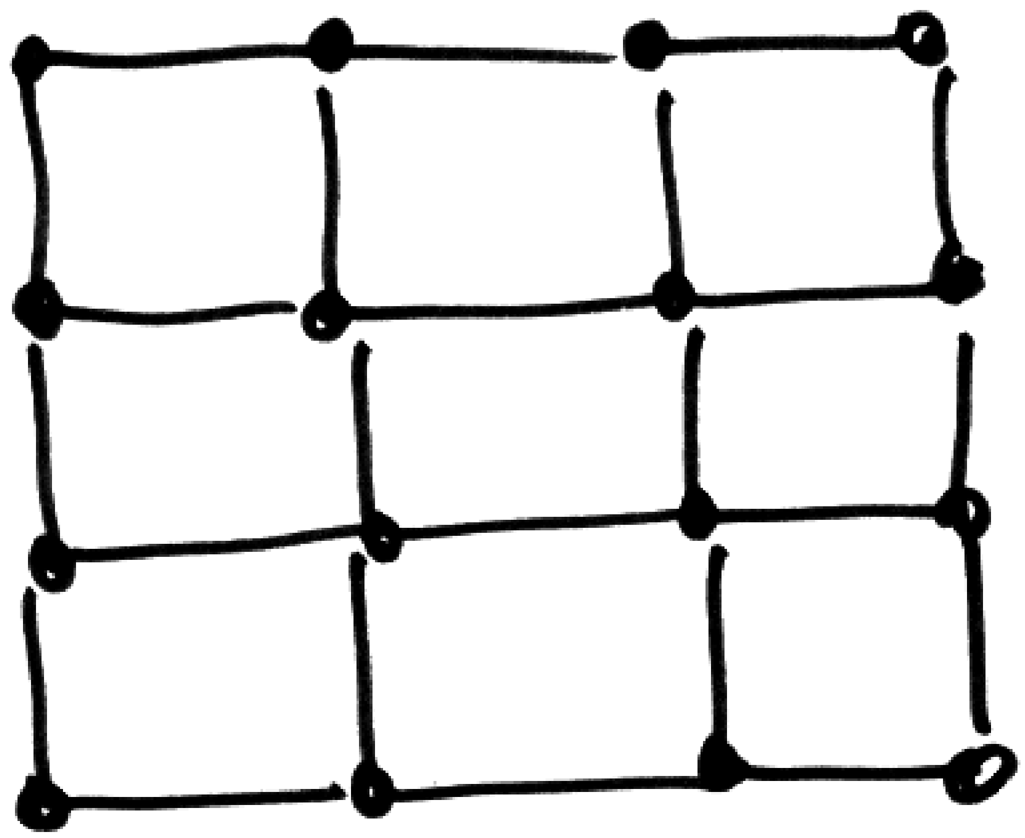


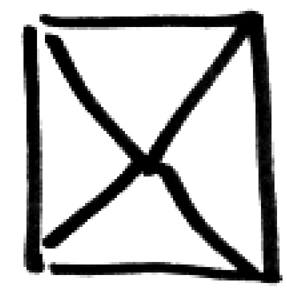
potentially $H = S_6$ size 6!

In fact $|H| = 120$

The graph for the 15-puzzle

Γ:





Wilson's theorem says that

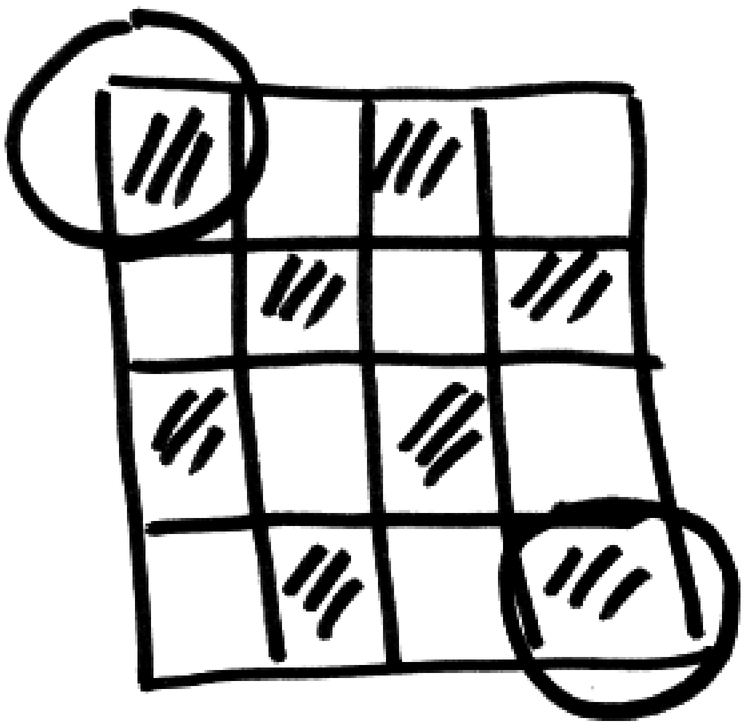
$$H = A_{15}\,, \quad |H| = 15!/2$$

This implies

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

cannot be solved.

Parity



Dominoes place them so that the two corners are left uncovered

Can't be done since tile down covers one and .

For permutations we have a sign function

$$S_n \longrightarrow \{\pm 1\}$$
$$\sigma \longmapsto \mathrm{sgn}(\sigma)$$

$$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$$

homomorphism.

σ <u>even</u> if $\mathrm{sgn}(\sigma) = +1$

σ <u>odd</u> if $\mathrm{sgn}(\sigma) = -1$

<u>permutations</u>

even $\circ$ even = even

even $\circ$ odd = odd

odd $\circ$ even = odd

odd $\circ$ odd = even

What is

sgn(σ) ?

Transposition gets

sgn $= -1$

swapping two numbers is <u>odd</u> permutation.

$$\mathrm{sgn}((i\,j)) = -1$$

$$\mathrm{sgn}((1\,2\,3)) = \mathrm{sgn}((1\,3)) \cdot \mathrm{sgn}((1\,2))$$

$$= (-1)\cdot(-1)$$

$$= +1$$

$$(1\,2\,3) = (1\,3)(1\,2)$$

$$\operatorname{sgn}((1234)) = -1$$

$$(1234) = (14)(13)(12)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$-1 \cdot -1 \cdot -1$$

$$= -1$$

$$\vdots$$

$$\operatorname{sgn}((12\dots r)) = (-1)^{r-1}$$

$$(12\dots r) = \underbrace{(1r)(1\,r{-}1)\dots(12)}_{r-1}$$

$$\operatorname{sgn}((12)(345)) = (-1)\cdot(+1)$$

$$= -1.$$

But σ can be written as a product of transpositions in different ways.

<u>Claim</u> The parity of the number of transpositions needed is always the same.

The sign function is well defined.

$$A_n = \{ \underline{\text{even}} \text{ permutations} \}$$
$$= \{ \sigma \mid \operatorname{sgn}(\sigma) = +1 \}$$

$$\sigma, \tau \in A_n \Rightarrow \sigma \cdot \tau \in A_n$$

<u>subgroup</u> of S_n

$\sigma \in A_n$

$\Rightarrow \sigma^{-1} \in A_n$

$$\operatorname{sgn}(\sigma \cdot \sigma^{-1}) = \operatorname{sgn}(1) = +1$$

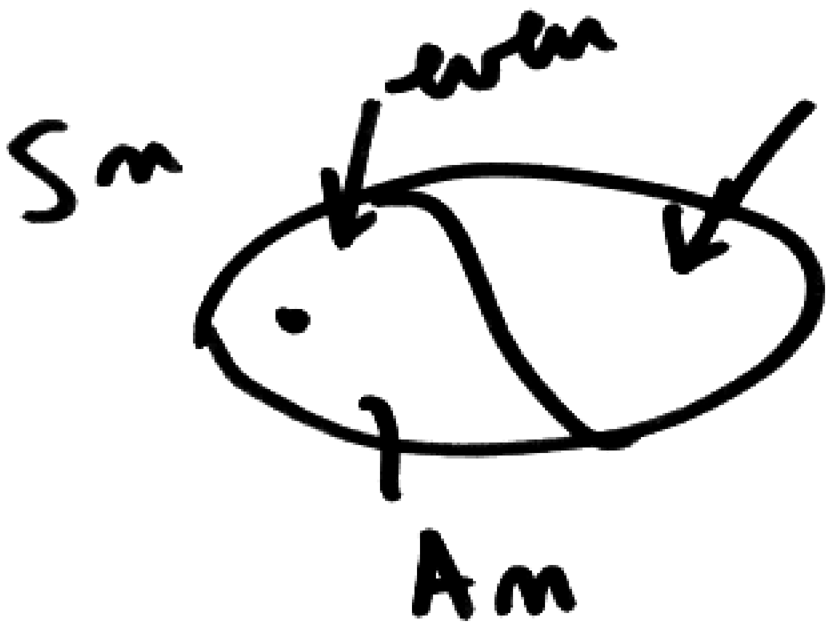
$$\operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\sigma^{-1}) = +1$$

$\downarrow$

$+1 \Longrightarrow +1$

$$|S_n| = n!$$

$$|A_n| = n!/2$$



$\sigma \in S_n$

$\sigma \in A_n$

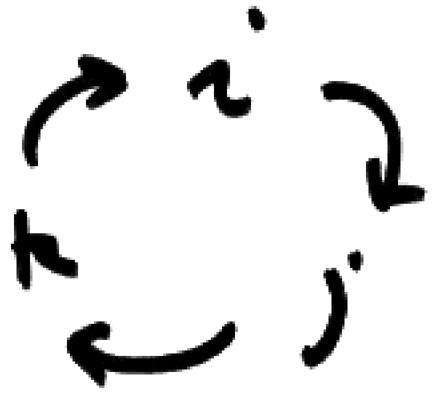
or

$\sigma = (1\,2)\tau$

$\tau \in A_n$

3-cycles

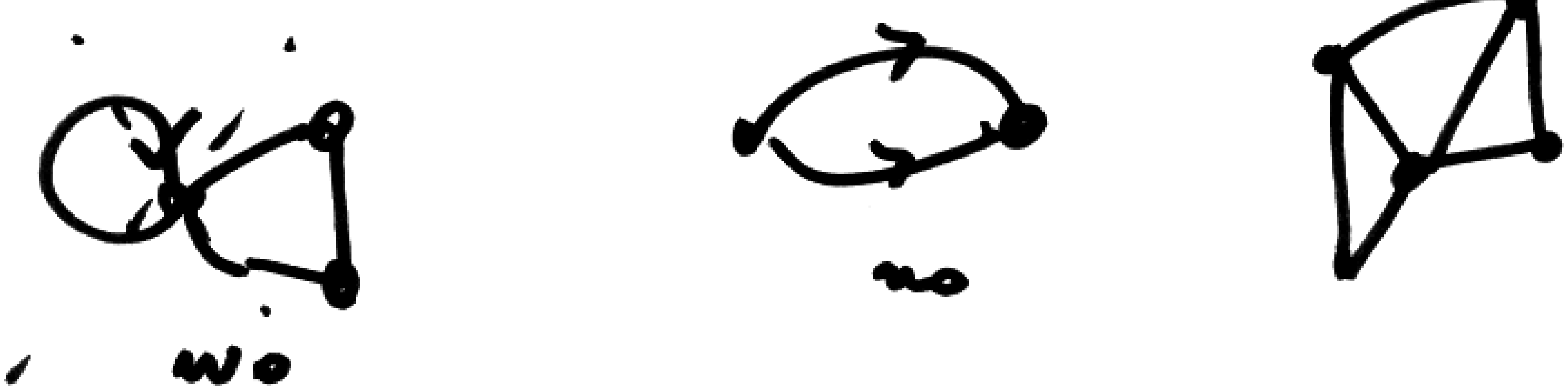
$(i\ j\ k)$



quintessential <u>even</u> permutations.

<u>THM</u> (Wilson '74)

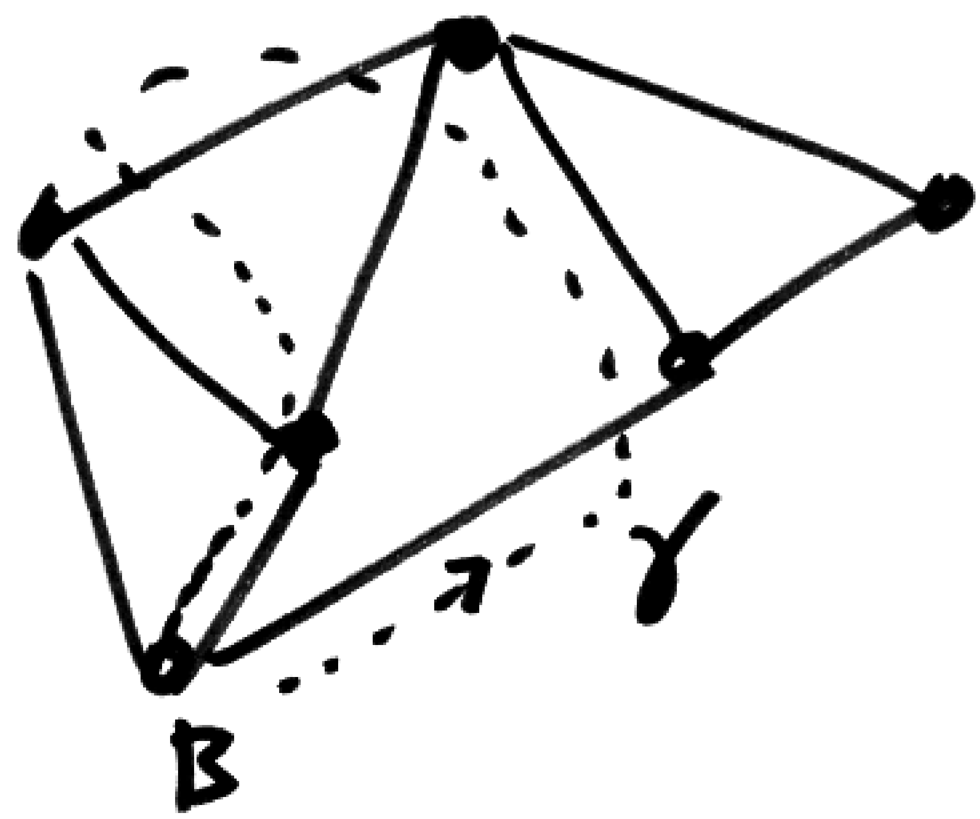
Γ simple graph



$H = \langle U \rangle$

subgroup of permutations obtained by playing this puzzle (with blank

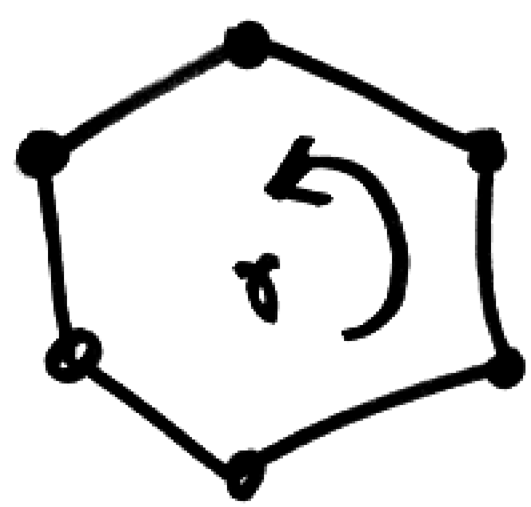
returning to the original (8)
position)



$\gamma \longmapsto \sigma$ permutation of the numbering of vertices

H is one of the following.

1)

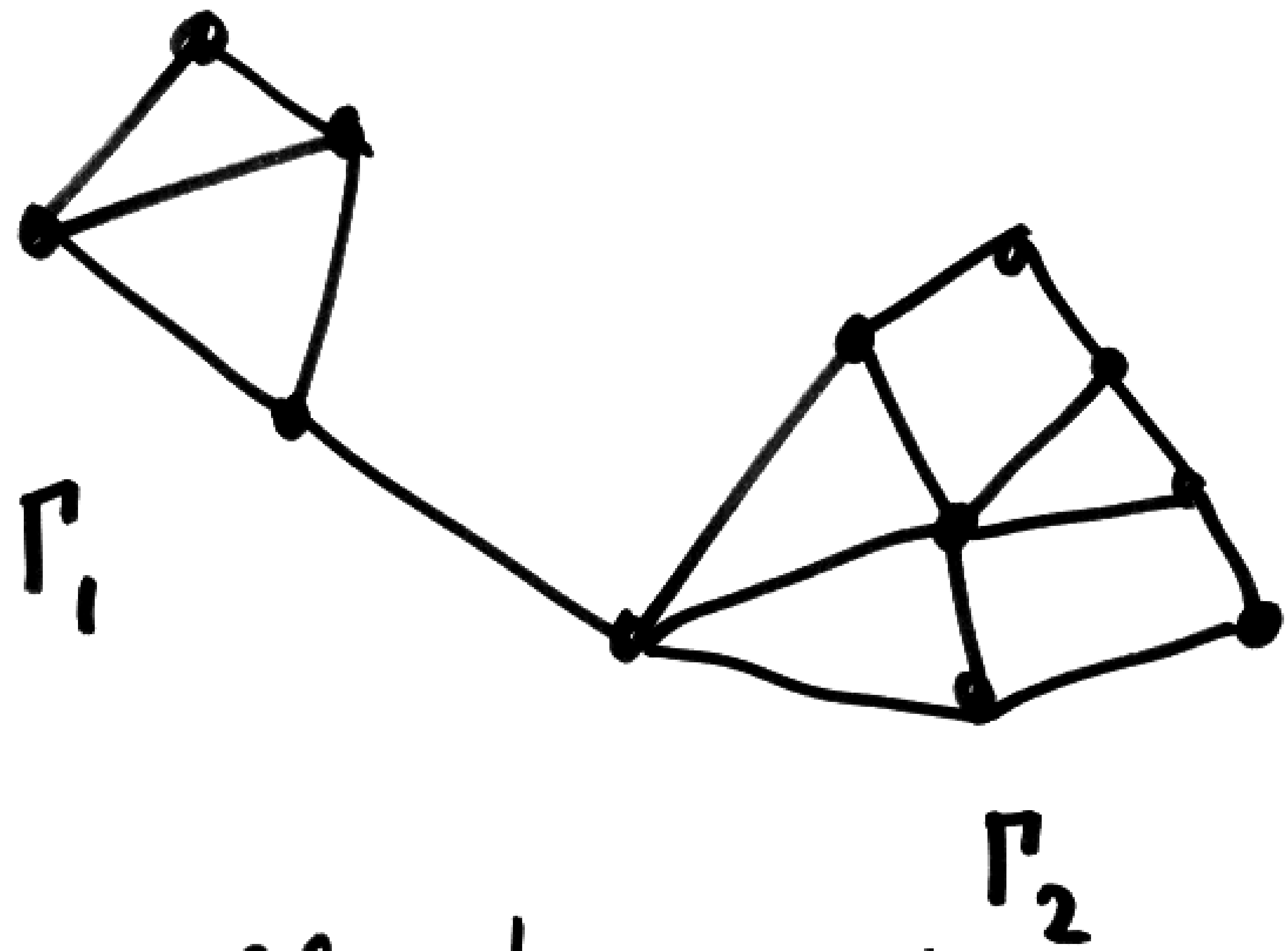


Γ

$\gamma \longmapsto u$

$H = \langle u \rangle$ cyclic group

2)



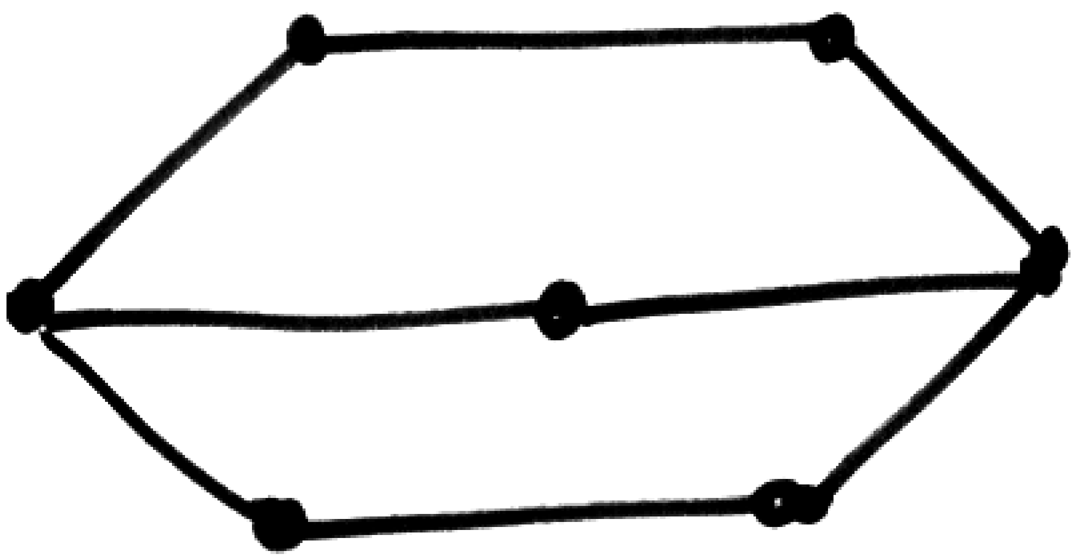
we really have <u>two</u> such puzzles together

$H = H_1 \times H_2$

A graph like this is called <u>separable</u>. Is one where removing one vertex leaves a disconnected graph.

3) $H = \begin{cases} S_n \\ A_n \end{cases}$

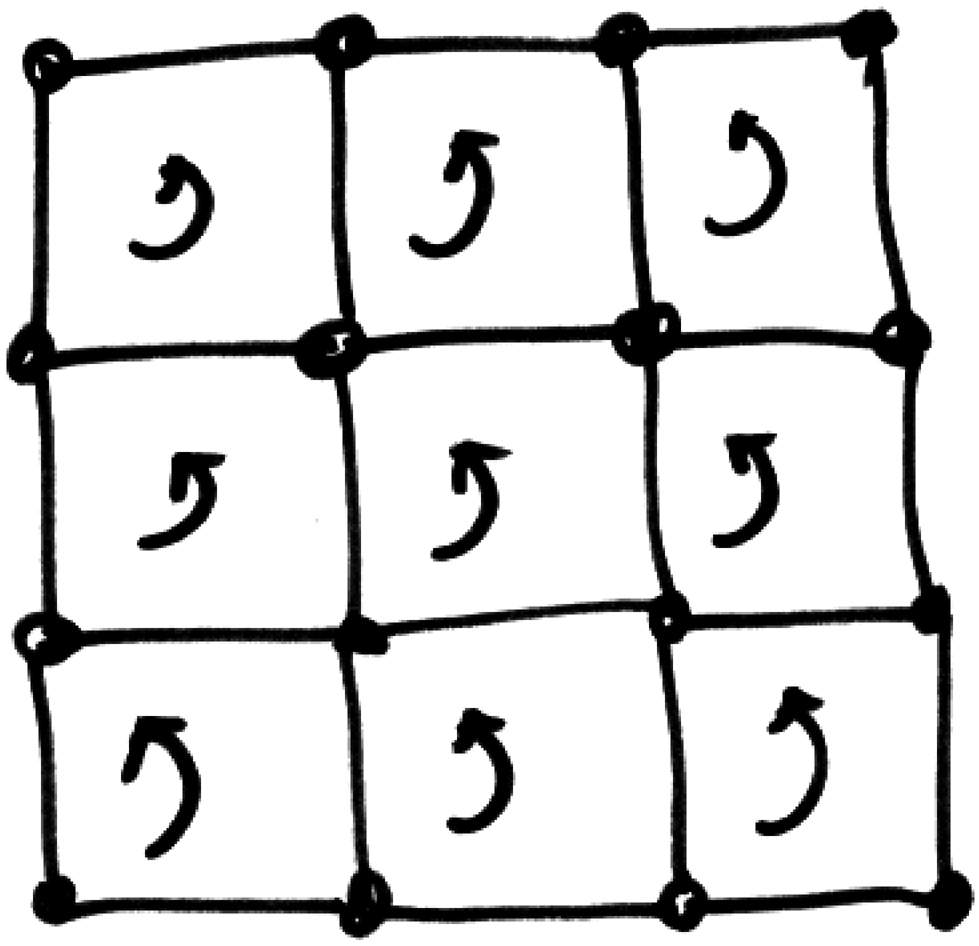
OR



$|H| = 120$

(group is related to the icosahedron)

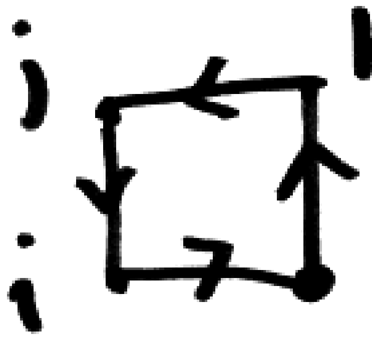
How do we tell apart S_n vs A_n?



15-puzzle

$\sigma = (i\,j\,k)$

$H = A_{15}$



$= \gamma$

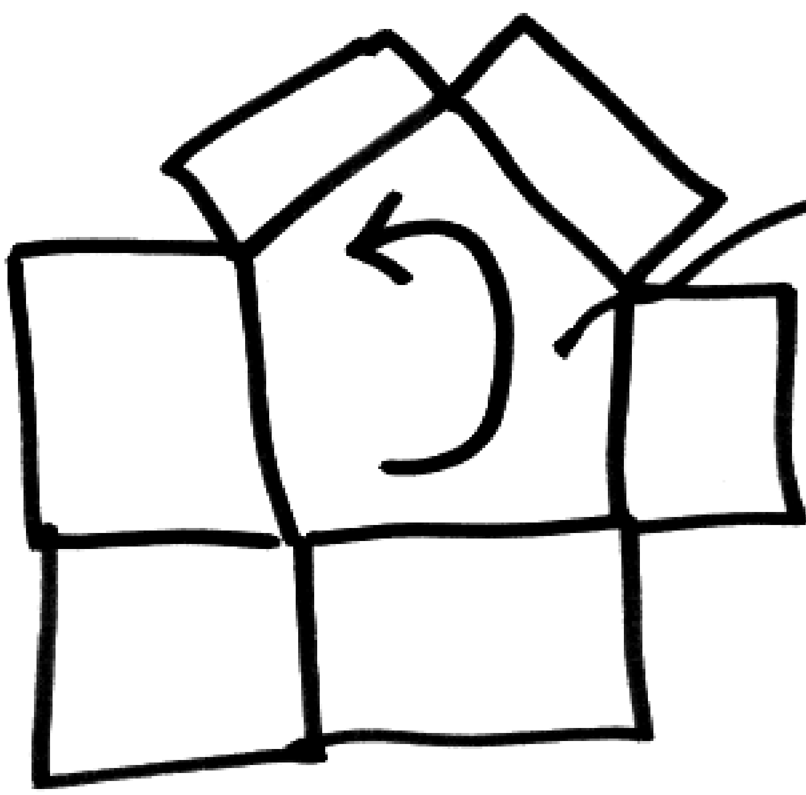
$\begin{array}{cc} j & K \\ i & B \end{array} \Big| \quad \begin{array}{cc} j & B \\ i & K \end{array} \Big| \quad \begin{array}{cc} B & j \\ i & k \end{array} \Big| \quad \begin{array}{cc} i & j \\ B & K \end{array} \Big|$

$\Big| \begin{array}{cc} i & j \\ K & B \end{array} \Big|$

 $\longleftrightarrow$ 3-cycle
even

Since H is either S_{15} or A_{15} we must have

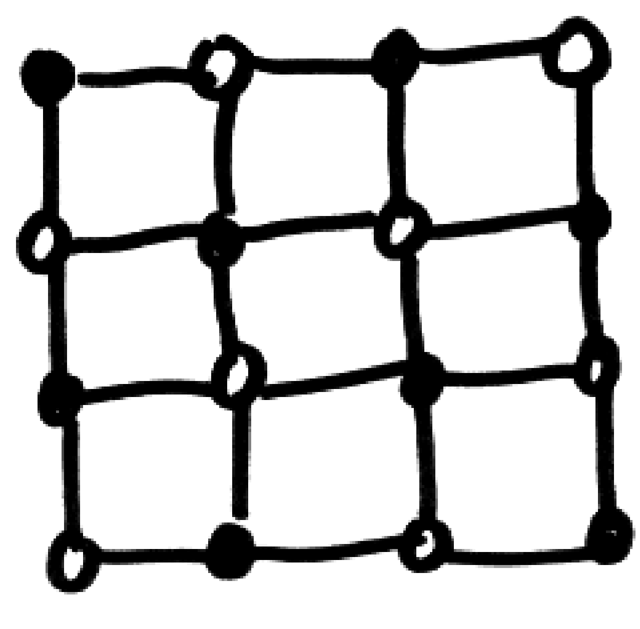
$S = A_{15}$

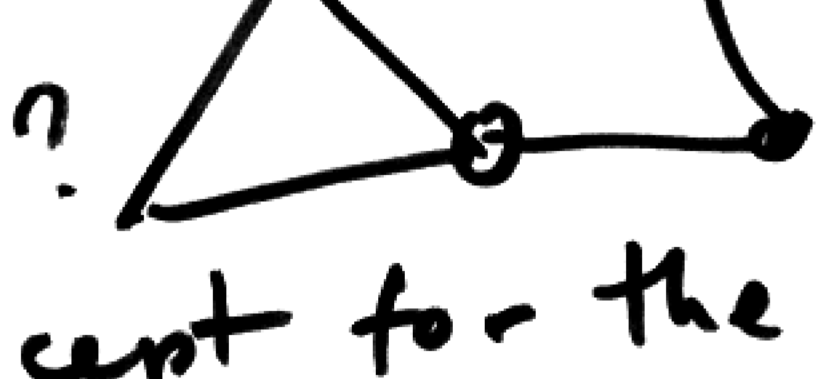
 4-cycle
odd

H contains <u>at least</u> one odd permutation.

Wilson's theorem implies it must be all of S_n.

We get A_n if all loops in Γ have an even number of sides. Such a graph is called <u>bipartite</u>





Except for the exceptions
S_n non bipartite graphs
A_n for bipartite.

$|H|$ = # possible states in the puzzle.

Wilson's theorem says # positions is as large as can be.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	.

unsolvable.

RATE
YOUR
MIND
PLA → RATE
YOUR
MIND
PAL

$(A_2 L)$ <u>not</u> possible

$R_1 A_1 T E$

$Y O U R_2$

$M I N D$

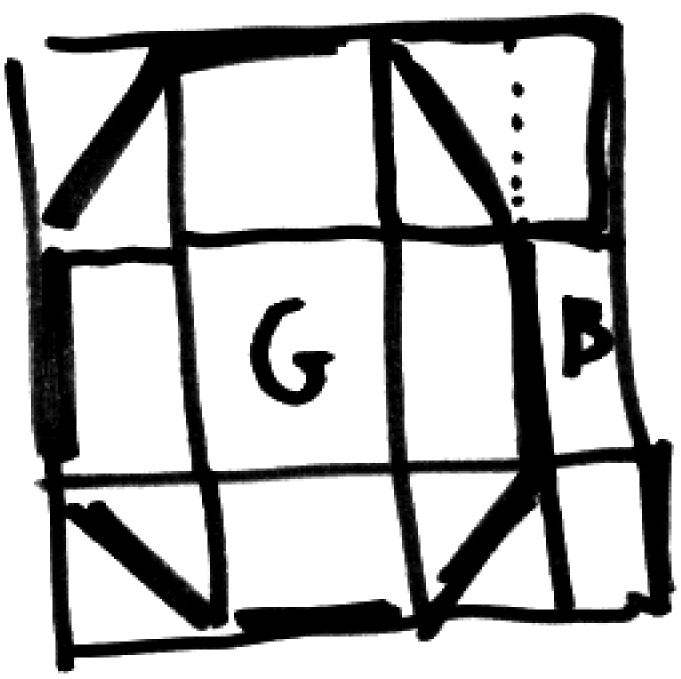
$P L A_2$

$(A_1 A_2)\ (A_2 L)$ possible

$(R_1 R_2)\ (A_2 L)$ "

same idea

Get your goat



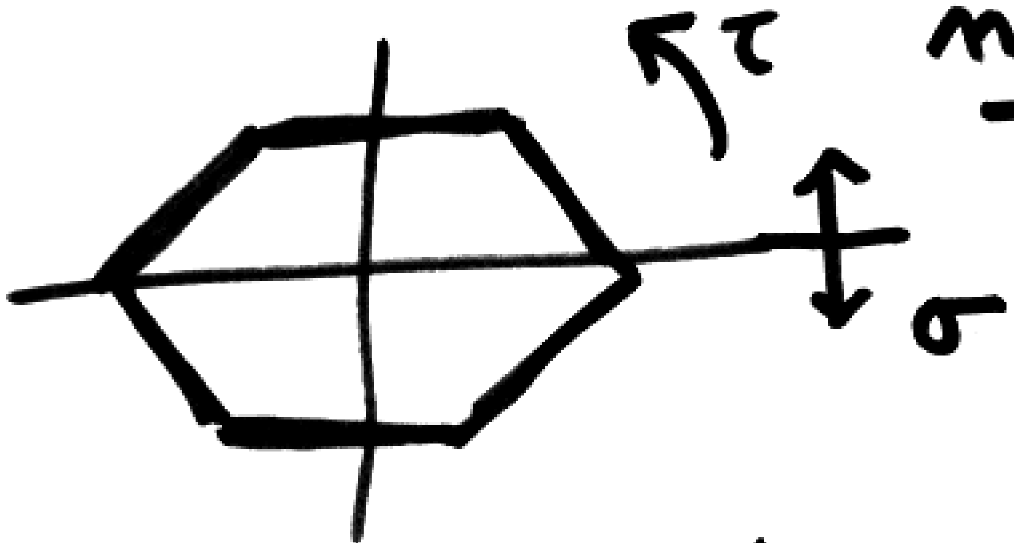
Groups

(Geometric flavor)

Symmetries

Dihedral group

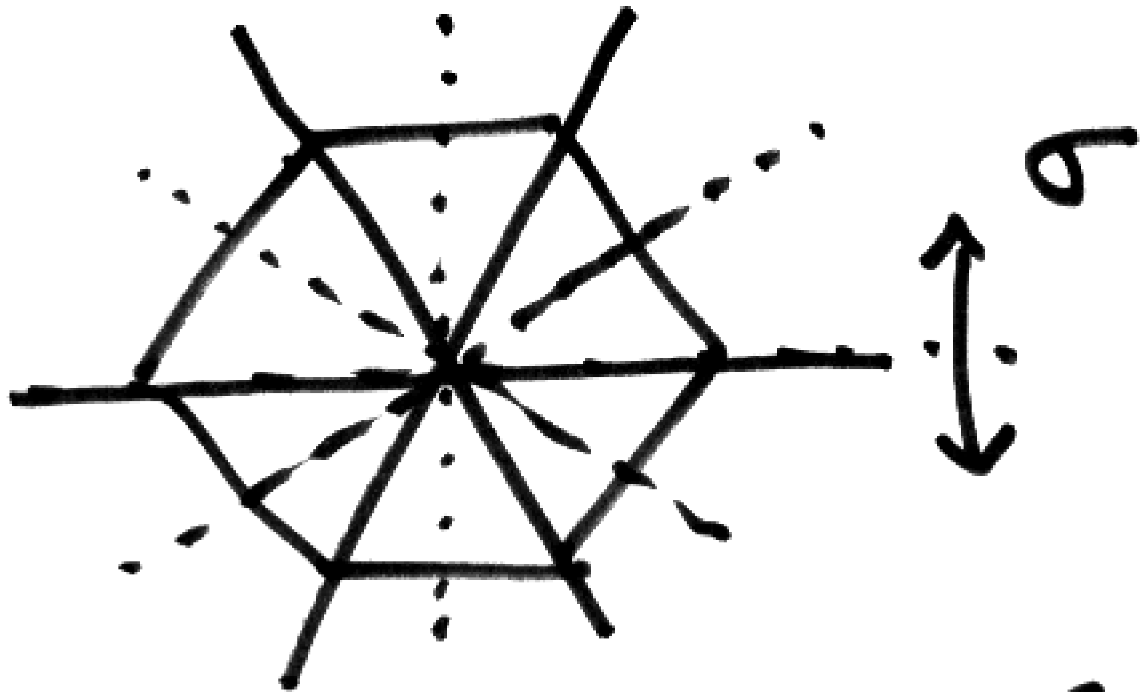
gp of symmetries of
a regular n-gon



$\underline{n=6}$

Rotations: $\{1, \tau, \tau^2, \tau^3, \tau^4, \tau^5\}$

subgroup



σ is a reflection

$\sigma^2 = 1$

D_6 dihedral group
order 12

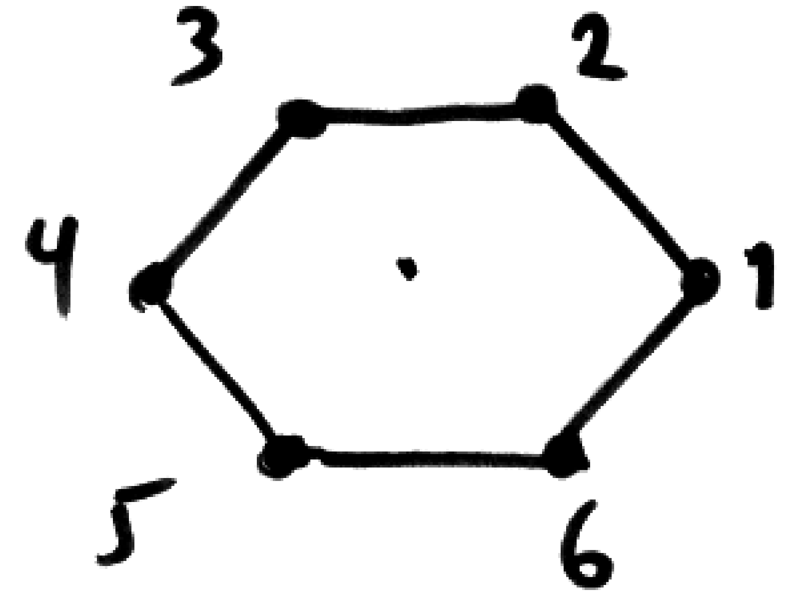
6 rotations &
6 reflections

$= \langle \sigma, \tau \rangle$

$\sigma, \sigma\tau, \sigma\tau^2, \sigma\tau^3, \sigma\tau^4, \sigma\tau^5$

reflections

Dihedral group $\boxed{IX}$



$n = 6$

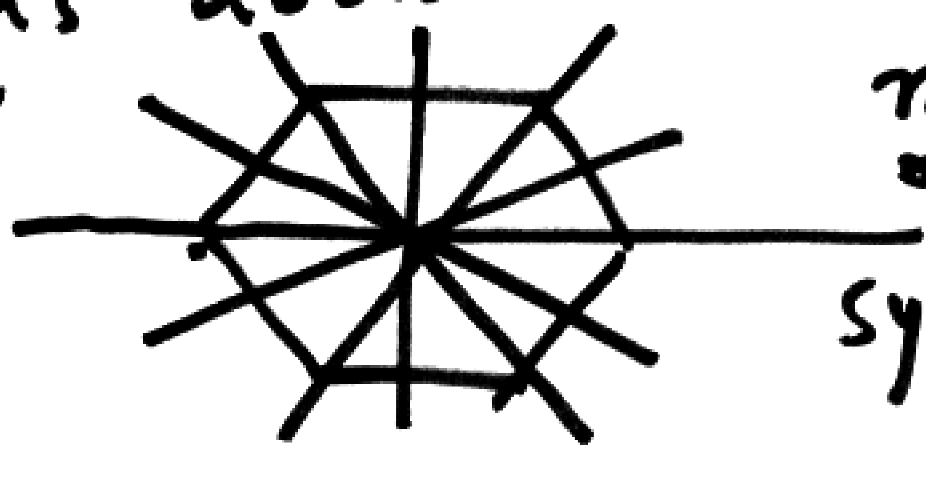
Regular n-gon

D_n = symmetries of the n-gon.

τ: Rotation $2\pi/n$

$1, \tau, \tau^2, \ldots, \tau^{n-1}$

Reflections about axis of symmetry



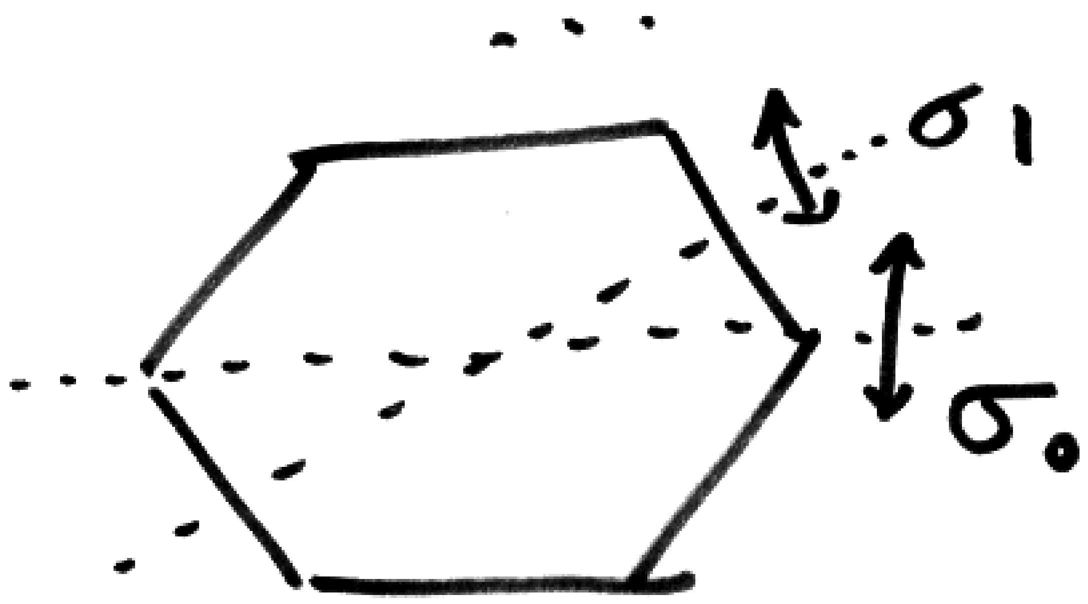
n-axes of symmetry

Claim

1) There are $2n$ symmetries

Rotations: $1, \tau, \tau^2, \dots, \tau^{n-1}$

Reflections: $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$



2) D_n is generated by τ, and σ $\quad (\sigma = \sigma_0)$

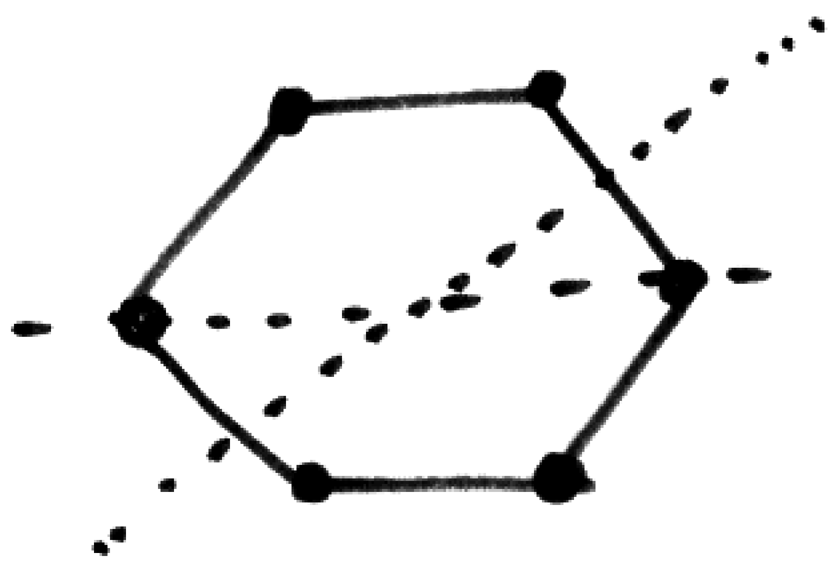
$D_n = \langle \tau, \sigma \rangle$

$\sigma^2 = id$

$g \in D_n, \quad g = \dots \sigma \tau^{k_1} \sigma \tau^{k_2} \dots$

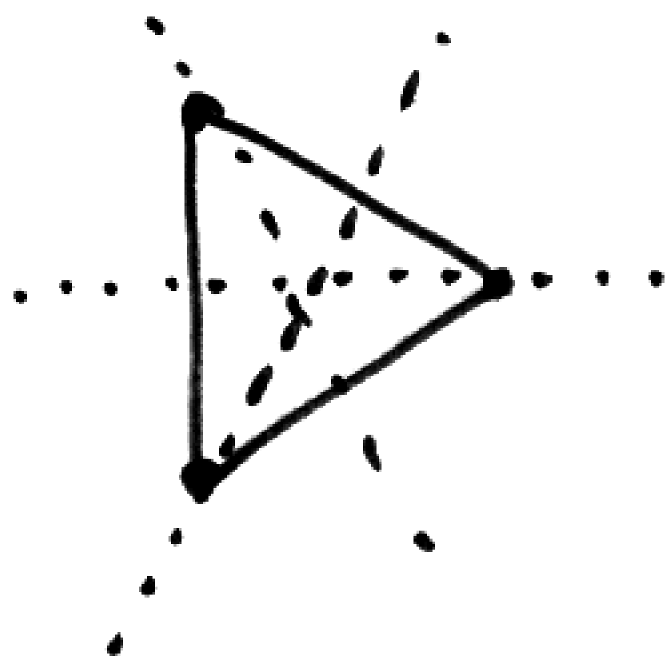
n even

n=6

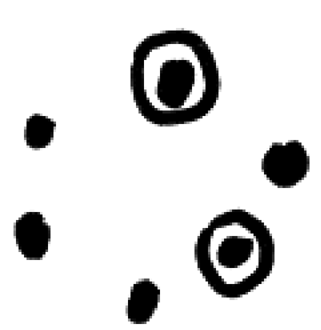


n odd

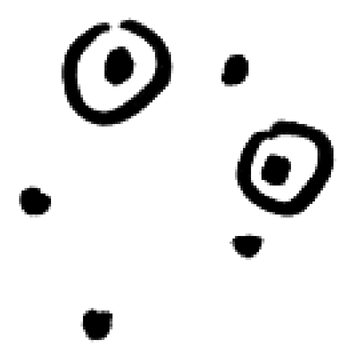
n=3



Q: Necklace n beads of 2 colors
How many different necklaces are there?



=



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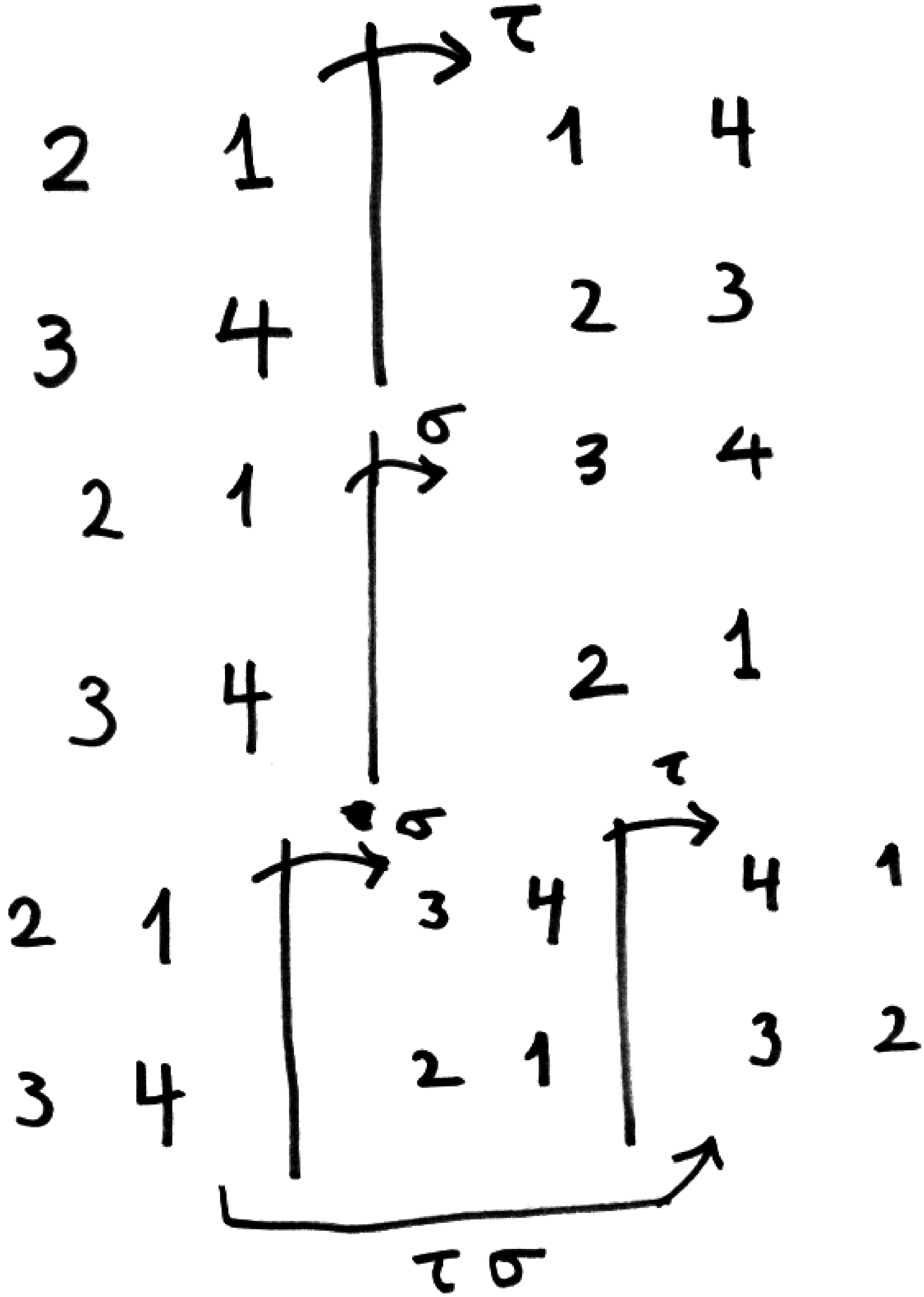
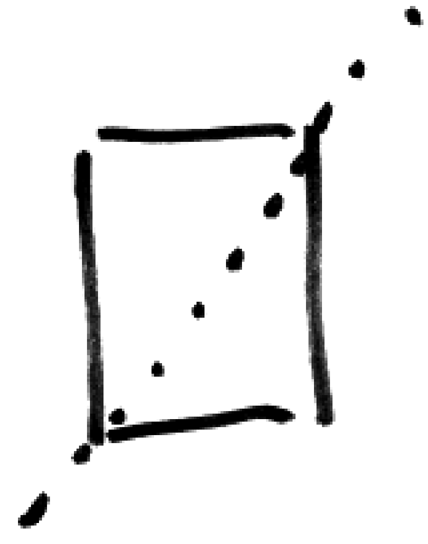
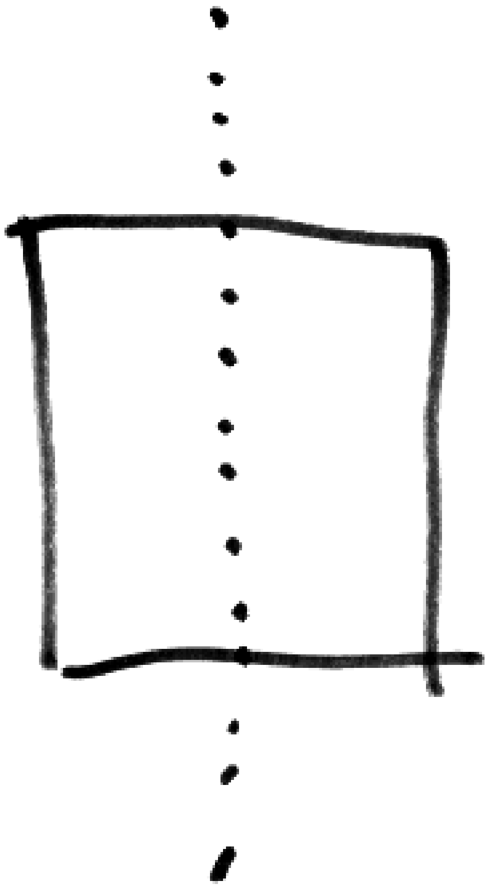


$\tau\sigma = \sigma_1$

reflection about



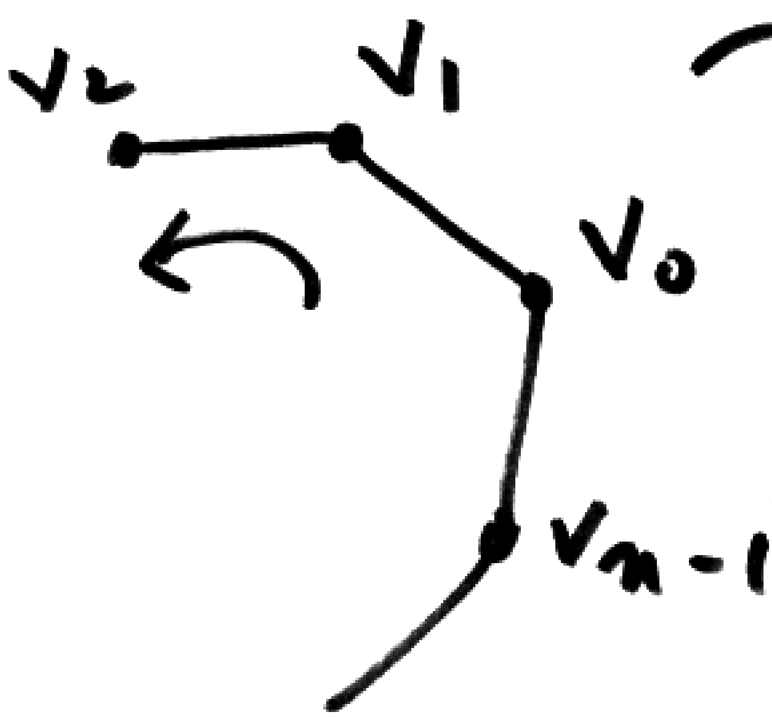


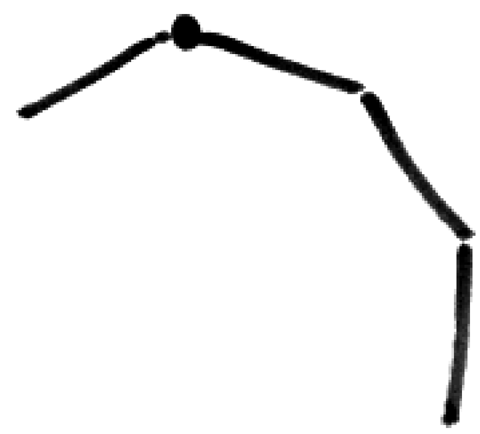
$$\sigma_2 = \tau\sigma_1$$
$$= \tau\tau\sigma$$
$$= \tau^2\sigma$$

$$\sigma_k = \tau^k \sigma$$

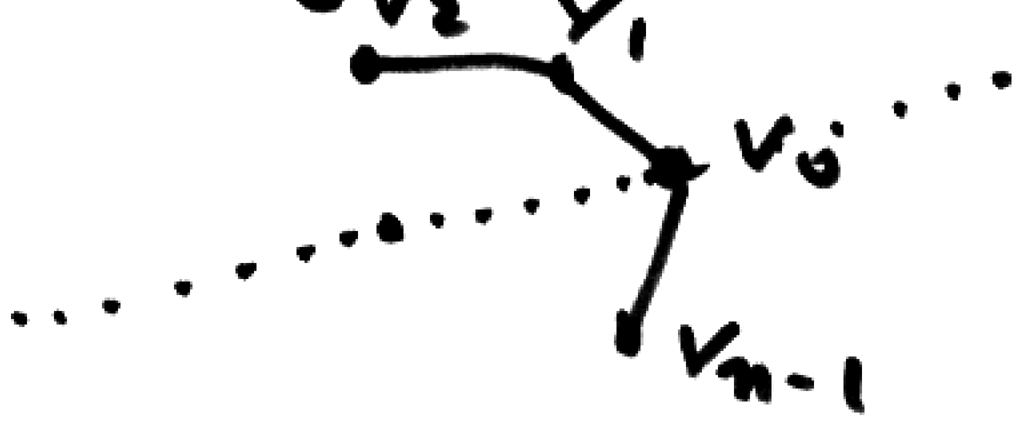
$g \in D_n$

$v_0 \longmapsto v_k$





$\tau^{-k} g : v_0 \longmapsto v_0$



fixes axis as well

Hence

$$\tau^{-k} g = \begin{cases} \text{identity} \\ \text{reflection about this axis} = \sigma \end{cases}$$

$$\tau^{-k} g = \sigma \text{ or } 1$$

$$\boxed{g = \tau^{k} \sigma} \quad \text{Reflection}$$

$$\boxed{g = \tau^{k}} \quad \text{Rotation}$$

$\{1, \tau, \tau^2, \dots, \tau^{n-1}\}$
subgroup of D_n of order n

$$g = \tau^{k} \sigma^{j} \qquad \begin{array}{c} 0 \leq k \leq n-1 \\ j = 0, 1 \end{array}$$

uniquely.

τ[redacted] σ, τ

$$\tau\sigma \cdot \tau = \tau^{k}\sigma^{j}$$

some k, j.

$$\boxed{\sigma\tau \neq \tau\sigma}$$

group is <u>not</u> commutative (abelian)

$$\boxed{\sigma\tau = \tau^{-1}\sigma}$$

Basic relation between our generators.

$$(\tau\sigma)\cdot\tau = \tau(\sigma\tau)$$
$$= \tau\tau^{-1}\sigma$$
$$= \sigma$$

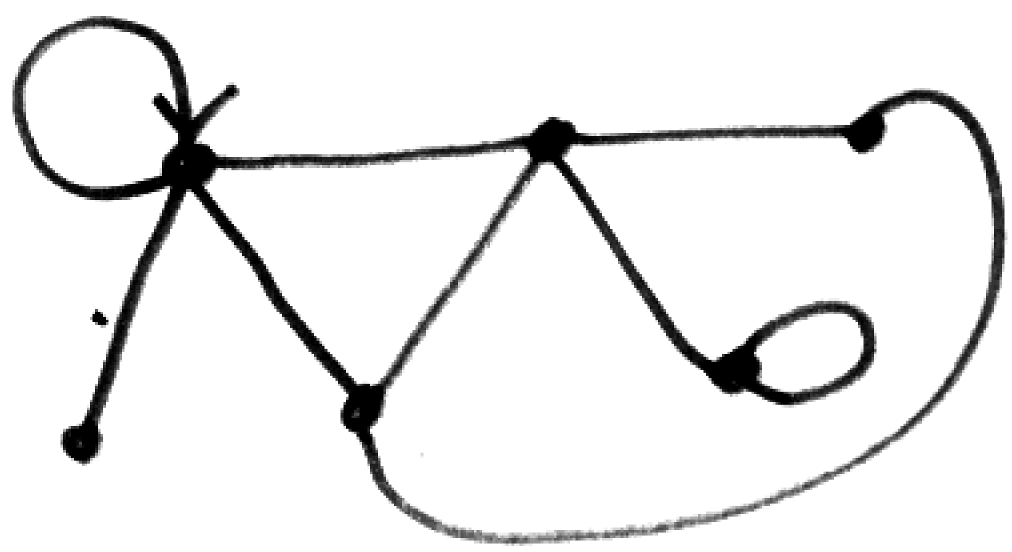
$$\tau\sigma\tau^2 = \tau\underbrace{\sigma\tau}\tau$$
$$= \tau\tau^{-1}\sigma\tau$$
$$= \sigma\tau$$
$$= \tau^{-1}\sigma$$

$$\tau^k\sigma^j \cdot \tau^{k'}\sigma^{j'} = \tau^{k''}\cdot\sigma^{j''}$$

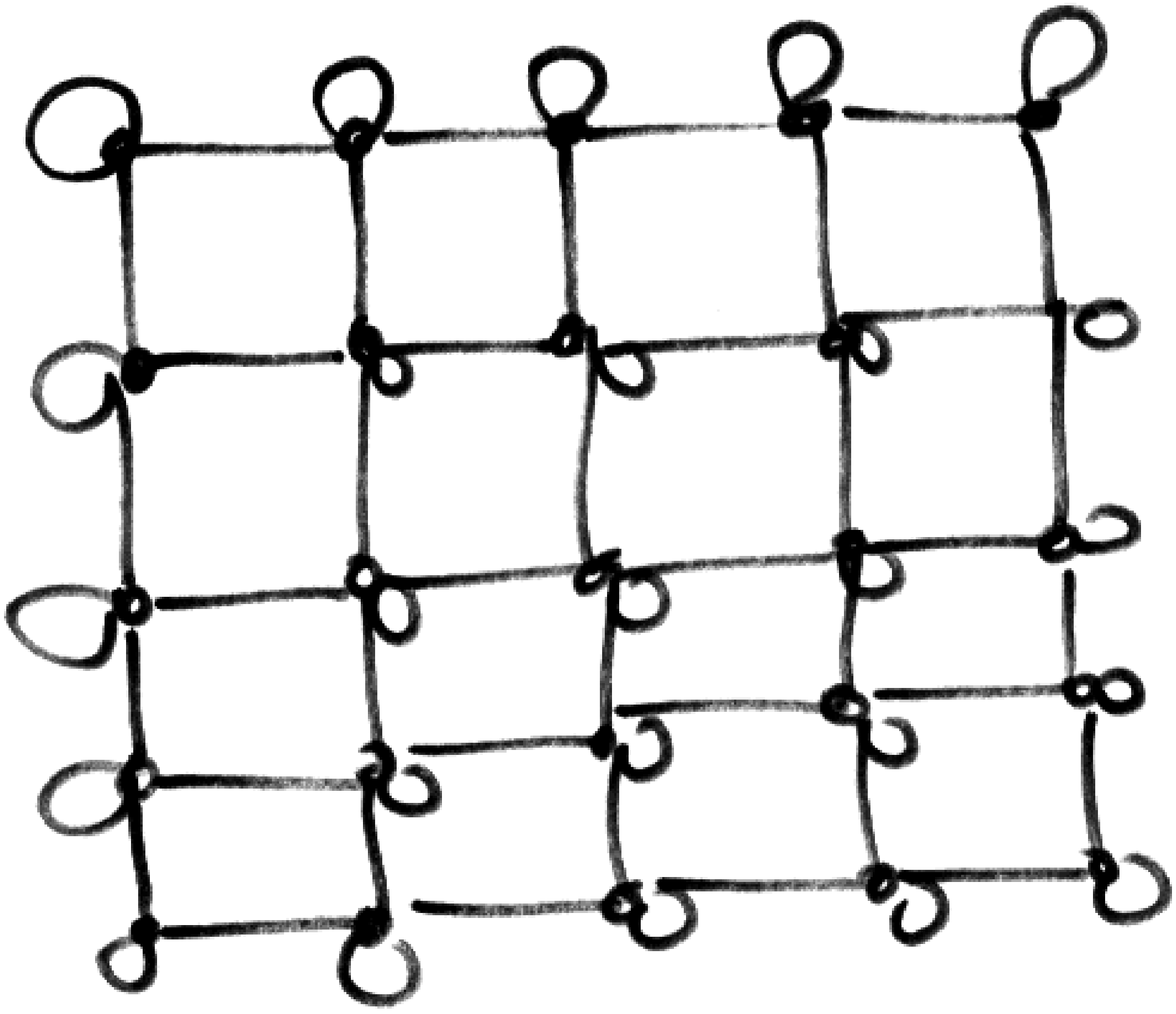
$$(k,j) \quad (k',j') \longrightarrow (k'',j'')$$

Lights Out

Play on any graph



Commercial version



Label the vertices

$v_1, v_2, \ldots, v_n$

<u>State</u> of puzzle

$S = (s_1, s_2, \ldots, s_n)$

$s_i = 0, 1.$

<u>Move</u>

Click on v_j

changes some v_i's.

$v_i \mapsto v_i + 1 \mod 2$

$\begin{bmatrix} 0 \mapsto 1 \\ 1 \mapsto 0 \end{bmatrix}$

The whole puzzle is encoded in a matrix

$$A = \begin{array}{cc} \downarrow j & \\ \begin{pmatrix} & \vdots & \\ & 1 & \\ & \vdots & \end{pmatrix} & \leftarrow i \\ c_j & \end{array}$$

1 in spot $a_{i,j}$
if clicking on v_j affects v_i

$$s_I \longmapsto s_I + c_j \mod 2$$

click on v_j

A solving puzzle

$$s_I + c_{j_1} + c_{j_2} + \dots + c_{j_N} \equiv 0 \mod 2$$

Encode sequence of moves in a vector

$t = (t_1, \ldots, t_n) \quad t_i = 0, 1$

$$\boxed{S_I + At = 0}$$

click v_j gives adds c_j

$$A \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} = c_j$$

$$t = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + t_n \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

$$At = t_1 c_1 + t_2 c_2 + \ldots + t_n c_n$$

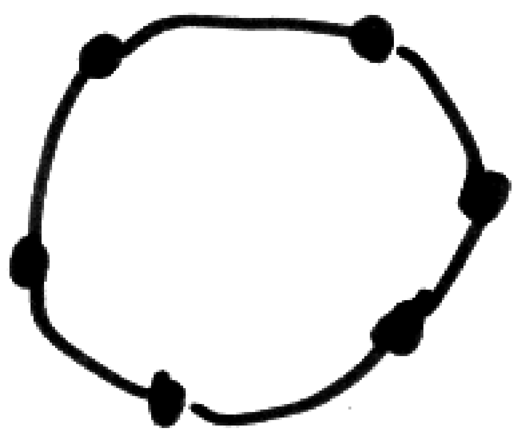
Solving puzzle is to find t and this is a linear algebra problem.

$$\boxed{s_I + At = 0}$$

<u>Q</u>: When can we always solve puzzle?

Light Out

graph



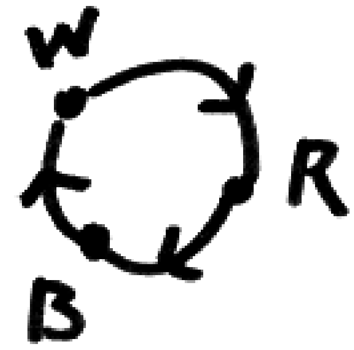
Hexa

clicking on a button changes cyclically colors of those connected to it.

$m :=$ # colors

($m=3$ in the example)

W R B
0 1 2



<u>Status</u> of puzzle

$$s : \Gamma \longrightarrow \mathbb{Z}/m\mathbb{Z}$$

Γ ↑ graph

$\mathbb{Z}/m\mathbb{Z}$ ↑ integers modulo m

Encoding vertex i has color $s(i)$.

Total number of possible states $= m^n$

$n = \# \Gamma$

(Example $n = 6$)

3^6 states.

Clicking on button w

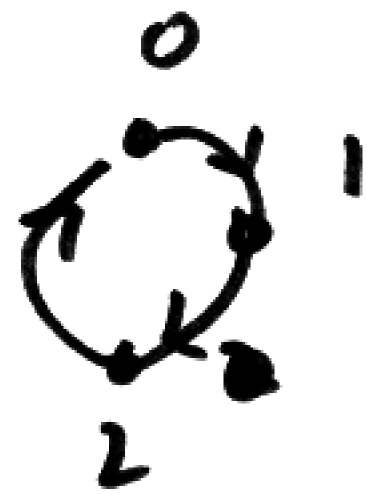
$$\boxed{s \longmapsto s + c_w}$$

$$c_w(u) = \begin{cases} 1 & w \leftrightarrow u \\ 0 & \text{otherw.} \end{cases}$$

$$s(u) \longmapsto \begin{cases} s(u) + 1 & w \leftrightarrow u \\ s(u) & \text{otherw} \end{cases}$$

Adding 1 mod m to the color cyclically permutes them

s :	0	1	2
s+1 :	1	2	0



Solve puzzle (4)

$$S + c_{w_1} + c_{w_2} + \dots + c_{w_N} \equiv 0 \mod m$$

operation of clicking a button takes place in a commutative group (adding mod m).

- Order does not matter
- clicking any given button m times is like not clicking it at all.

$$\underbrace{c_w + c_w + \dots + c_w}_{m} \equiv 0 \mod m$$

Label the vertices

$w_1, w_2, \ldots, w_n$

a solution is encoded

in $t = (t_1, t_2, \ldots, t_n)$

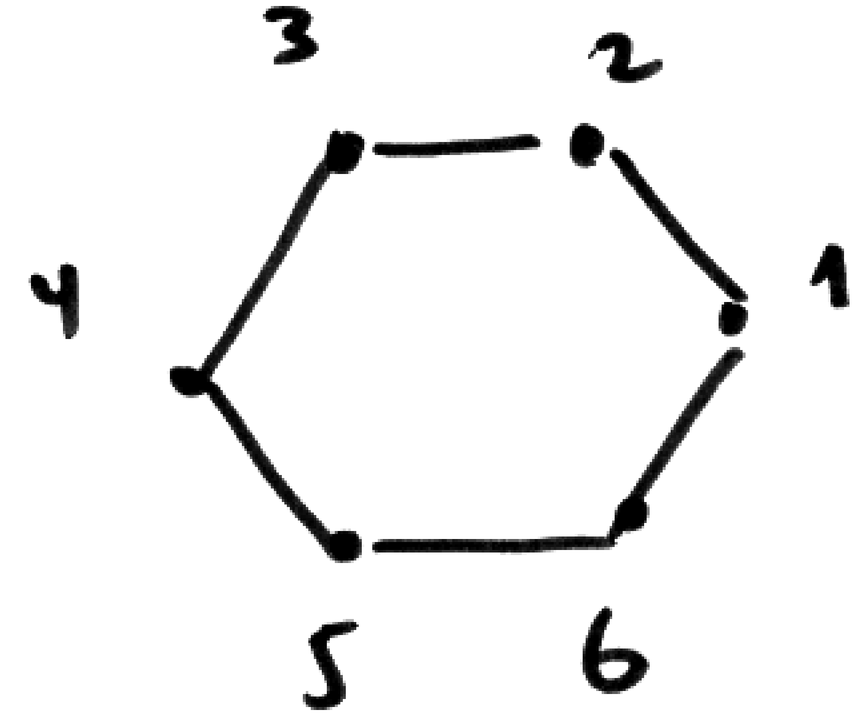
$t_i = 0, 1, 2, \ldots, m-1$

$t \leftrightarrow$ clicking button 1 t_1 times, button 2 t_2 times, ...

$$\boxed{s + At = 0}$$

where A is a matrix having columns $c_{w_1}, c_{w_2}, \ldots, c_{w_n}$

For Hexa



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Clicking on 3rd button adds column 3 of A to the state.

$$\text{column 3 of } A = A \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} = t_1 C_1 + t_2 C_2 + \dots + t_n C_n$$

$C_i = i^{\text{th}}$ column of A.

$$\boxed{s + At = 0}$$

Solving puzzle means finding t, i.e. linear algebra problem mod m.
We may solve this equation if A had an inverse

$$A^{-1} \cdot A \equiv I_n \mod m$$

$s + At = 0$

$$s + At = 0$$

multiply by A^{-1} on the left

$$A^{-1}s + \underbrace{A^{-1}A}_{I_n} t = 0$$

$$A^{-1}s + t = 0$$

$$\boxed{t = -A^{-1}s}$$

<u>Q</u>: How do we know when A has an inverse?

A = adjacency matrix of graph Γ

$$a_{ij} = \begin{cases} 1 & i \leftrightarrow j \\ 0 & \text{otherwise.} \end{cases}$$

$d := \det A$

A has an inverse mod m

iff $\gcd(d, m) = 1$

suppose $m = 6$

a:	0	1	2	3	4	5
a^{-1}:	•	1	•	•	•	5

$a \cdot a^{-1} \equiv 1 \bmod 6$

$2 \cdot 3 \equiv 0 \bmod 6$

" $2^{-1} \cdot 2 \cdot 3 \equiv 0 \bmod 6$

$3 \equiv 0 \bmod 6$ "

$$s = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

$$t \equiv -A^{-1}s \mod 3$$

$$-A^{-1} = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$

$$t = (2, 1, 0, 2, 1, 0)$$

solution
(unique, optimal).

puzzle

$$s \longmapsto s + At$$

solving puzzle

$$s' \longmapsto s' - A^{-1}t'$$

no s

$$0 \longmapsto -A^{-1}s = t$$

solution to original

If d is relative prime to m then A has inverse mod m and we can solve for t

$$t = -A^{-1}s$$

- unique.

<u>Any</u> s can be solved

$$t \equiv -A^{-1}s \mod m$$

If A is <u>not</u> invertible mod m.

$$s + At = 0$$

$$\boxed{At = -s}$$

then not every s has a t
if you can there are more than one solutions.
For puzzle: it may not be solvable and it will be solvable when possible in more than one way.

$$A^*A = -2$$

$$s + At \equiv 0 \mod 2$$

$$A^*(s + At) = A^*s + A^*At$$
$$= A^*s - 2t$$

$$t = \frac{1}{2} A^*s$$

$$\uparrow$$

puzzle

solving puzzle

$$0 \mapsto A^*s$$

can find t if A^*s has only even entries

Label colors :

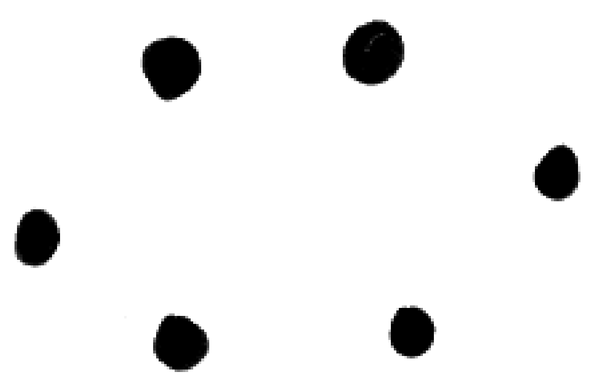
0	1	2	3
W	B	R	B

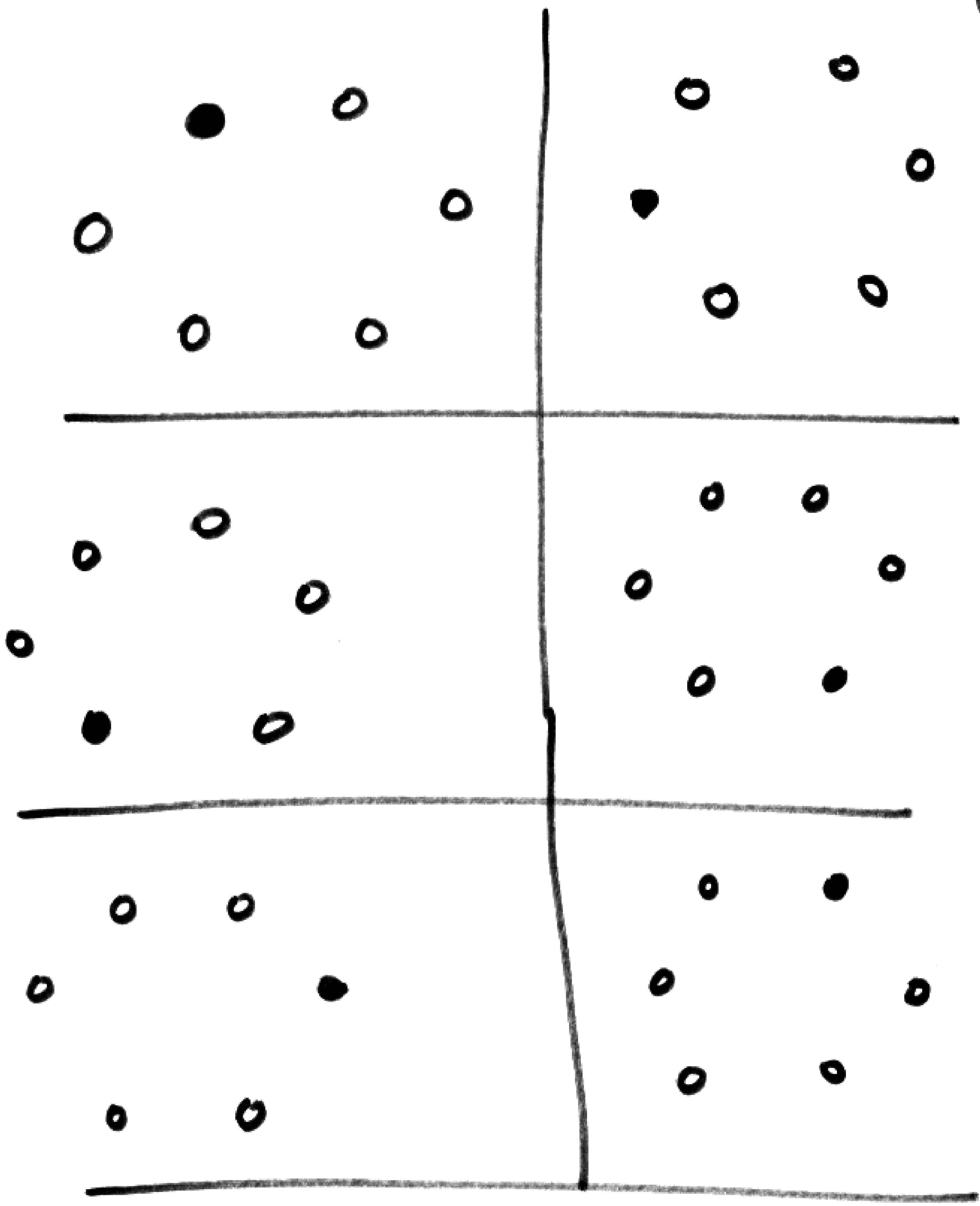
Polya's theory of counting

Count different necklaces

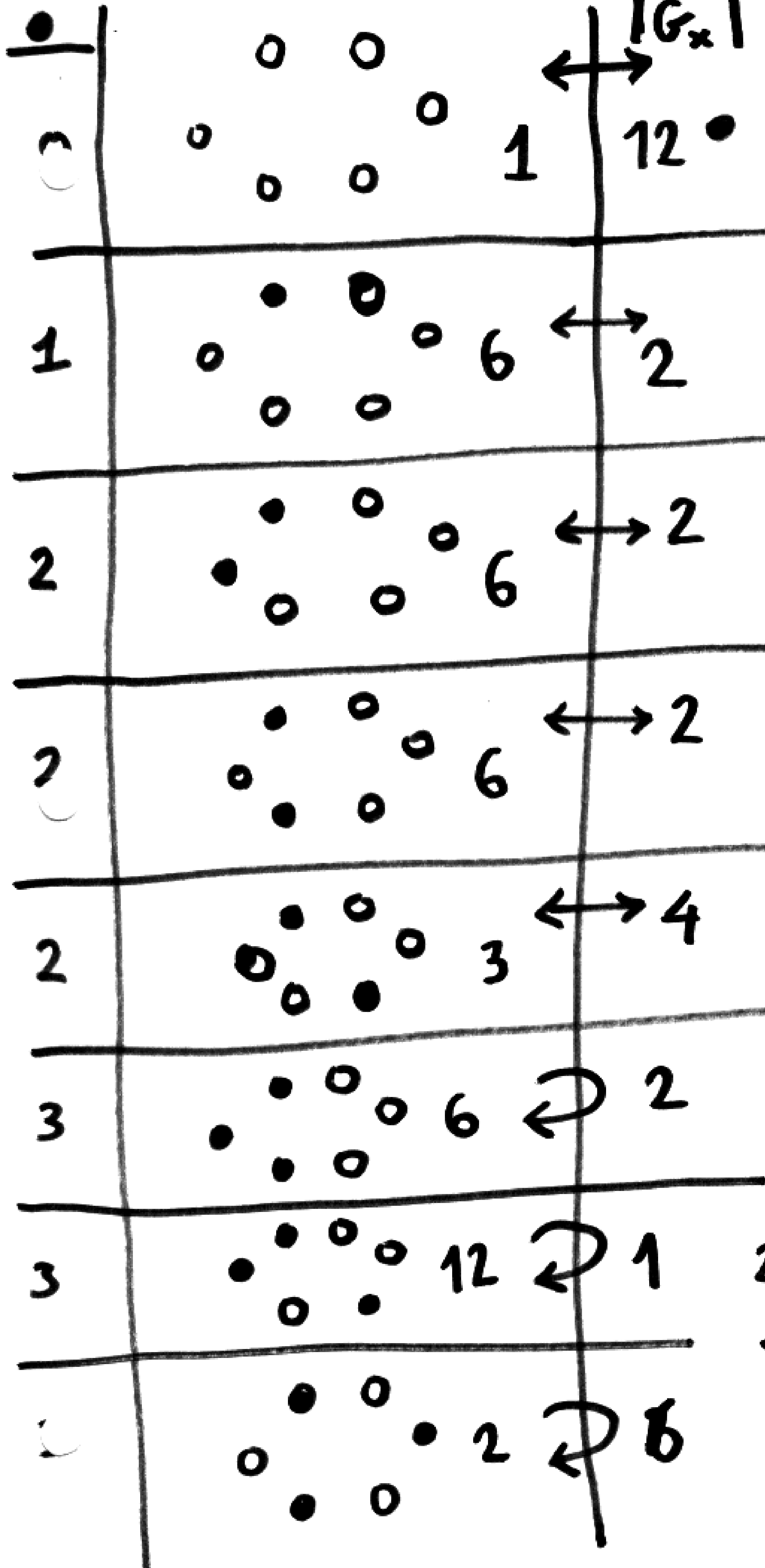
$n = 6$ beads

$m = 2$ colors





3



$\bullet \times \bullet = 12$
$= |G|$

$\boxed{\text{TOTAL} = 13}$

$2 \times (1+6+6+6+3)$
$+ 6+12+2 = 64$
$= 2^6$

group G

acting on a set X

Example $\quad G = D_6$

symmetries of hexagon

$X =$ colorings of vertices of hexagon

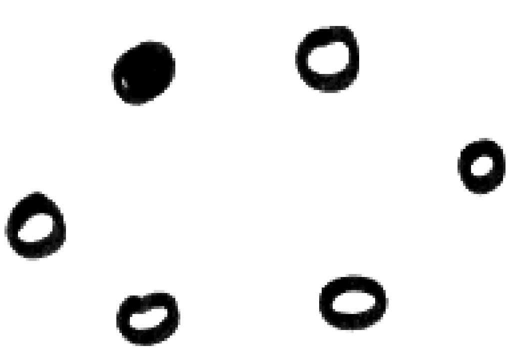
$\# X = 64$

Each $g \in G$ gives a permutation

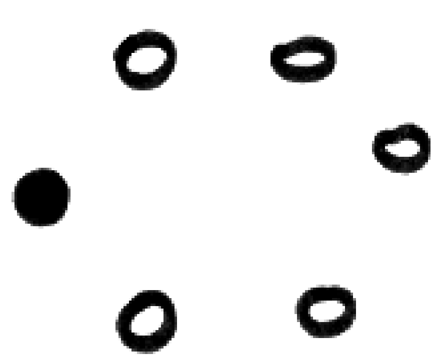
$$X \to X$$
$$x \mapsto g \cdot x$$

E.g. $g = \tau$ rotation $\frac{2\pi}{6}$

$x =$



$\tau \cdot x$



$$g_1(g_2 x) = (g_1 g_2) x$$

consistency rule.

Stabilizer

$x \in X$

$$G_x = \{ g \in G \mid gx = x \}$$

subgroup of G

$$\left.\begin{array}{l} g_1 x = x \\ g_2 x = x \end{array}\right\} \Rightarrow \begin{aligned} (g_1 g_2) x &= g_1 (g_2 x) \\ &= g_1 x \\ &= x \end{aligned}$$

$$gx = x \Rightarrow g^{-1} x = x$$

$$g^{-1}(gx) = g^{-1} x$$

$$= (g^{-1} g) x$$

$$= x$$

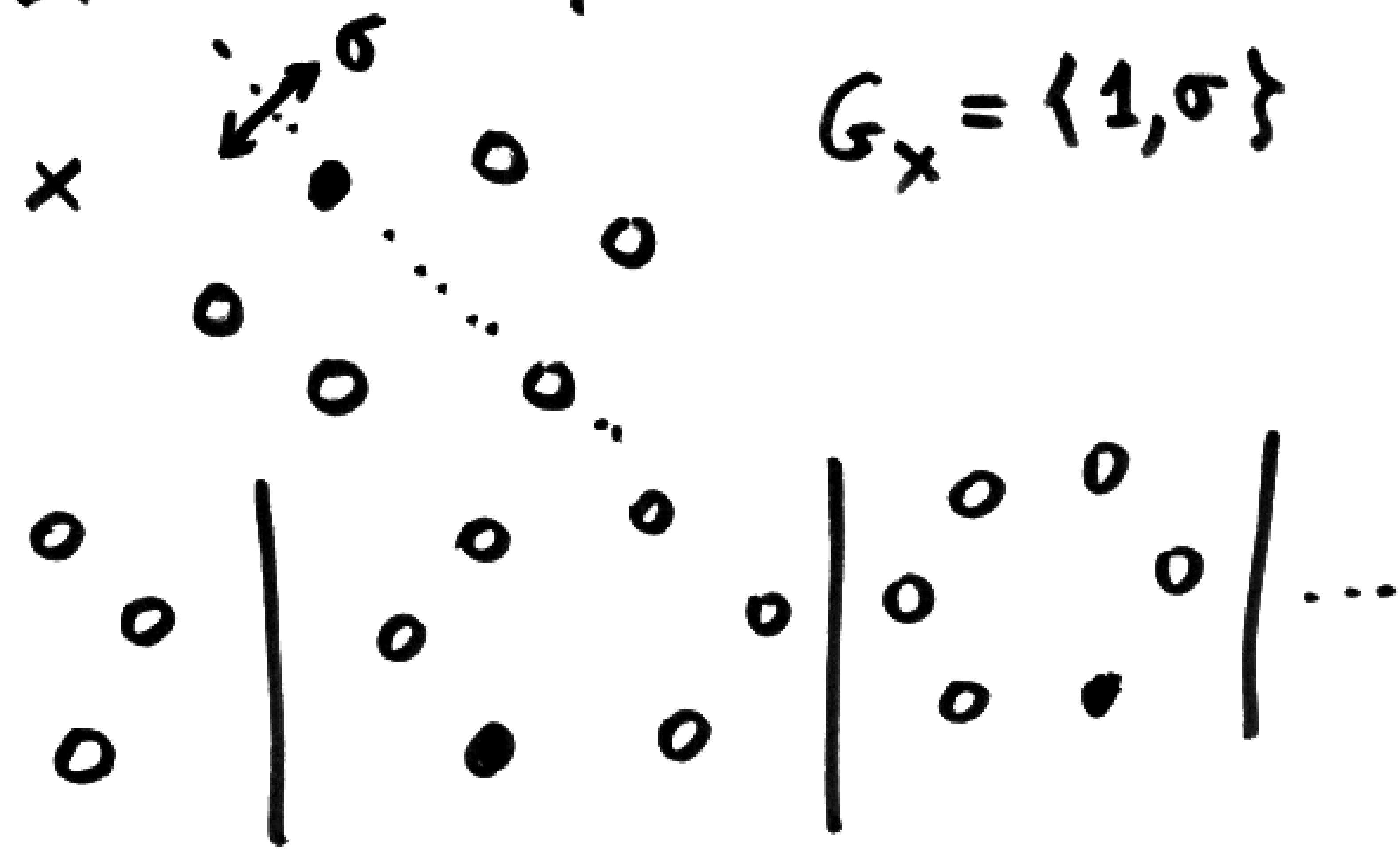
$G \cdot x = \{ gx \mid g \in G \}$

for a fixed x is called the <u>orbit</u> of x.

<u>THM</u>

$$\underbrace{\# Gx}_{\text{size of orbit}} \cdot \underbrace{|G_x|}_{\text{size of stabilizer}} = |G|$$

E.g.



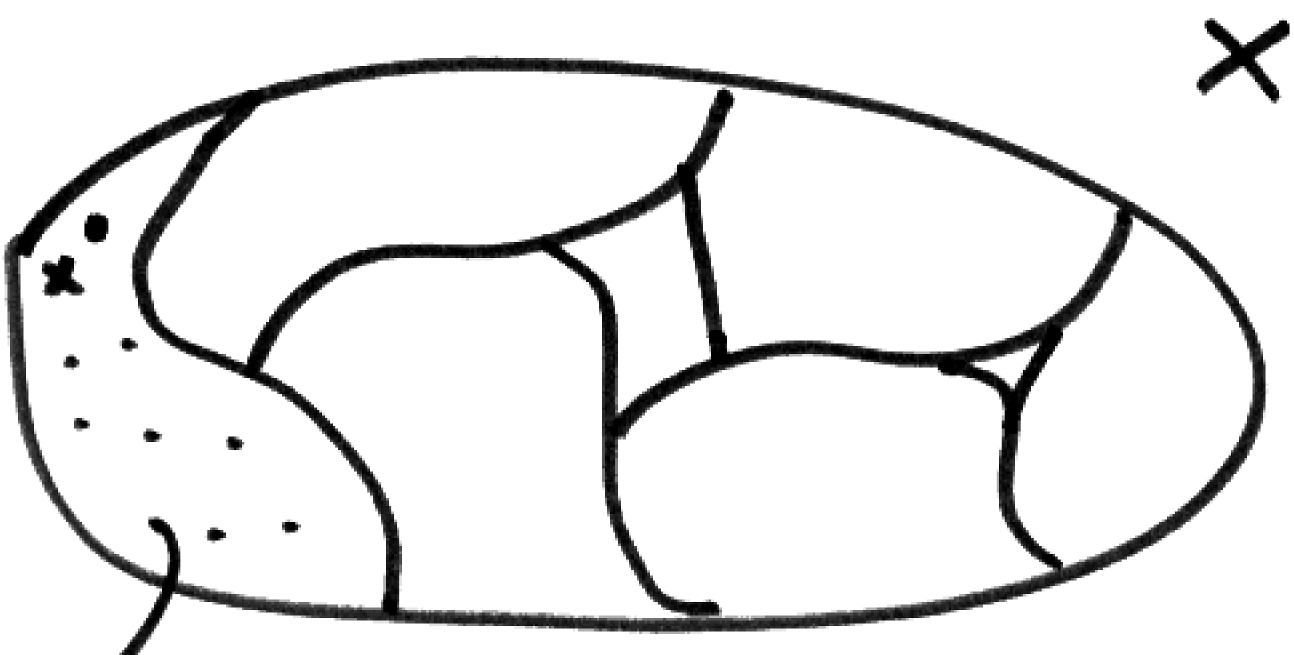
Gx

Size of orbits divides the size of G.

on X we can define an equivalence relation

$x \sim y$

if $y = gx$ for some $g \in G$



equivalence class = orbit

<u>Want:</u> # of orbits

Burnside's Lemma

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} F(g)$$

$$F(g) := \#\{ x \in X \mid gx = x \}$$

"average of # fixed points"

Proof

$\cancel{x}y \in X$ is counted on the rhs for every $g \in G_y$

$|G_y|$

Each element in orbit of y

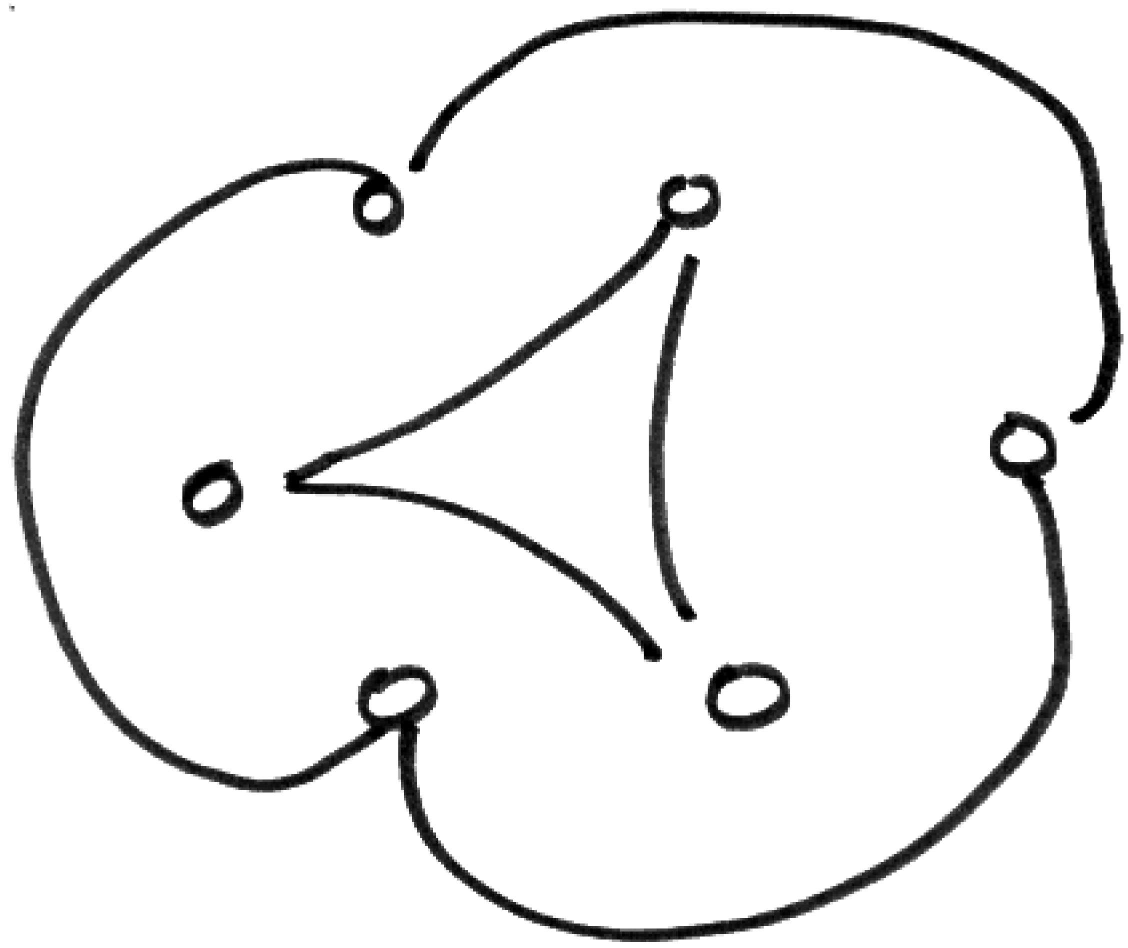
[redacted] $\# Gy \cdot |G_y| = |G|$

each orbit gets counted $|G|$ □

$D_6 = G$

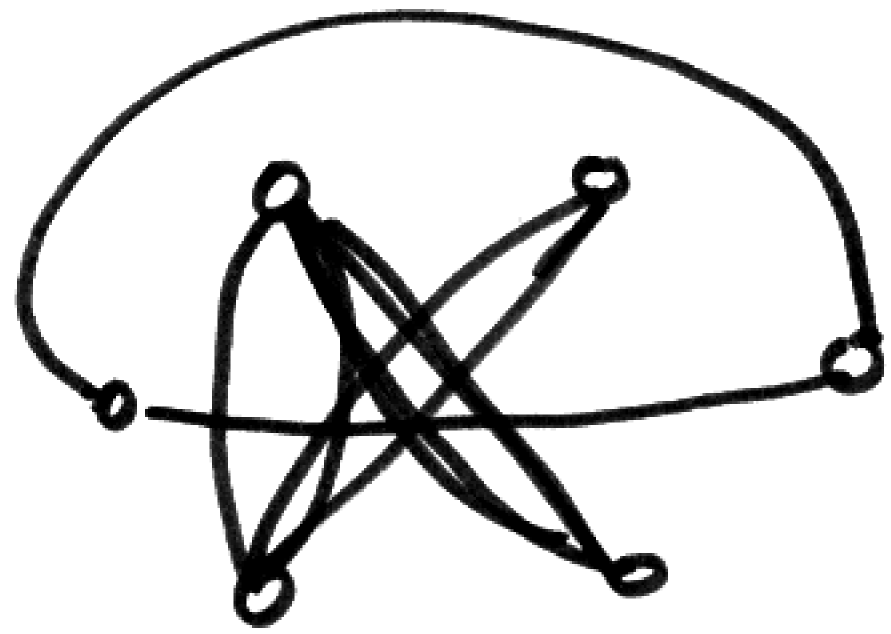
g	$F(g)$	
1	64	m^6
τ	2	m
τ^2	4	m^2
τ^3	8	m^3
τ^4	4	m^2
τ^5	2	m
σ_1	2^4	m^4
σ_2	2^3	m^3
σ_3	2^4	m^4
σ_4	2^3	m^3
σ_5	2^4	m^4
σ_6	2^3	m^3

$$\frac{1}{12}\left(64+2+4+8+\dots+2^4+2^3\right)=13$$

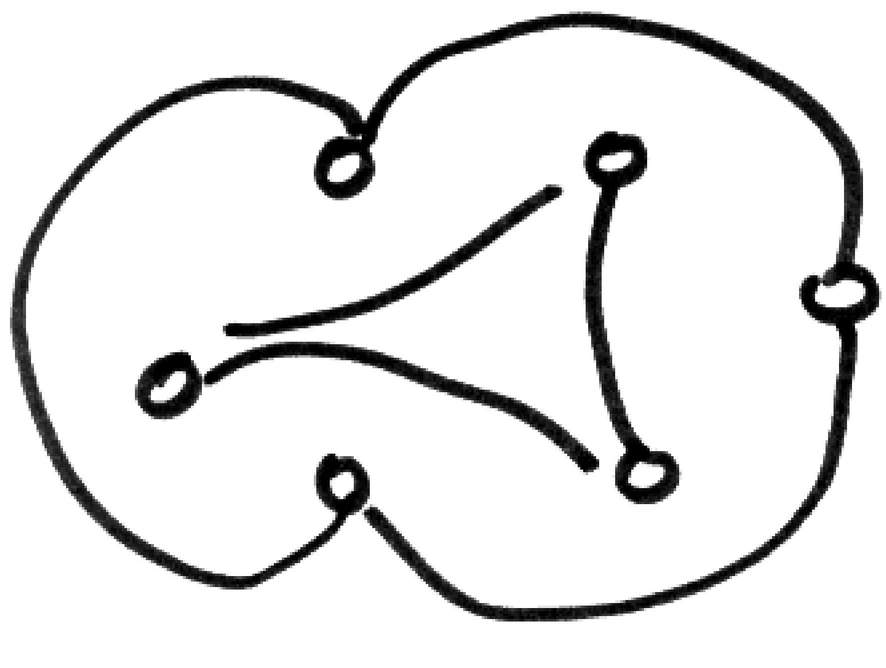


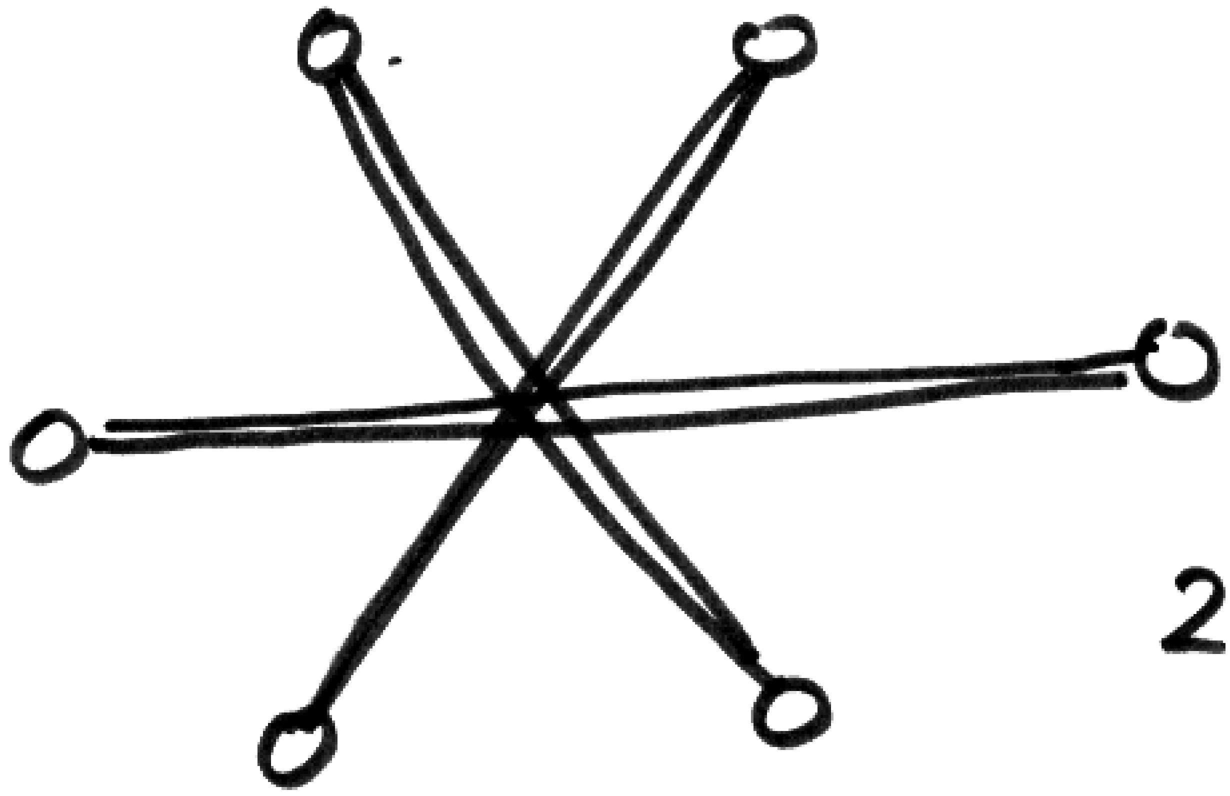
τ^2

τ^3



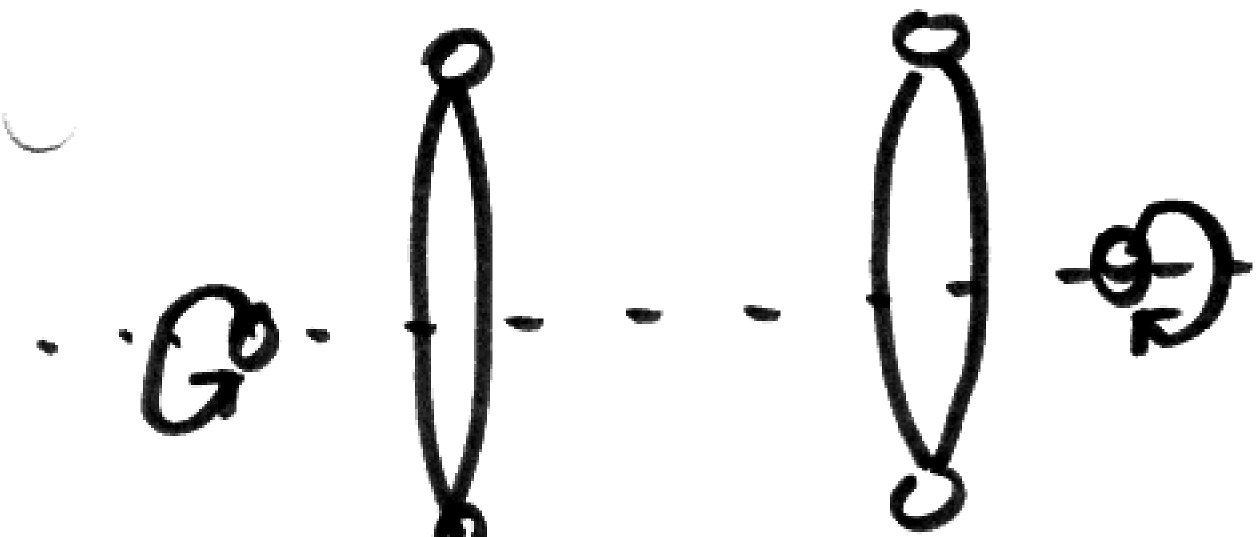
τ^4





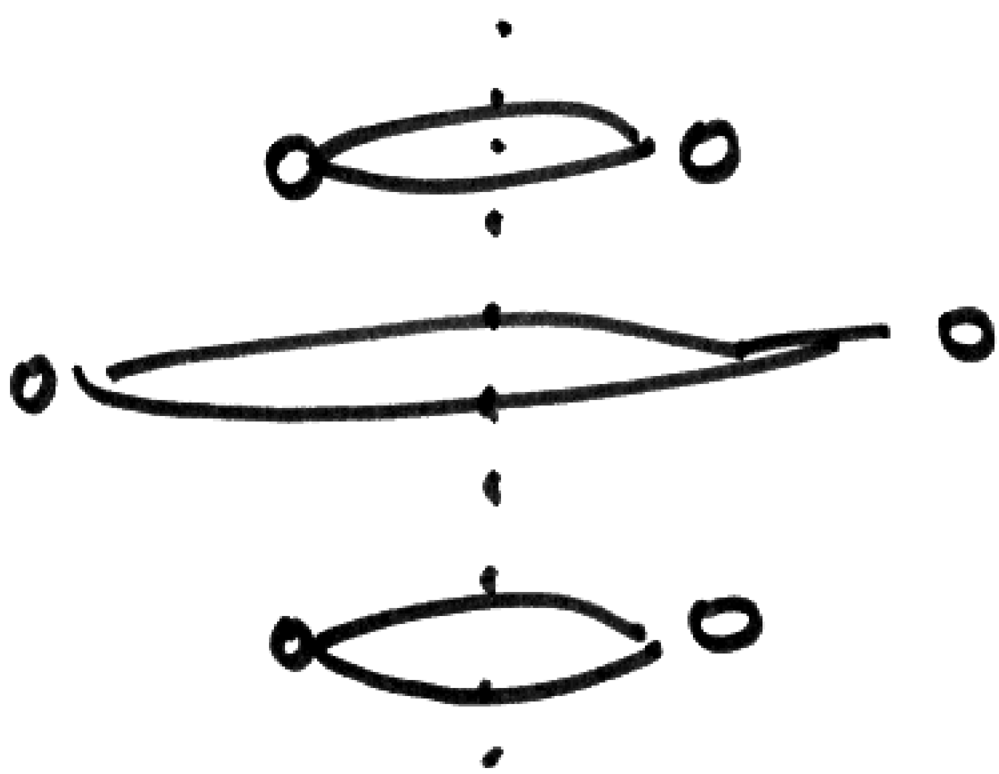
τ^3

$2 \cdot 2 \cdot 2 = 2^3 = 8$



σ_1

2^4



2^3

$$\boxed{\#\text{ necklaces} = \frac{1}{12}\left(m^6 + 2m + 2m^2 + 4m^3 + 3m^4\right)}$$

<u>main term</u>

$$\frac{1}{12} m^6$$

---

polynomial depends only on how $g \in G$ act on vertices of the polygon.

Colorings

G finite group

acting on a set V

$(G \subseteq S(V))$

Coloring

$$\lambda : V \longrightarrow C$$

set of colors

$$X = \{ \lambda \text{ coloring} \}$$

$$= V^{C}$$

G acts on X

$$\boxed{(g\lambda)(v) = \lambda(g^{-1}v)}$$

$$v \in V$$

Want to use Burnside's Lemma. to compute the number N of orbits.

$$\boxed{N = \frac{1}{|G|} \sum_{g \in G} F(g)}$$

$F(g) =$ # fixed points of g acting on X

$X =$ set of colorings

$F(g) =$ $\#\{\lambda \mid g\lambda = \lambda\}$

When is λ fixed by g?

$$g\lambda = \lambda$$

$$(g\lambda)(v) = \lambda(v)$$

for every vertex $v \in V$

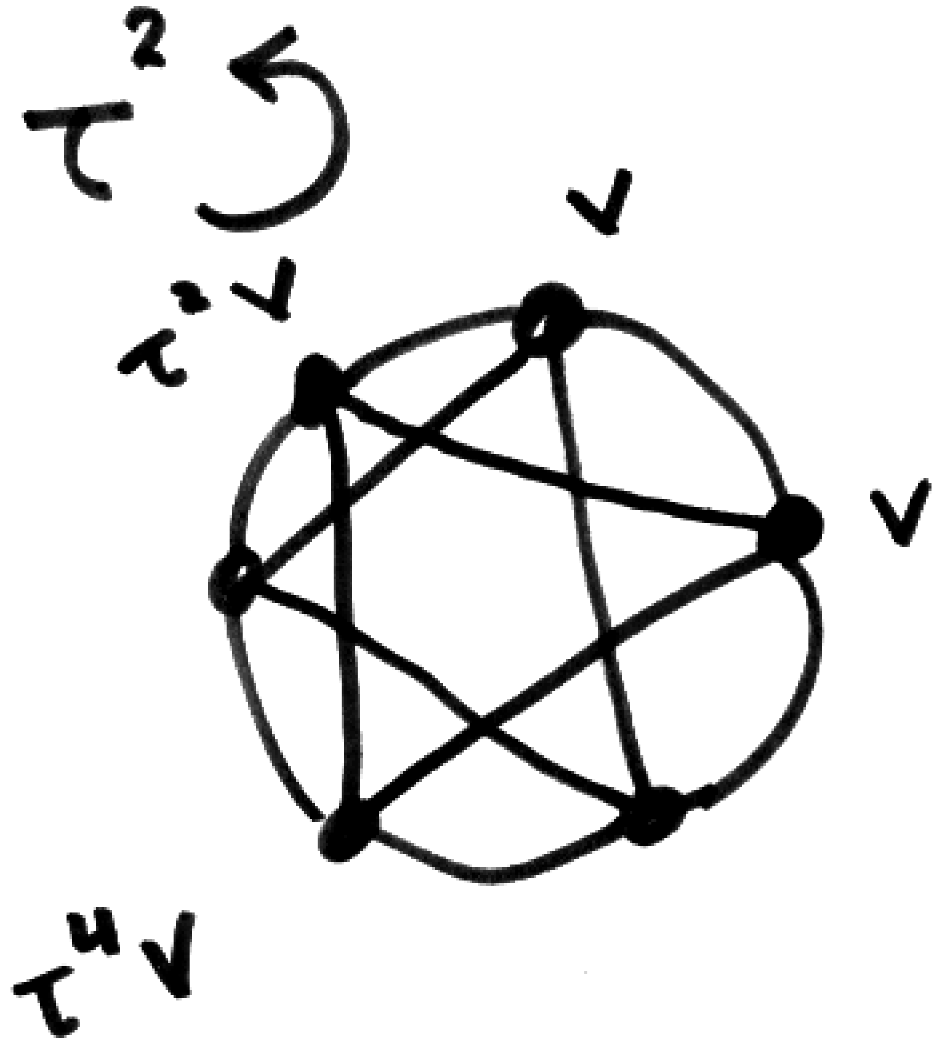
$$\boxed{\lambda(g^{-1}v) = \lambda(v)}$$

$\Rightarrow$ $\lambda(v) = \lambda(gv)$
$= \lambda(gv) = \lambda(g^2v)$
$= \lambda(g^3v) = \lambda(g^4v) \dots$

$\Rightarrow$ λ is <u>constant</u> on an orbit of g

$v, gv, g^2v, g^3v, \dots$

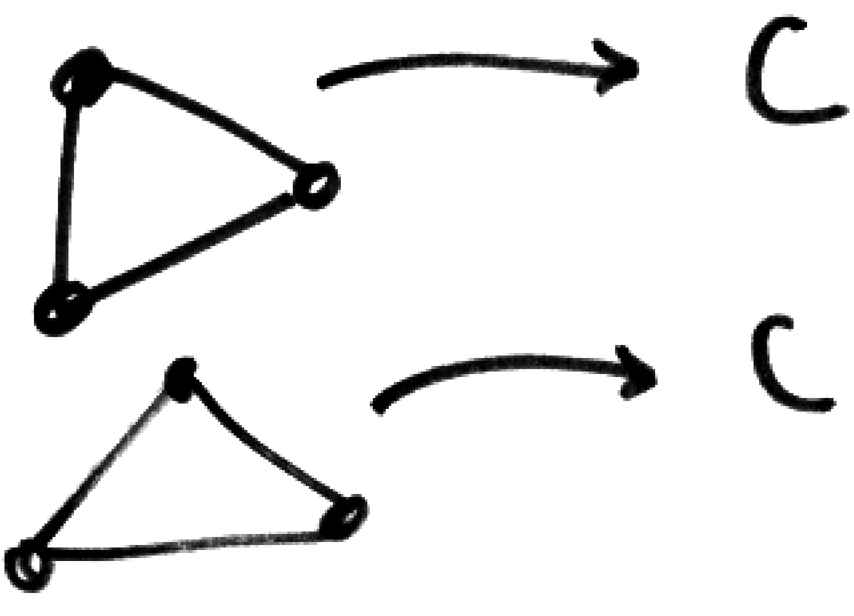
E.g. Necklaces



$n = 6$

λ is fixed by τ^2

$\Leftrightarrow$ λ has constant value in the three vertices of each triangle.



5

To describe fixed colorings λ (fixed by g.) it's ~~enough~~ equivalent to give a color on each orbit of g.

$$F(g) = m^{\ell(g)}$$

$\ell(g) :=$ # of orbits of g
cycles

$m = \# C$

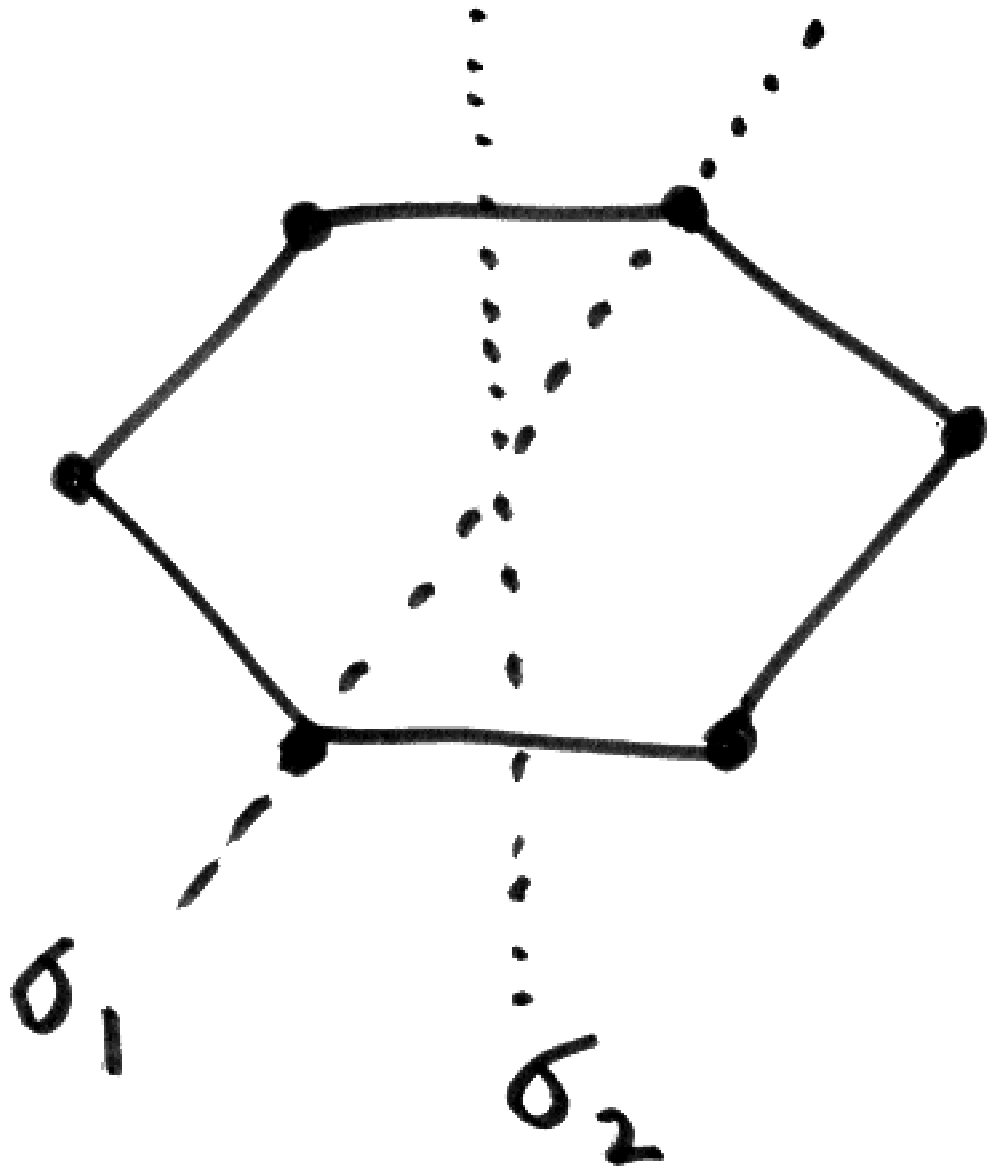
$$\boxed{N = \frac{1}{|G|} \sum_{g \in G} m^{\ell(g)}}$$

cycle indicator

6

Dihedral group D_6 acting on V = regular hexagon $n = \# V = 6$

g	cycle decomp	$\ell(g)$
1	$(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)$	6
τ	$(\cdot\cdot\cdot\cdot\cdot\cdot)$	1
τ^2	$(\cdot\cdot\cdot)(\cdot\cdot\cdot)$	2
τ^3	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$	3
τ^4	$(\cdot\cdot\cdot)(\cdot\cdot\cdot)$	2
τ^5	$(\cdot\cdot\cdot\cdot\cdot\cdot)$	1
σ_1	$(\cdot)(\cdot)(\cdot\cdot)(\cdot\cdot)$	4
σ_2	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$	3
σ_3	$(\cdot)(\cdot)(\cdot\cdot)(\cdot\cdot)$	4
σ_4	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$	3
σ_5	$(\cdot)(\cdot)(\cdot\cdot)(\cdot\cdot)$	4
σ_6	$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$	3



$$N = \frac{1}{12}\left(m^6 + 3m^4 + 4m^3 + 2m^2 + 2m\right)$$

$\uparrow$

cycle indicator for the action of D_6 on vertices of the hexagon.

$G \circlearrowright X$

$F(g) = \#$ fixed pts of g

$N = \#$ orbits

$$\boxed{N = \frac{1}{|G|} \sum_{g \in G} F(g)}$$

- $|G_x| \cdot \# Gx = |G|$

- $y = gx \qquad g \in G$

$$\boxed{G_y = g\, G_x\, g^{-1}}$$

$h \in G_x$

$ghy = ?$

$ghg^{-1}y = ghx$

$= gx = y$

$$\Rightarrow |G_x| = |G_y|$$

<u>**Proof**</u>.

$$\sum_{g \in G} F(g) = \sum_{x \in X} |G_x|$$

$x \in X$ counted in $F(g)$

if $gx = x$

i.e. $|G_x|$ times in the whole

sum

$$\sum_{x \in X} |G_x| = \sum_{Gx} \sum_{y \in Gx} |G_y|$$

orbits

$$= \sum_{G_x} |G_x| \sum_{y \in G_x} 1$$

$$= \sum_{G_x} \underbrace{|G_x|\,|G_x|}_{|G|}$$

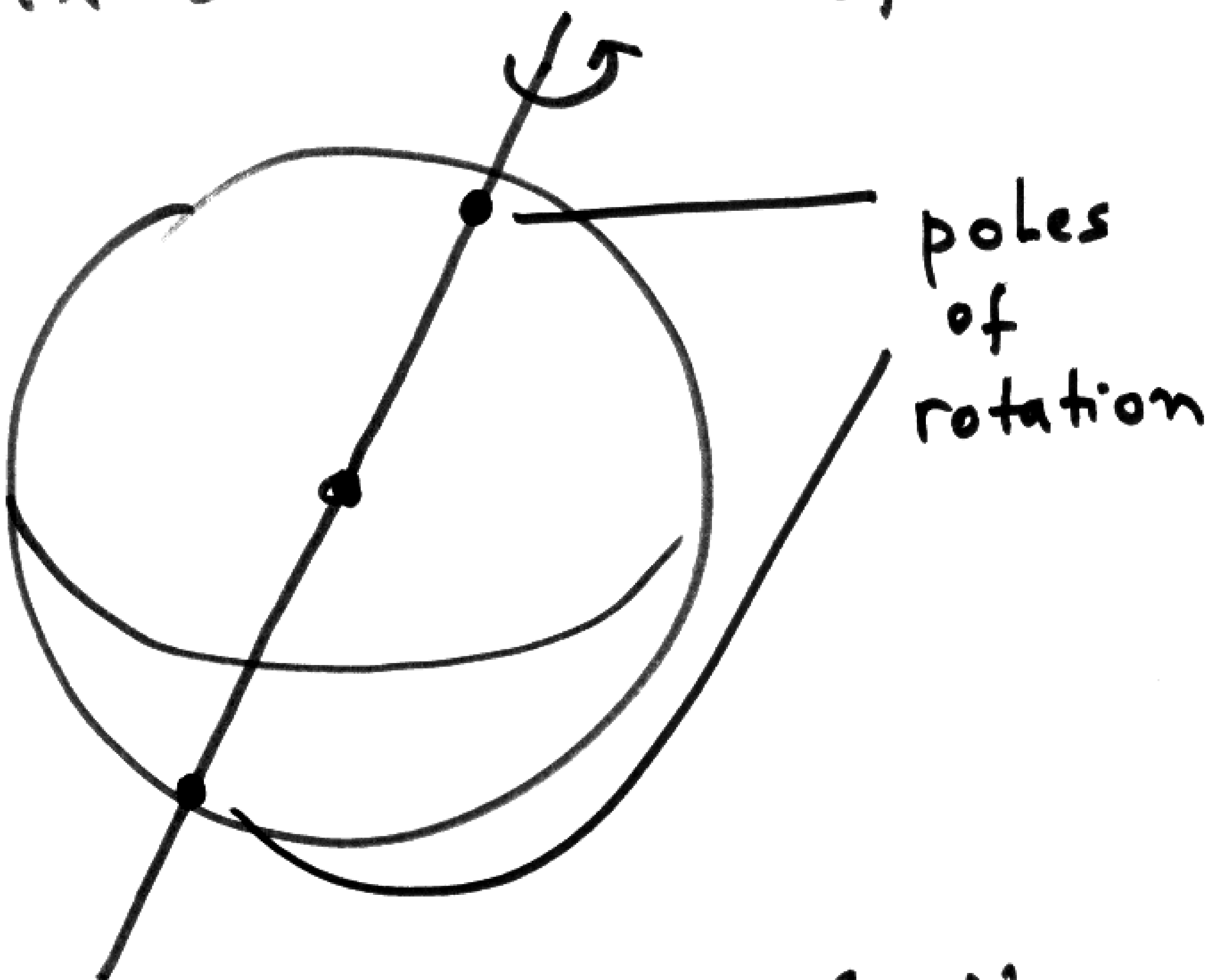
$$= |G| \cdot \sum_{G_x} 1$$

$$= |G| \cdot N \qquad \square$$

Rotations in $\mathbb{R}^3$

Finite groups

A rotation in $\mathbb{R}^3$ fixes the unit sphere



Assume we have a finite group G of rotations $|G| > 1$.

Let $\mathcal{P} = \{$ poles of rotations in $G \}$

finite set.

- G acts on $\mathcal{P}$.

Suppose $P \in \mathcal{P}$

$gP = P$ for some $g \in G$

$h \in G$

<u>Claim</u> hP is also in $\mathcal{P}$

$$\underbrace{hgh^{-1}}_{\in G}(hP) = hP \quad \square$$

$N :=$ # orbits of G acting on $\mathcal{P}$

$$N = \frac{1}{|G|} \sum_{g \in G} F(g)$$

$$F(g) = \begin{cases} \# \mathcal{P} & g = 1 \\ 2 & g \neq 1 \end{cases}$$

$$N = \frac{1}{|G|} \left(\# \mathcal{P} + 2 \left(|G| - 1 \right) \right)$$

$$\boxed{(N-2)\,|G| = \# \mathcal{P} - 2}$$

Since $|G| > 1 \Rightarrow \# \mathcal{P} \geq 2$

$\Rightarrow \underline{N \geq 2}$

If $N = 2$ then $\# \mathcal{P} = 2$

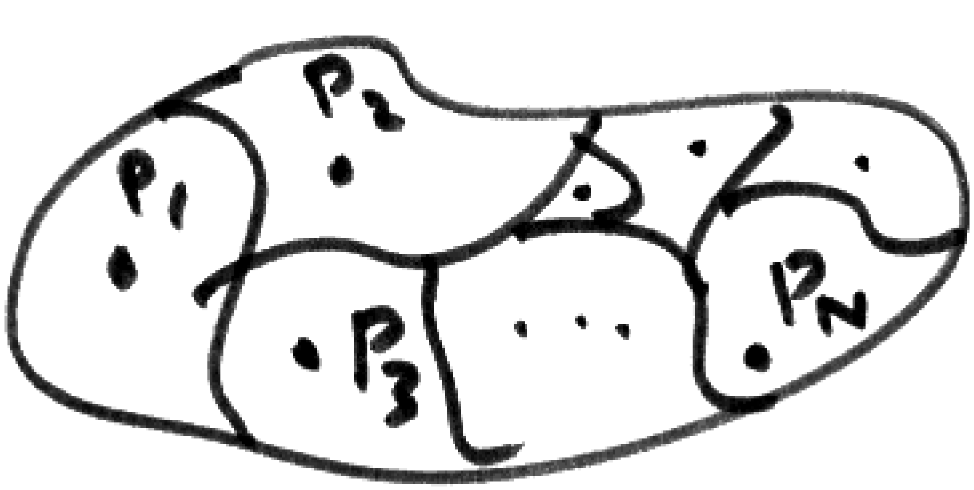
$\Rightarrow$ G cyclic

$1, \tau, \tau^2, \dots, \tau^{n-1}$

If $N = 3$ then $\# \mathcal{P} > 2$

Let $P_1, P_2, \dots, P_N$ be a pole per orbit

$\mathcal{P}$



orbit of P_i has size

$$\frac{|G|}{|G_i|} = \# G P_i$$

where $G_i :=$ Stabilizer of P_i

$$\# \mathcal{O} = \sum_{i=1}^{N} \frac{|G|}{|G_i|}$$

$$\boxed{\frac{\# \mathcal{P}}{|G|} = \sum_{i=1}^{N} \frac{1}{|G_i|}}$$

$$\sum_{i=1}^{N} 1 = N = \frac{\# \mathcal{P}}{|G|} + 2\left(1 - \frac{1}{|G|}\right)$$

Substracting these eqns

$$\boxed{\sum_{i=1}^{N}\left(1-\frac{1}{|G_i|}\right)=2\left(1-\frac{1}{|G|}\right)}$$

(Riemann - Hurwitz formula)

$\left(|G|>1 \Rightarrow |G|\geq 2\right)$

rhs < 2

terms in lhs are at least $1/2$

$|G_i|\geq 2$

$\Rightarrow \quad N=2$ or 3

<u>$N=3$</u>

$G_1, \quad G_2, \quad G_3$

$n_1, \quad n_2, \quad n_3$ sizes

$2 \leq n_1 \leq n_2 \leq n_3$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$|G| \geq 2$

$\Rightarrow \quad n_1 = 2$

XIII

1

$G \subset$ rotations of $\mathbb{R}^3$

$\uparrow$

finite

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$N=3$ orbits of poles

$n_1 \leq n_2 \leq n_3$

$n_i := |G_{p_i}|$ stabilizer of i^{th} orbit of poles

$n_i \geqslant 2$

$|G| > 1$

We can't have $n_1, n_2, n_3 \geqslant 3$ because otherwise

lhs < 1

whereas rhs > 1.

Hence

$$n_1 = 2$$

$$\frac{1}{2} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$$\frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} + \frac{2}{|G|}$$

If $n_2 = 2$ then n_3 could be anything say n

$$\frac{1}{2} + \frac{1}{n} = \frac{1}{2} + \frac{2}{|G|}$$

$$|G| = 2n$$

There is such a G of order $2n$
$G = D_n$ dihedral gp of order $2n$ matches this solution.

$n_1 = n_2 = 2$ $\quad n_3 = n$ $\quad |G| = 2n$

③

$$\frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} + \frac{2}{|G|}$$

If $n_2 > 2$ $\quad (n_3 \geq n_2)$

then $n_2 = 3$ $\quad$ Otherwise

$$lhs \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

whereas

$$rhs > \frac{1}{2}$$

$$\boxed{n_2 = 3}$$

$$\frac{1}{3} + \frac{1}{n_3} = \frac{1}{2} + \frac{2}{|G|}$$

$$\frac{1}{n_3} = \frac{1}{6} + \frac{2}{|G|}$$

$n_3 < 6$

$n_3 = 3, 4, 5$

The equation

$$1 + \frac{2}{|G|} = \boxed{\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1}$$

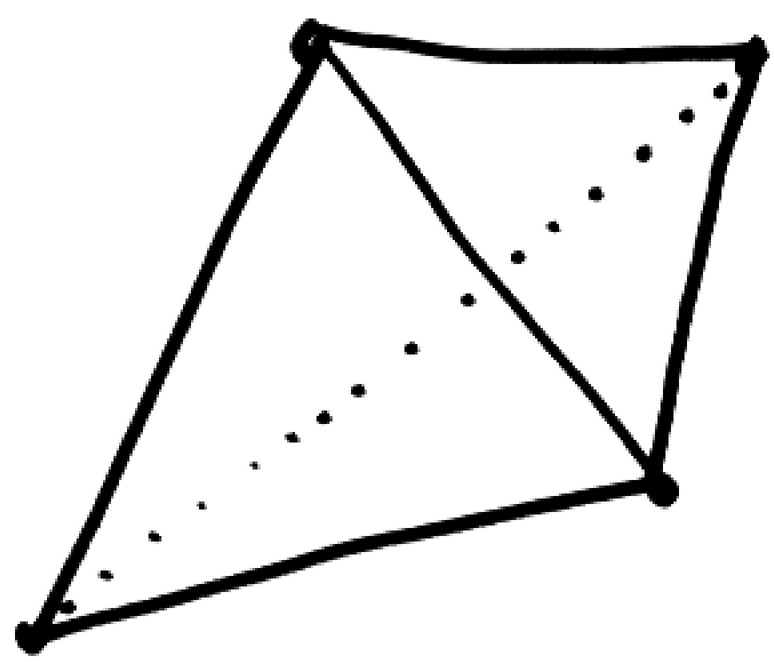
$n_i \geqslant 2$ n_i integer.

has solutions

	$\lvert G\rvert$	n_1, n_2, n_3	
D_n	$2n$	2, 2, n	Dihedral, n-gon
A_4	12	2, 3, 3	Tetrahedron
S_4	24	2, 3, 4	Cube / octahedron
A_5	60	2, 3, 5	Icosahedron / Dodecahedron

There is a corresponding group rotations fixing either a regular n-gon or one of the platonic solids

Tetrahedron



$R_V = R_F$ order 3, 4 vertices (4 faces)

total of 8

R_E order 2, 6 edges

total of 3

identity

$1 + 8 + 3 = 12$

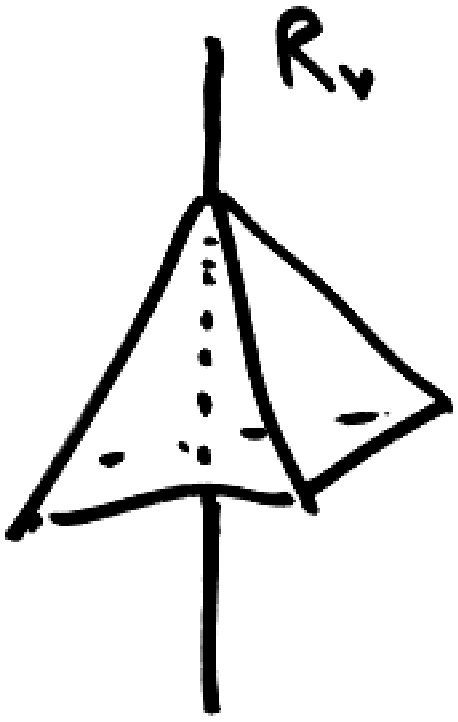
corresponds to $(2,3,3)$

$G \simeq A_4$

(6)

$G \simeq A_4$

comes from viewing G as acting on vertices (or equivalently on faces)



$R_v \longleftrightarrow (123)$

$R_E \longleftrightarrow (12)(34)$

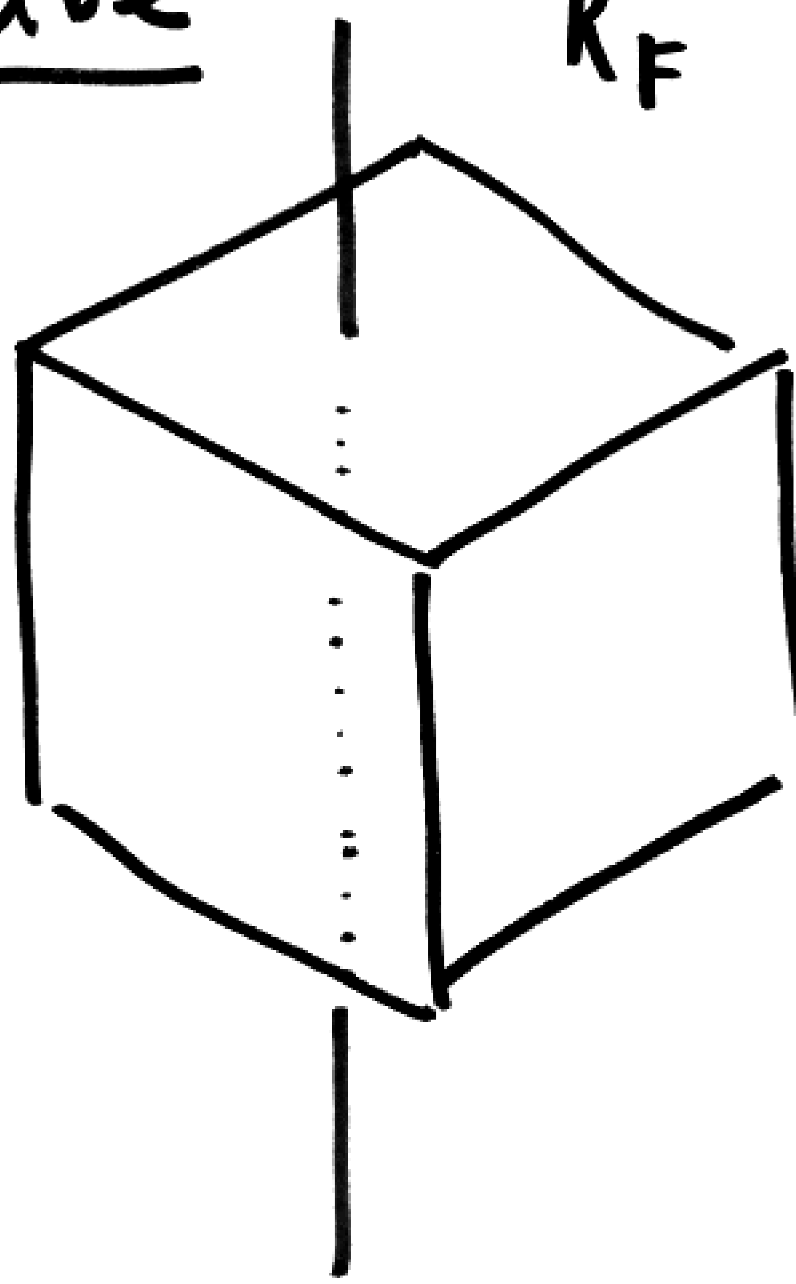
R_v's $\longleftrightarrow$ 3 - cycles

3-cyles $\begin{cases} (123) & (132) \\ (124) & (142) \\ (134) & (143) \\ (234) & (243) \end{cases}$

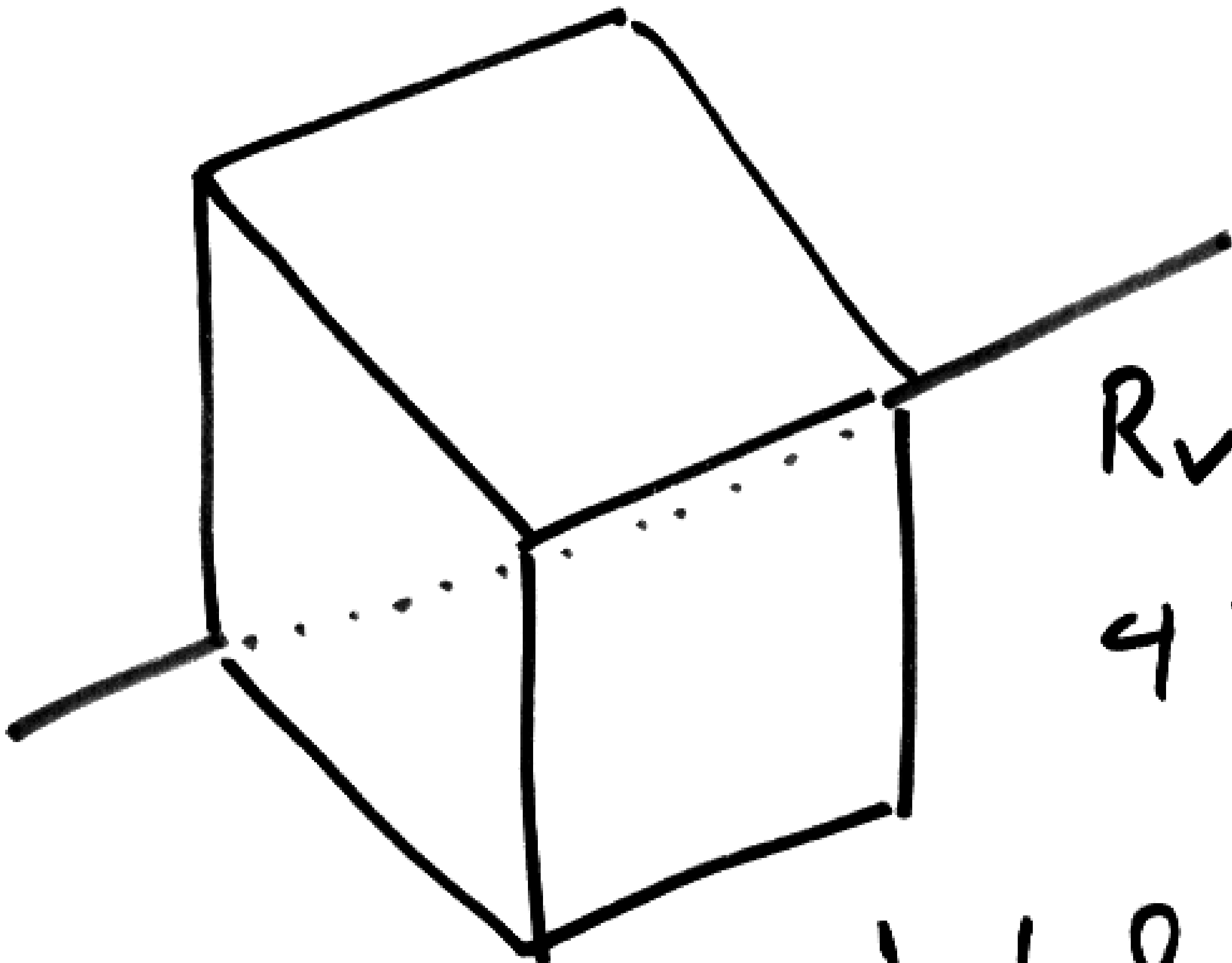
R_E's $\longleftrightarrow$ $(\cdot\cdot)(\cdot\cdot)$

$$\left\{\begin{array}{l}(12)(34)\\(13)(24)\\(14)(23)\end{array}\right.$$

<u>Cube</u>



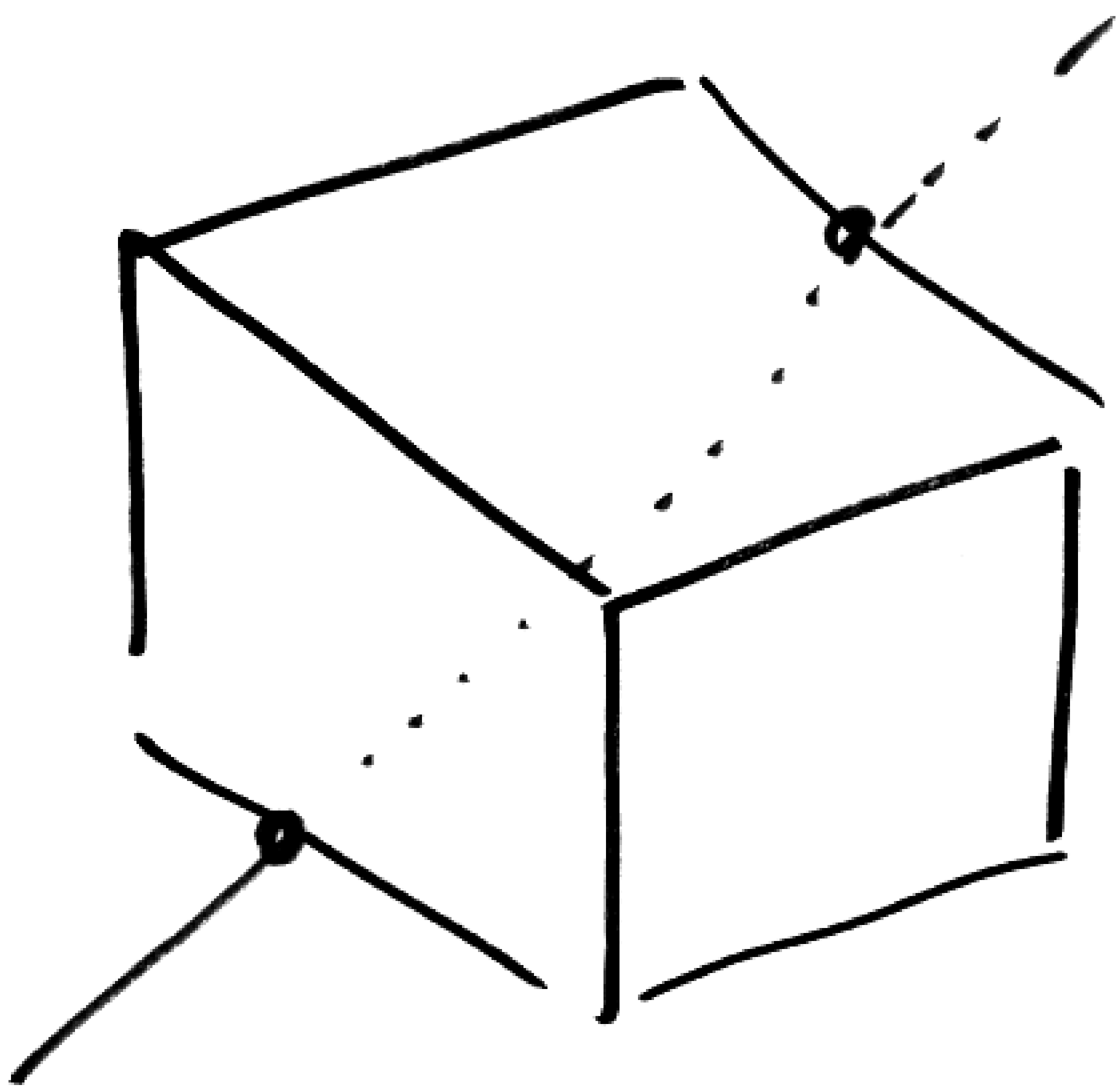
order 4
3 pairs of faces
total = 9



R_V order 3

4 pairs of opp vertices

total = 8



R_E order 2

6 pairs of opp. edges

total = 6

$\underline{1} + 9 + 8 + 6 = 24$

corresponds (2, 3, 4)

$G \simeq S_4$

We can see this by looking at the action of G on pairs of opposite vertices (or diagonals)

$R_F^2 \leftrightarrow (\cdot\cdot)(\cdot\cdot)$

$R_F \longleftrightarrow (\cdot\cdot\cdot\cdot)$

$R_V \longleftrightarrow (\cdot\cdot\cdot)$

$R_E \longleftrightarrow (\cdot\cdot)$

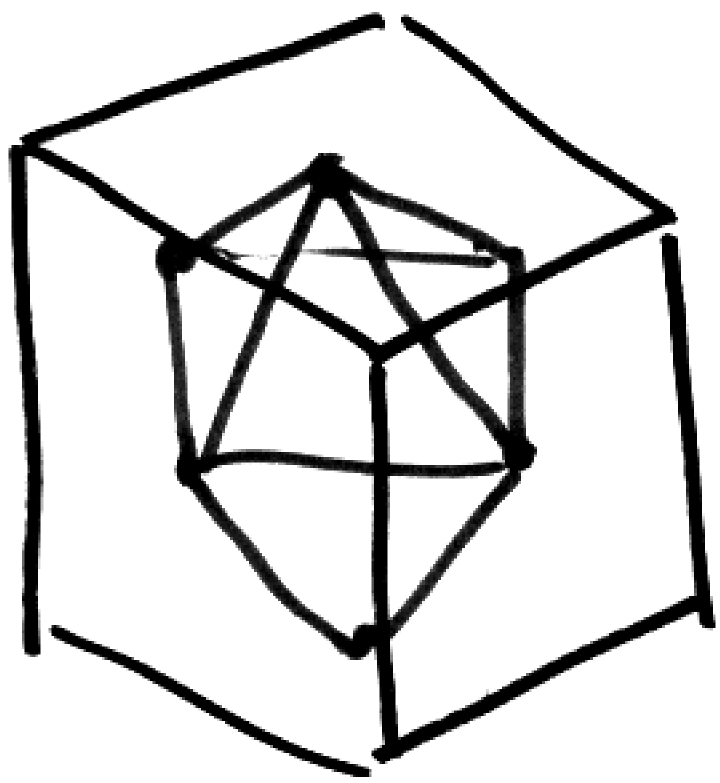
swaps diagonals connected to edge

(12) (13) (14)

(23) (24)

(34)

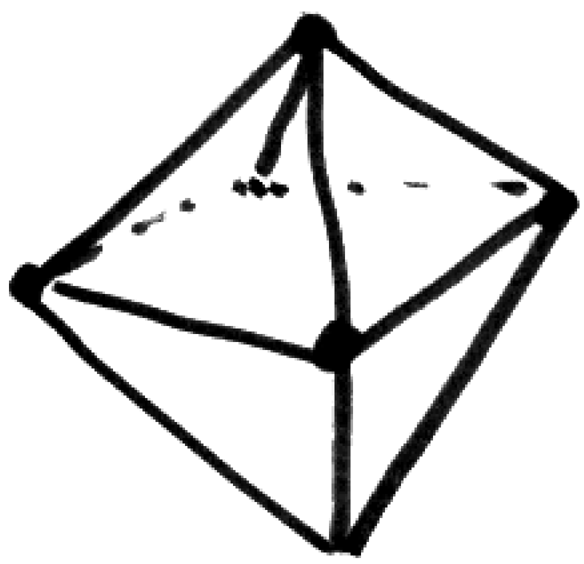
Platonic Solids



cube

$\updownarrow$

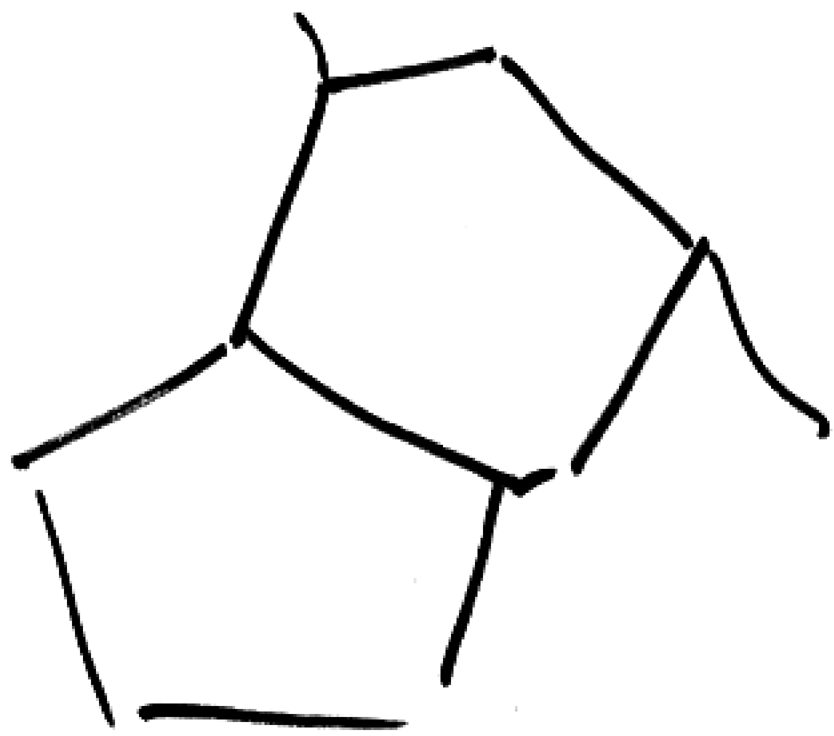
octahedron



share same group of rotations

(2, 3, 5) solution

corresponds dodecahedron
/icosahedron

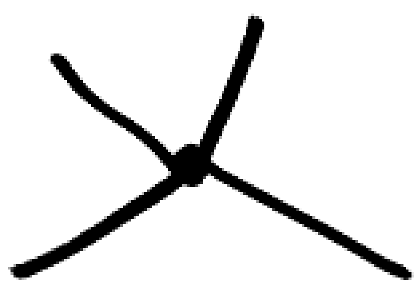


$G \simeq A_5$

Regular polyhedron

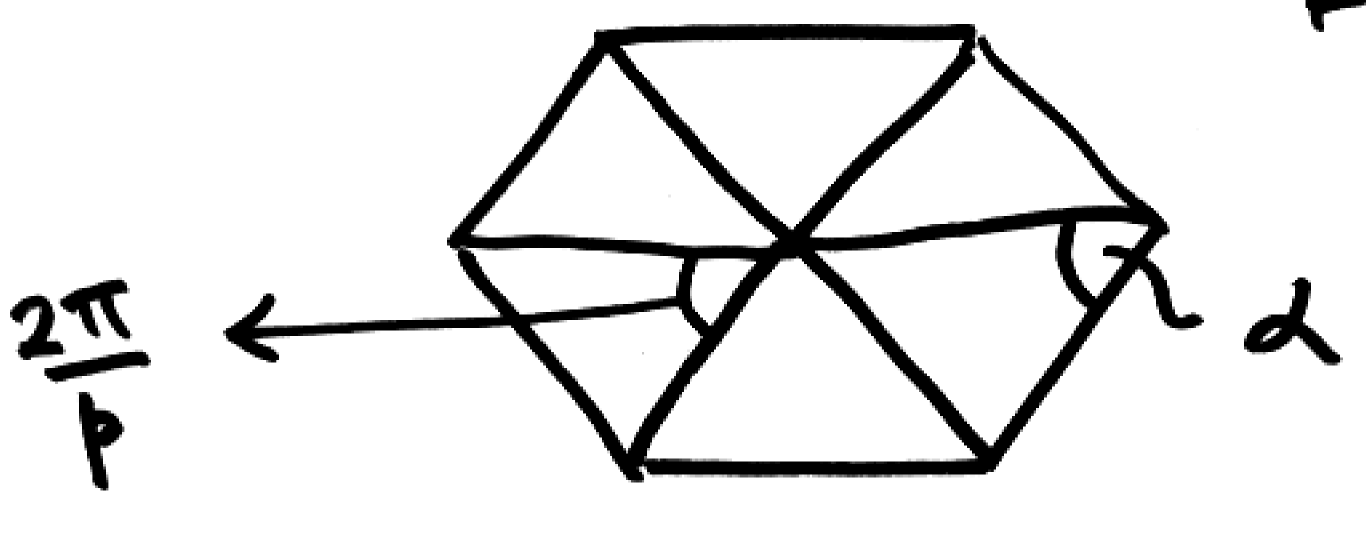
Putting together faces

- all faces are the same
- same number of faces per vertex

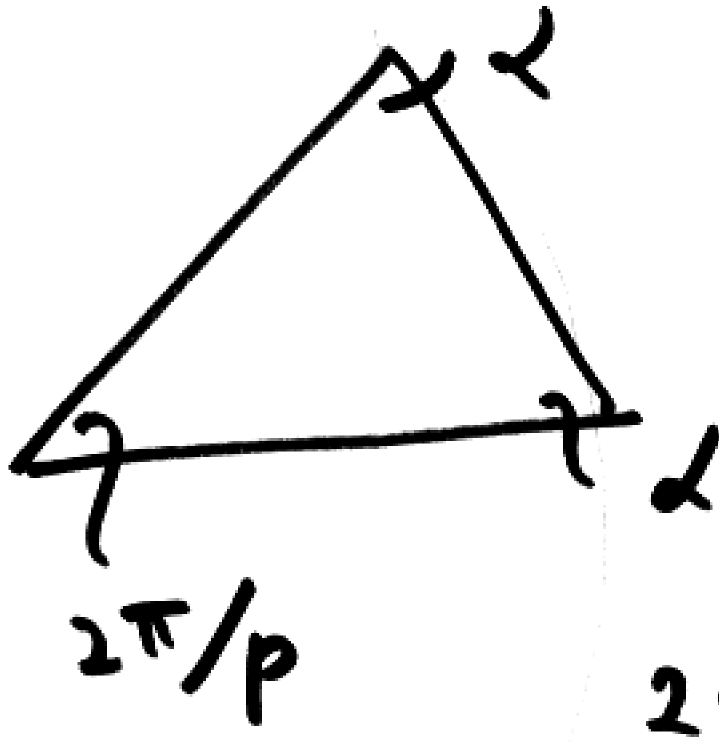


(As symmetrical as possible)

Face



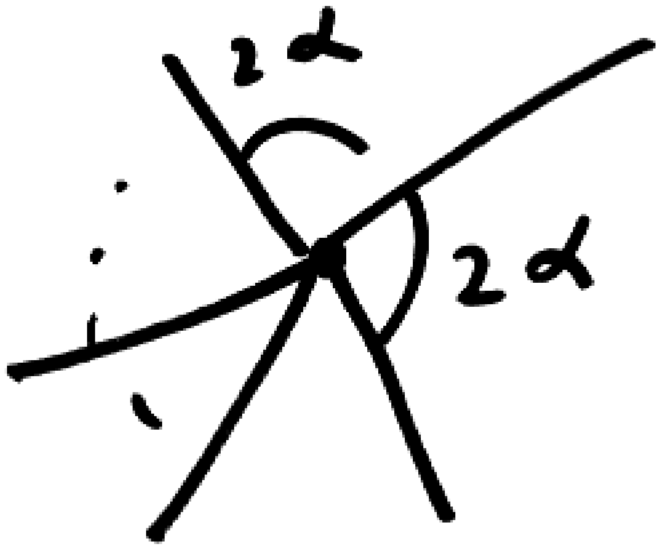
p = the number of edges on a face



$$2\alpha + \frac{2\pi}{p} = \pi$$

$$\Rightarrow \quad 2\alpha = \pi\left(1 - \frac{2}{p}\right)$$

At a vertex



q = # of faces at a vertex
= # of edges at a vertex

$$q\,2\alpha < 2\pi$$

$$q\pi\left(1-\frac{2}{p}\right) < 2\pi$$

$$q\left(1-\frac{2}{p}\right) < 2$$

$$q(p-2) < 2p$$

$$qp - 2q - 2p < 0$$

$$qp - 2q - 2p + 4 < 4$$

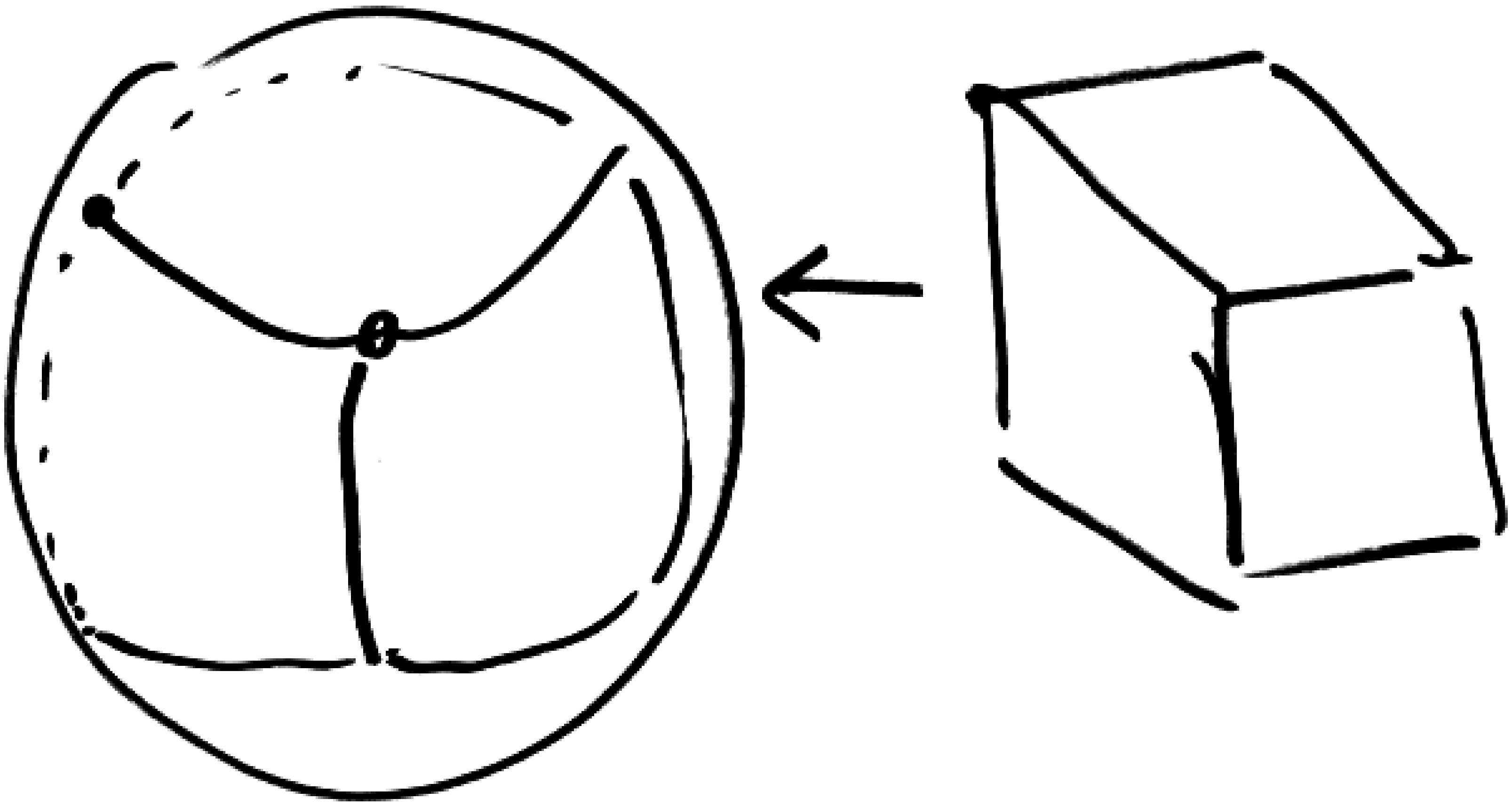
$$(q-2)(p-2) < 4$$

$$\Rightarrow \quad (p,q) = \begin{array}{l} \begin{array}{ll} p & q \end{array} \\ 3,\ 3 \longleftrightarrow T \\ \left[\begin{array}{l} 3,\ 4 \longleftrightarrow O \\ 4,\ 3 \longleftrightarrow C \end{array}\right. \\ \left[\begin{array}{l} 3,\ 5 \longleftrightarrow \\ 5,\ 3 \longleftrightarrow \end{array}\right. \end{array}$$

<u>Euler</u>

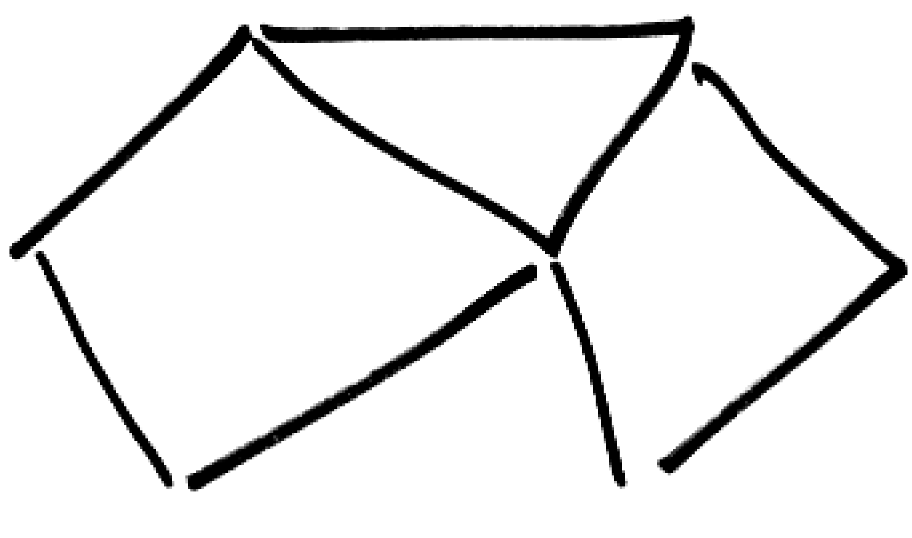
$$\begin{array}{ccccccc} V & - & E & + & F & = & 2 \\ 0 & & 1 & & 2 & & \end{array}$$

Take platonic solids
and pump air into them



Euler number = V − E + F
is a topological invariant
sphere has Euler number
2.

Take



Group fixing corners

$\simeq \{(\alpha_1, \dots, \alpha_8) \mid \alpha_1 + \dots + \alpha_8 \equiv 0 \mod 3$

$\alpha_i \mod 3$

$=$ angle of rotation at corner i

Moves that do little.

<u>Commutators</u> $x, y \in G$

$[x,y] = x y x^{-1} y^{-1}$

<u>Note</u> $xy = yx \Rightarrow xyx^{-1} = y$

$\Rightarrow xyx^{-1}y^{-1} = 1$

x, y commute

$[x,y] = 1$

Conversely if

$[x,y] = 1 \Rightarrow xy = yx$

G a group of permutations

$x, y \in G$

<u>Know</u>: x, y are disjoint (i.e. involve different sets of objects) they commute

e.g. $x = (123)(45)$

$y = (6789)$

$\Rightarrow \quad xy = yx$

$\Rightarrow \quad [x, y] = 1$

<u>Example</u> $x = (1234)$

$y = (1567)$

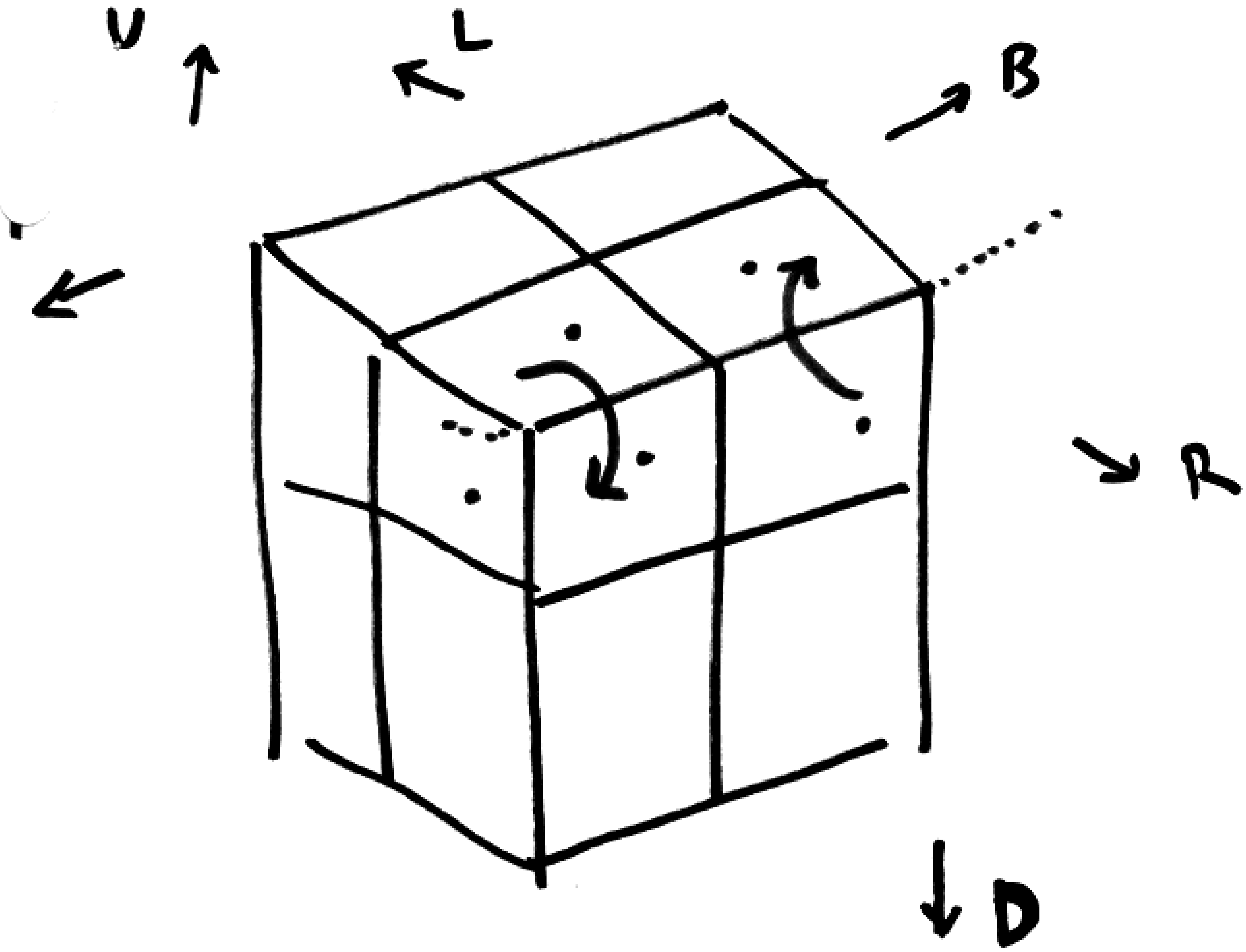
$[x,y] = xyx^{-1}y^{-1} = (xyx^{-1}) \cdot y^{-1}$

$xyx^{-1} = (2567)$

$xyx^{-1}y^{-1} = (2567)(7651)$

$= (125)$

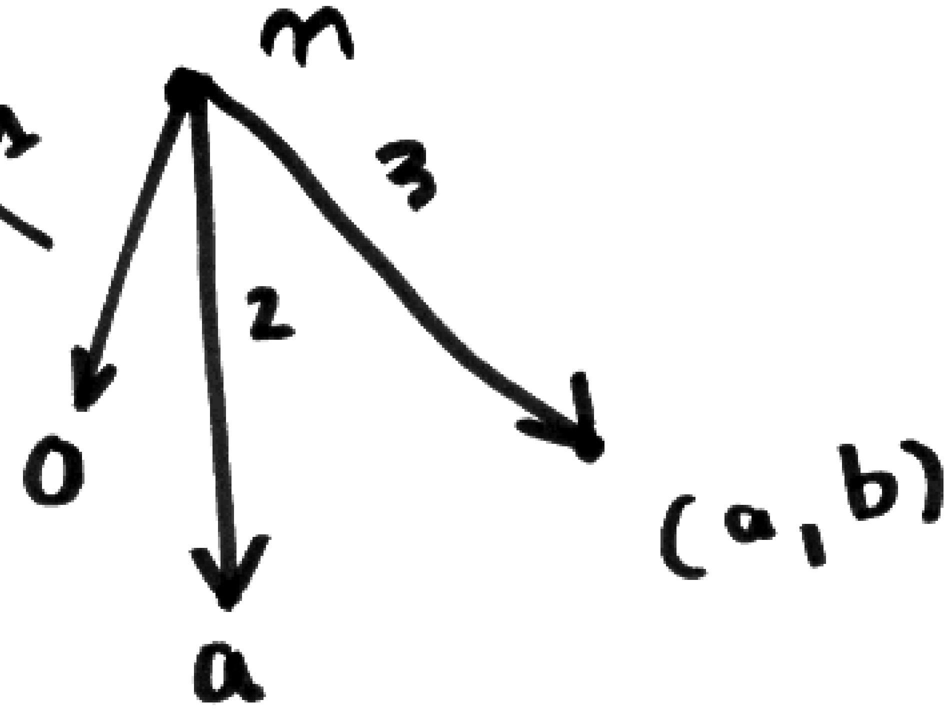
③



$$[F, D]^2\, U\, [D, F]^2\, U^{-1}$$

③

[illegible] only
[illegible] the H



$a < n$

$a < b < n$

$$\begin{array}{ccc} a & \quad & n \\ \times & & H \\ I & & \\ \downarrow & & \downarrow \\ x, H & & T \end{array}$$

$$G(n) = \operatorname{mex}\{0,\ G(a),\ G(a) * G(b)\}$$

$a < n$
$a < b < n$

n	1	2	3	4	5	6	7	8	9
	1	2	4	7	8	11	13	14	16

$$G(n+1) = \begin{cases} 2n \\ 2n+1 \end{cases}$$

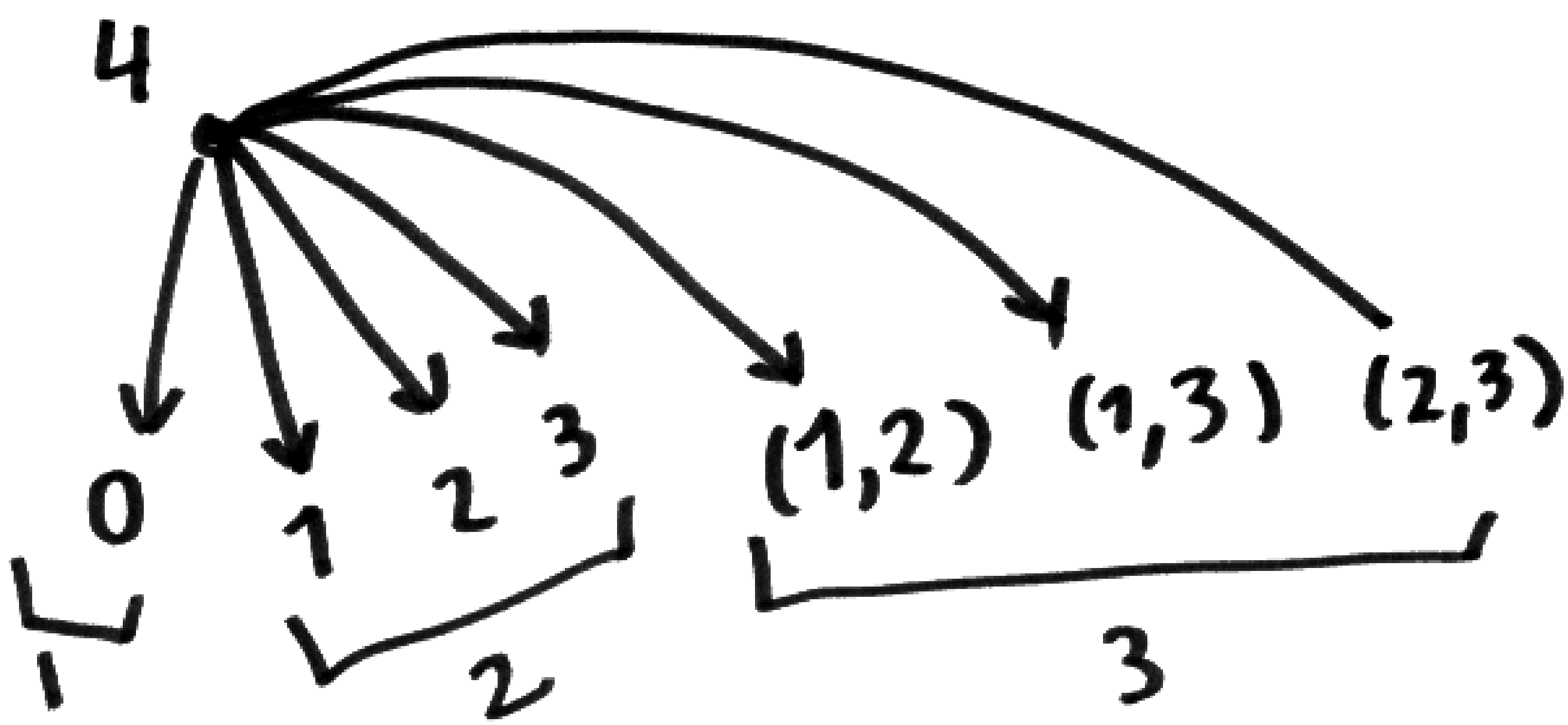
which one depends on the parity of the sum of the binary digits. Want the answer to have ~~[illegible]~~ sum to be odd.

1	2	3	4	5	6	7	8	9
H	T	T	H	T	H	H	T	T
1			7		11	13		

$$1 * 7 * 11 * 13 = 6 * 11 * 13$$
$$= 13 * 13$$
$$= 0$$

This is a P-position

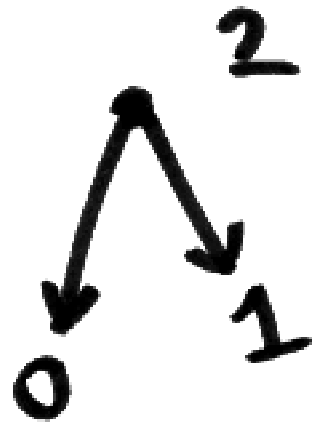
4



$$G(4) = \text{mex}\{0, 1, 2, 4, 3, 5, 6\} = 7$$

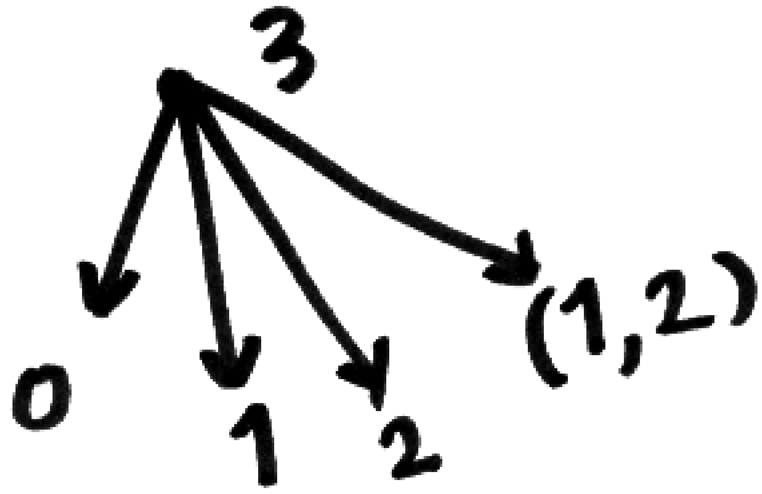
1 2 3 4
H T T T… ④

$G(1) = \text{mex}\{0\} = 1$



X H T T….

$G(2) = \text{mex}\{0, 1\} = 2$



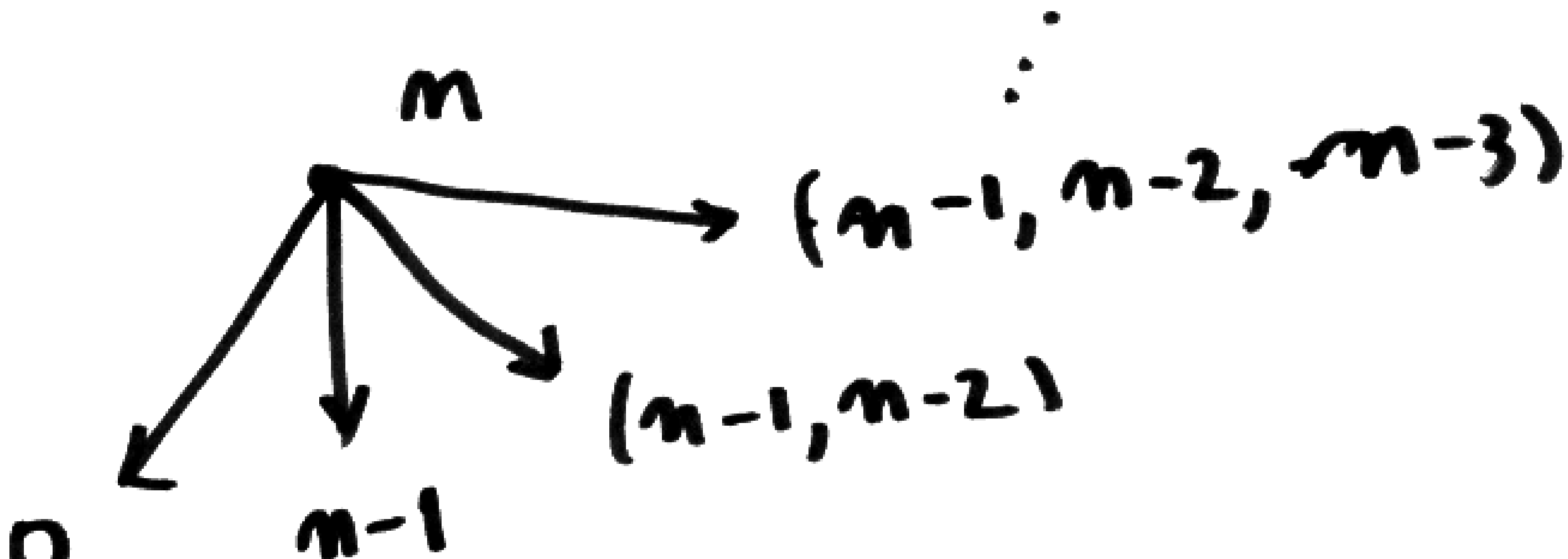
X Y H T….

$G(1,2) = G(1) \oplus G(2)$
$= 3$

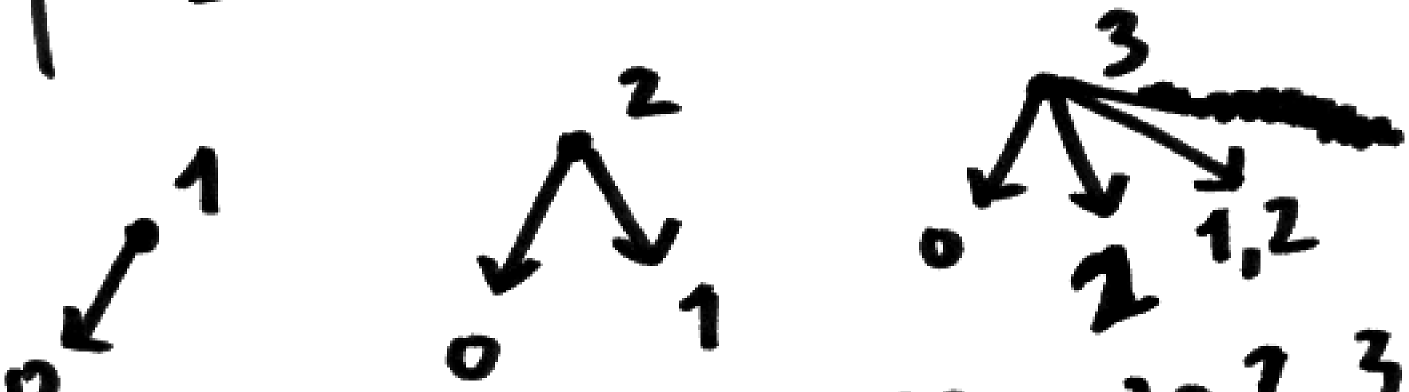
$G(3) = \text{mex}\{0, 1, 2, 3\} = 4$

Ruler Game

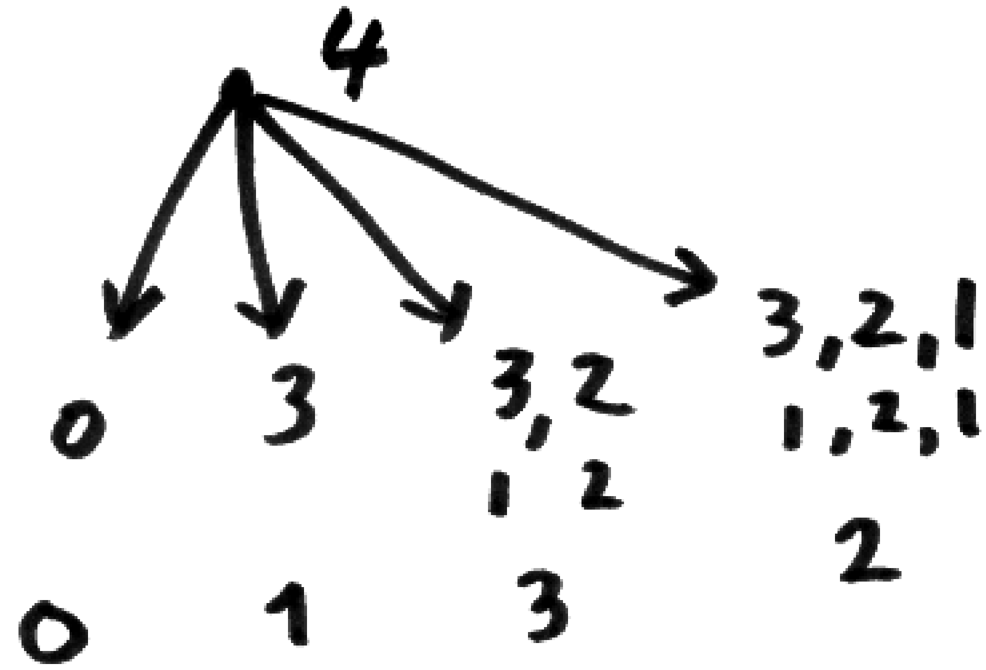
Turn any number of **consecutive** coins.
(Rightmost H → T)

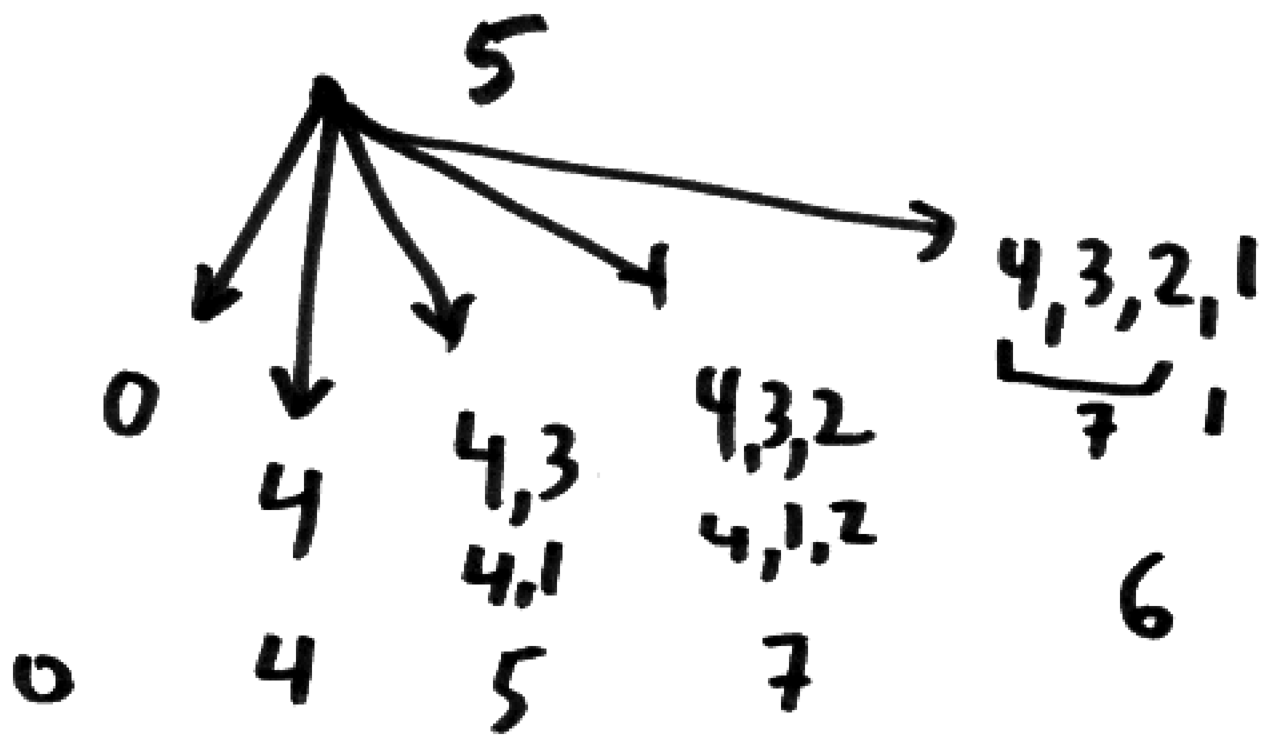


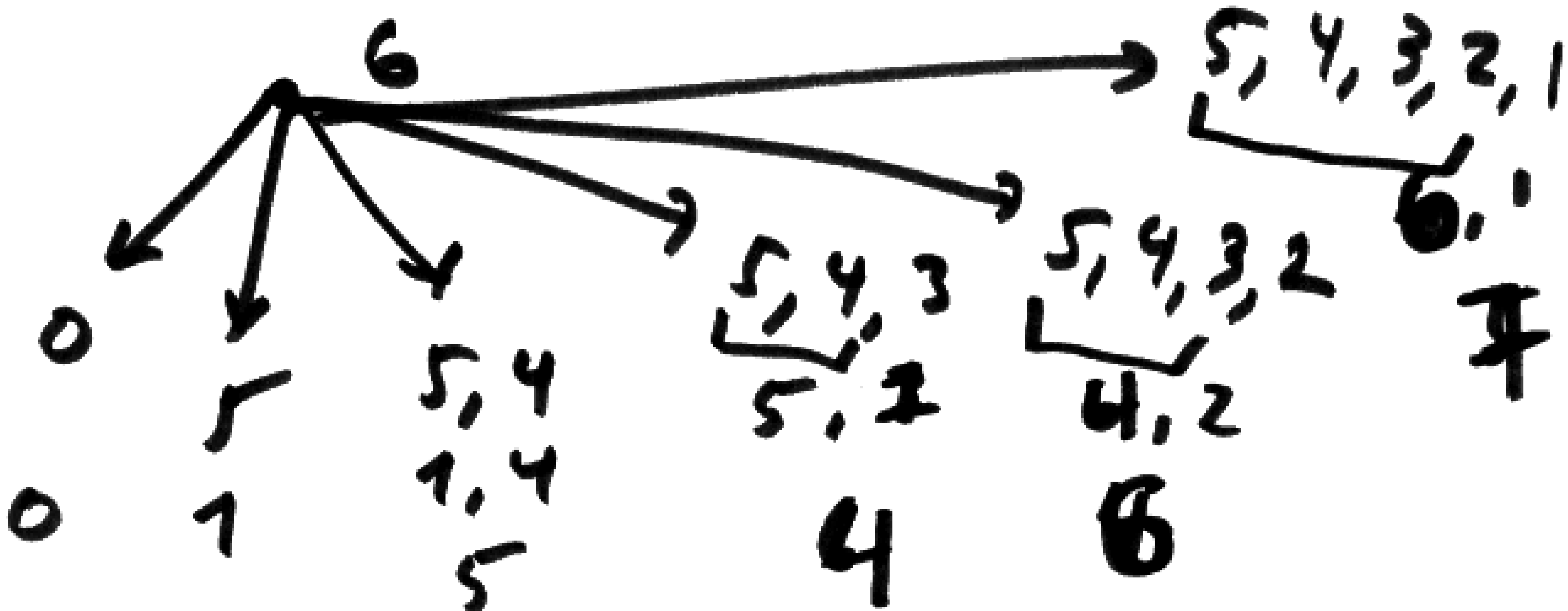
n	1	2	3	4	5	6	7	8
	1	2	1	4	1	2	1	8



mex {0, 2, 3}





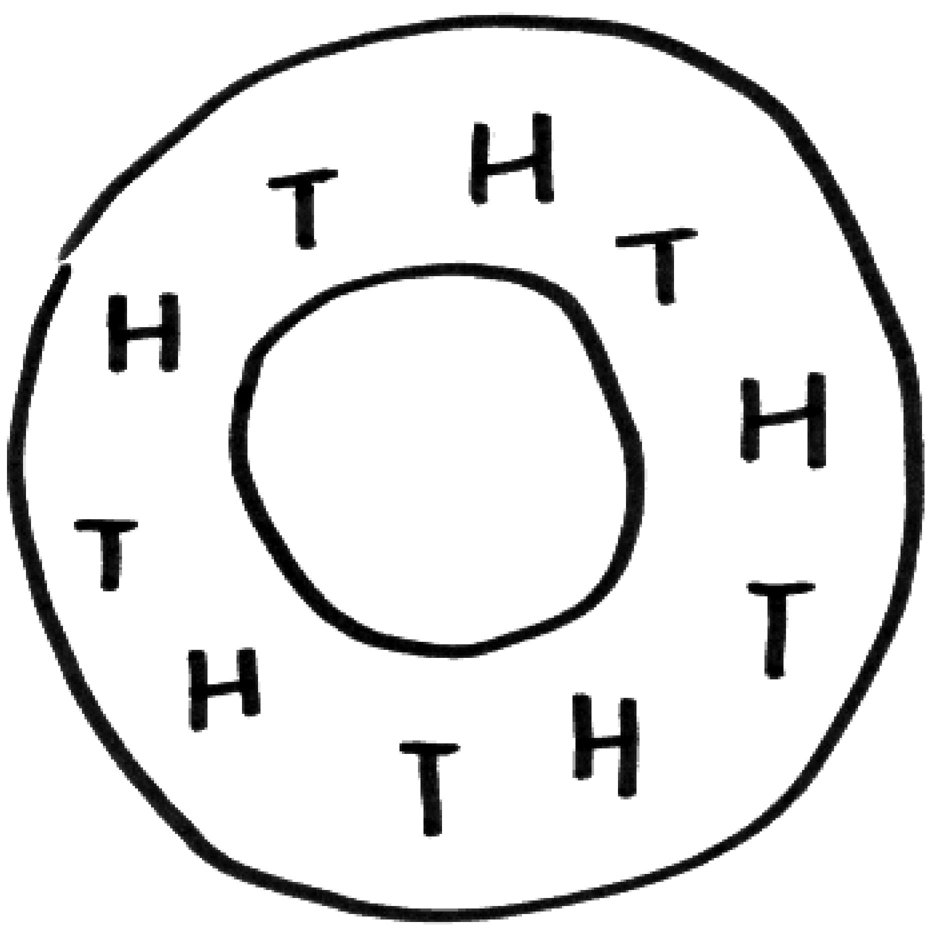


$$\boxed{G(n) = 2^{v_2(n)}}$$

Theory for Misére

person unable to play wins.

BLET



Rule

HTH $\longleftrightarrow$ THT

Goal Find the largest number of H's possible

Greedy play

THT → HTH

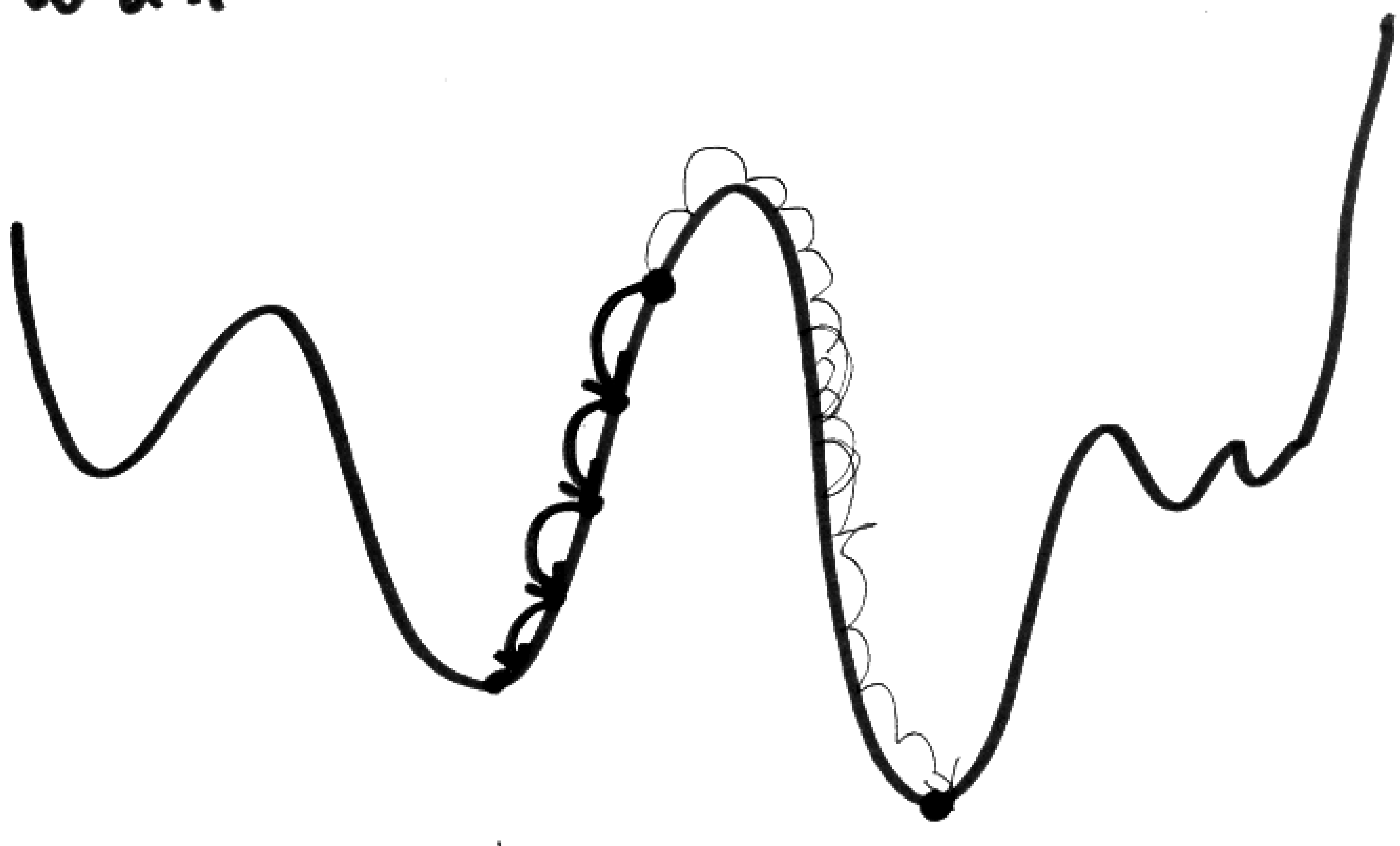
H
T
H
T
H

H
H
T
H
H

Greed fails

Simulated annealing

- Discrete space
- function
- want: minimum value



Pick neighbor
- If smaller value take it
- If not flip coin and take it or not according to result.

. Let the computer play the game by simulated annealing.

H T H → T H T

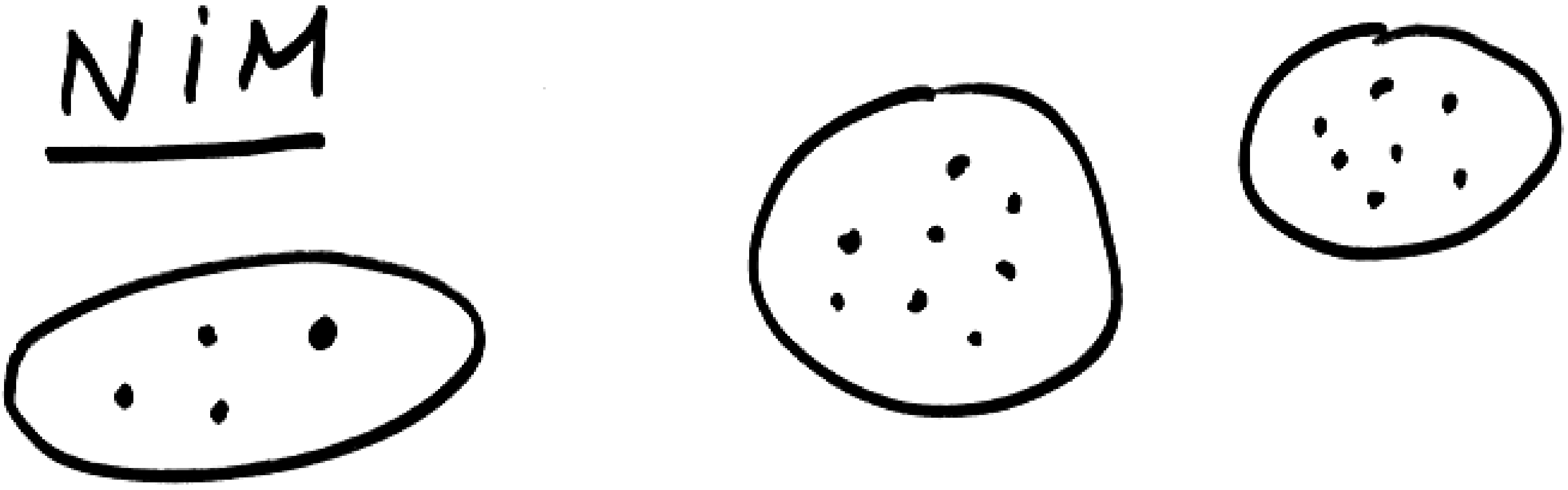
XVB 1

Conway, Berlekamp
Guy.

Winning Ways

<u>Impartial games</u>

<u>NiM</u>



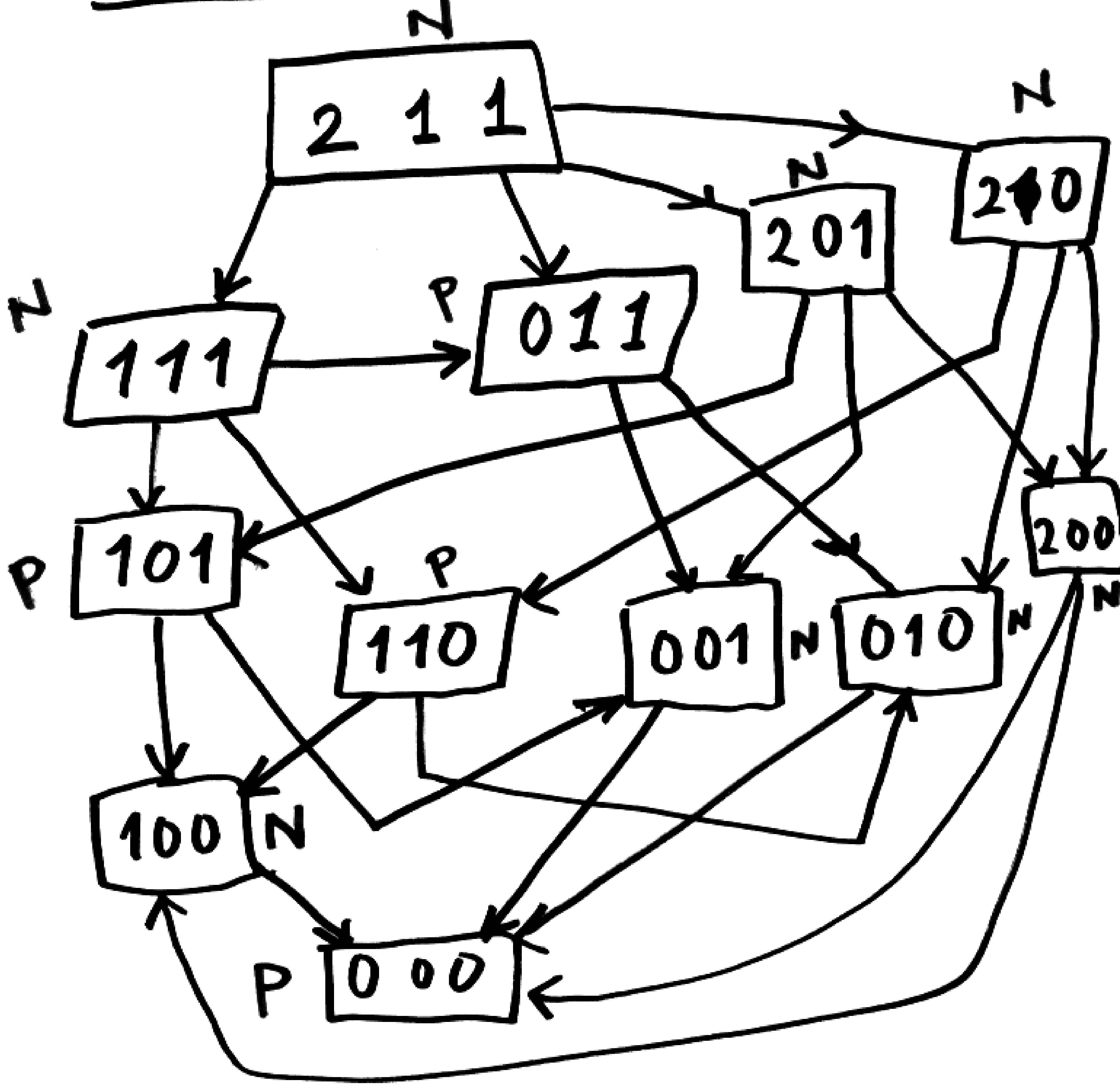
<u>Play</u>: Take away any number of coins from one pile.

<u>End</u>: If you can't play you lose

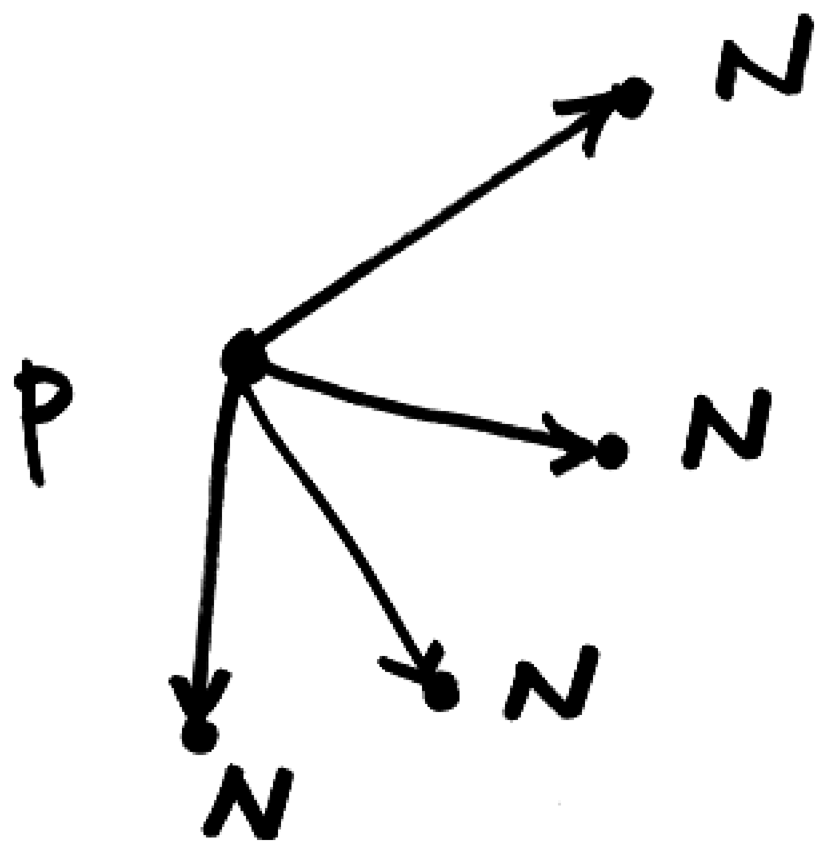
(Normal play)

Complete Analysis & Strategy

P/N - positions

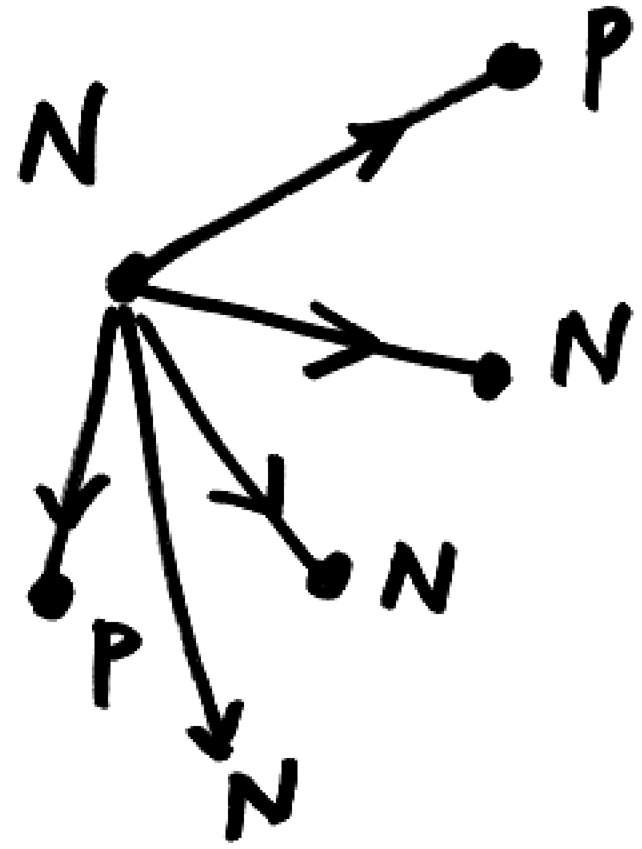


P - position



reach _only_ N

N - position



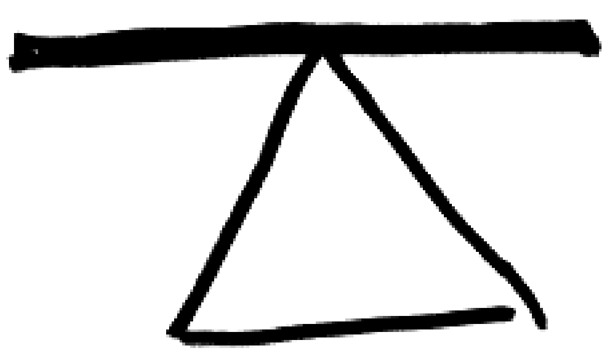
reach _at least_ one P

Strategy

- Reach P
- Opponent moves to N

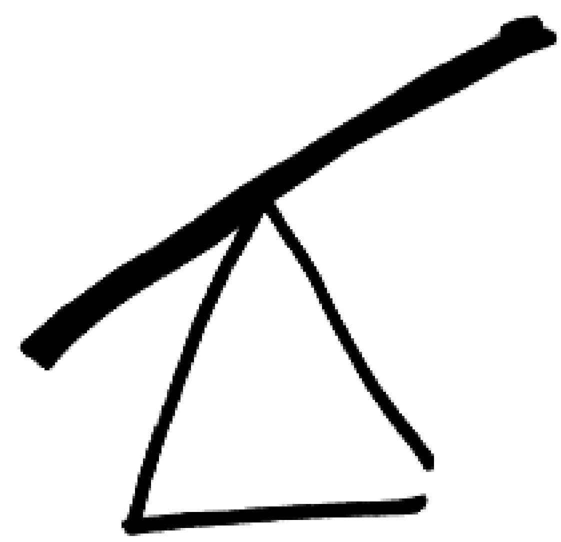
Analogy

P



equilibrium

N



not-equil.

For any directed graph (w/ no loops) we can define recursively P/N labelling of the vertices.

Ch. Bouton 1901-02

Ann. of Math. JSTOR

Nim addition

$5 * 7 = 2$

$$\begin{array}{r} 101 \\ 111 \\ \hline 010 \end{array}$$

Bouton

1) $(n_1, n_2, \dots, n_k)$

$n_i = 0, 1, 2, \dots$

\# of objects in ith pile

position is P

if and only if

$$\boxed{n_1 * n_2 * \dots * n_k = 0}$$

(2, 1, 1) N

$$\begin{array}{c} 10 \\ 01 \\ 01 \\ \hline 10 \end{array}$$

$n * n = 0$

2) If we have an N position we can move to P as follows.

E.g. $(7, 5, 6)$

$$
\begin{array}{c|c}
111 & 7 \\
101 & 5 \\
110 & 6 \\
\hline
100 & 4
\end{array}
$$

$7 * 5 * 6 = 4$

change $7 \to 011 = 3$

$$
\begin{array}{c}
011 \\
101 \\
110 \\
\hline
000
\end{array}
$$

$3 * 5 * 6 = 0$

8

$(15,5,13)$

1	1	1	1	15
0	1	0	1	5
1	1	0	1	13
0	1	1	1	7

N

$\downarrow$

$(15,2,13)$

1	1	1	1	15
0	0	1	0	2
1	1	0	1	13
0	0	0	0	0

P

$$\boxed{n_1 * \cdots\cdots * n_K = 0}$$

Impartial games

- Two players, alternate
- same moves
- No chance
- complete information
- no ties / endgames

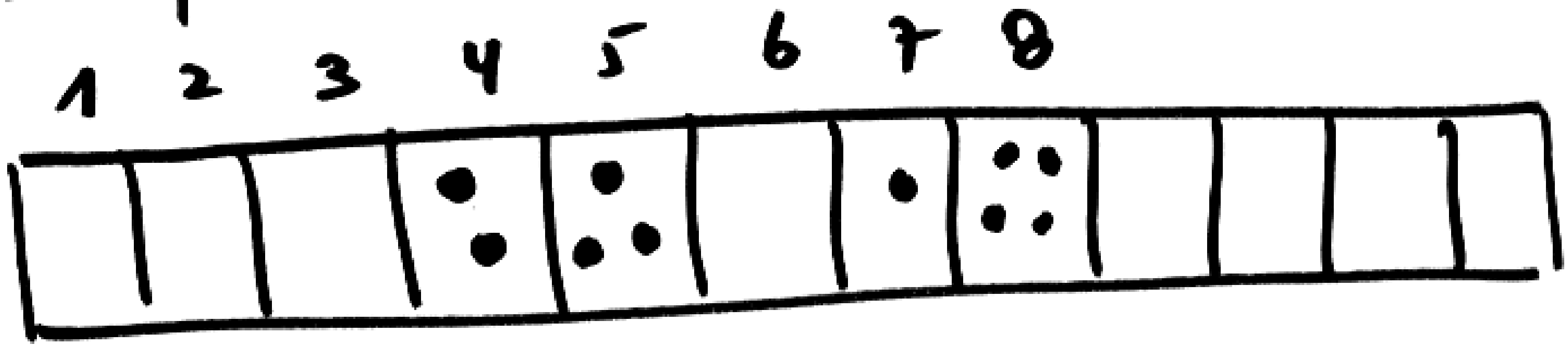
player unable to move loses
(Normal play)

(opposite ... wins)
Misére play

THM All impartial games are isomorphic to NIM.

1) **Nimble**

k pennies



Rule: take one penny and move it to the left.

penny $\longleftrightarrow$ pile

position $\longleftrightarrow$ # of objects in pile.

$n_1, \dots, n_k$ position of pennies

$P \Leftrightarrow n_1 * \dots * n_k = 0$

2) Turning turtles

H T H H T T H

Rule: Turn some H → T

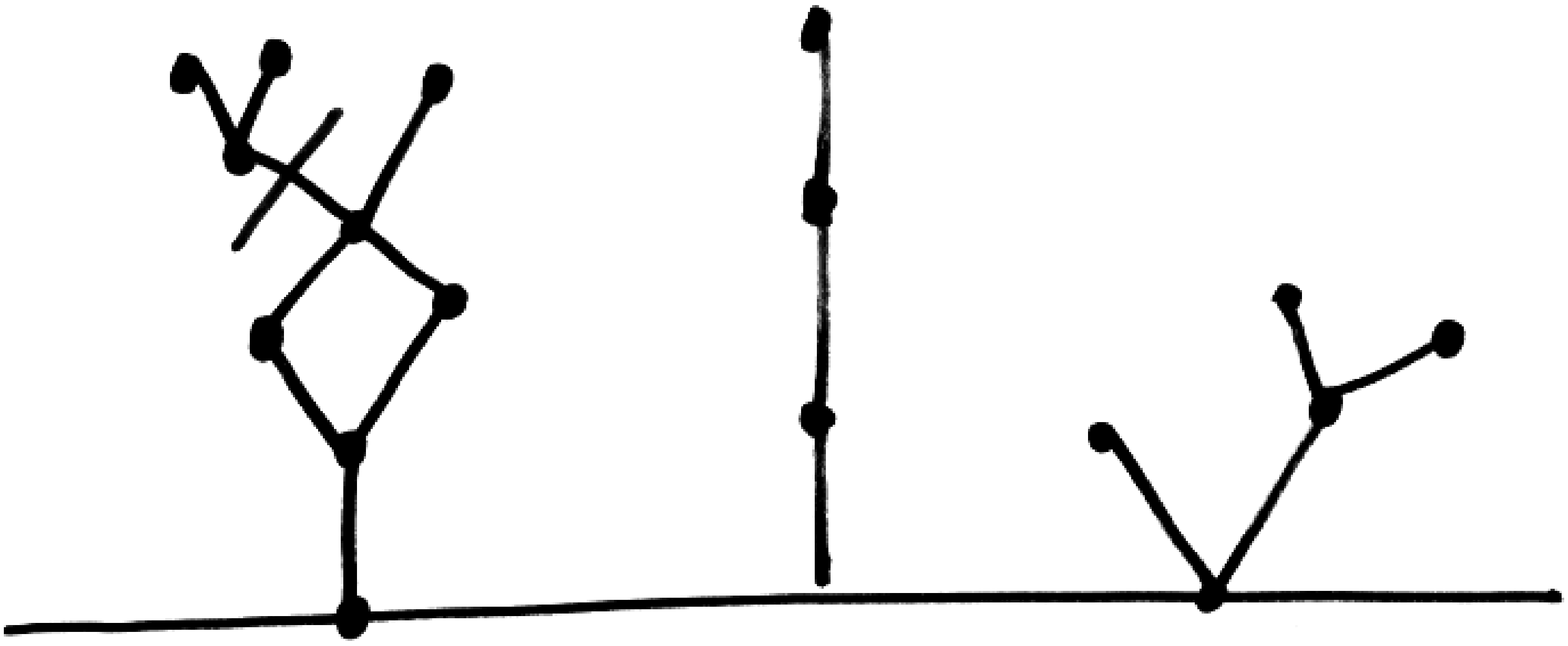
and if want turn any one coin to its left H ↔ T

E.g.

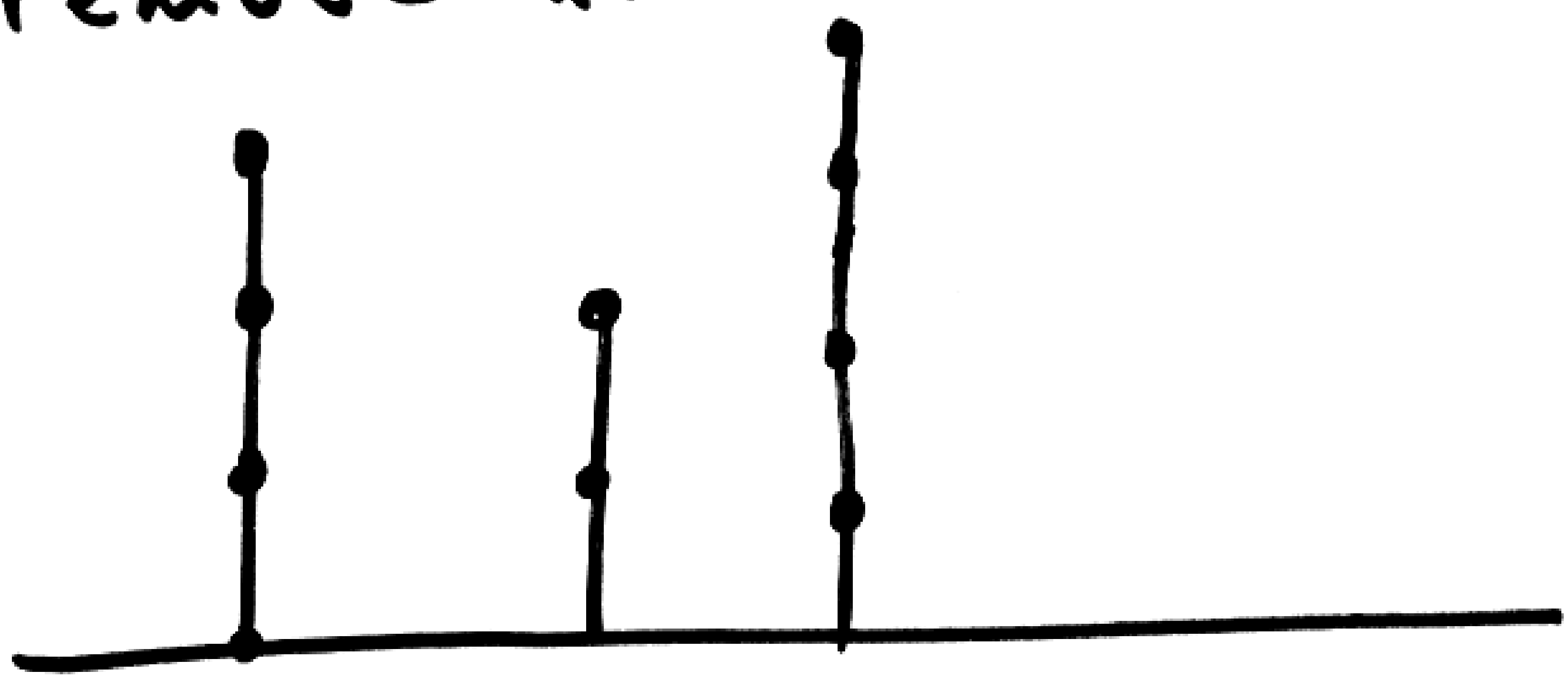
T → H   H → T

H H H T T T H

3) Hackenbush



Rule Hack any segment
remove all unrooted pieces



3 2 4

4) Grundy's game

Start pile n things

Rule Take any one pile and divide it into two unequal piles

6

5 1   4 2   3 3

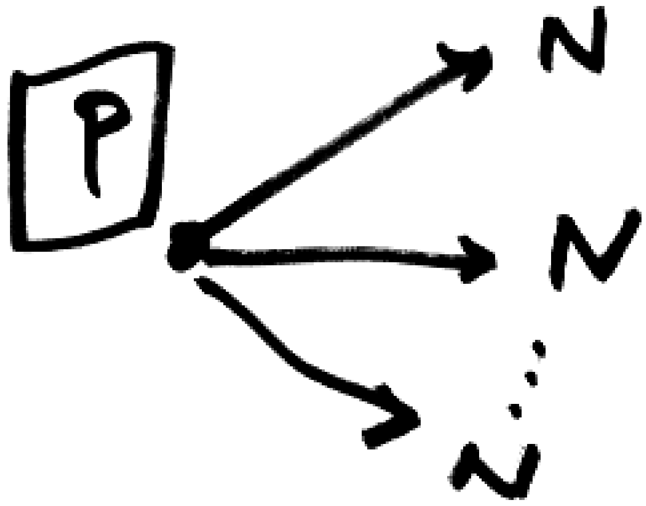
3 2 1

1 2 2 1

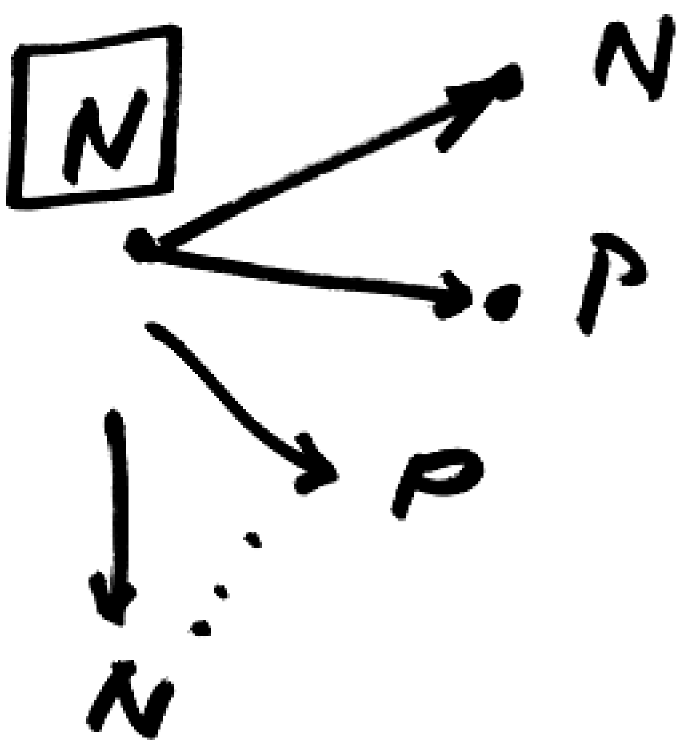
Impartial game

Directed graph w/ no loops

Labelling of the vertices



all N



at least one P

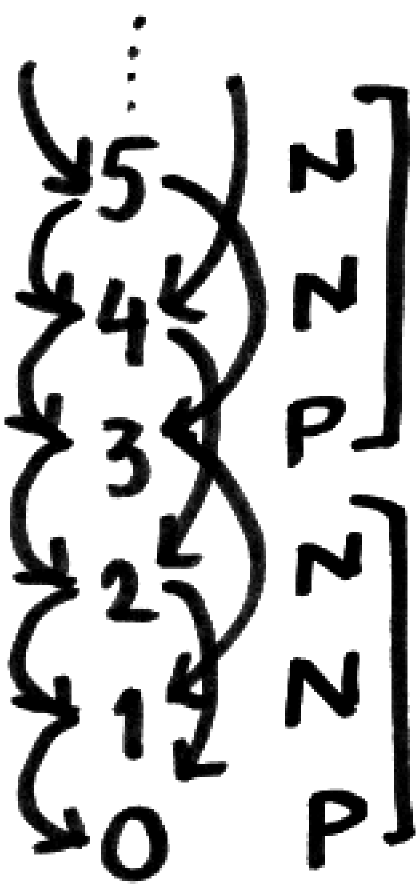
Substraction games

Start: A pile of n things

$$S \subseteq \mathbb{N} = \{1, 2, 3, \dots\}$$

Move: Take away s things from the pile where $s \in S$

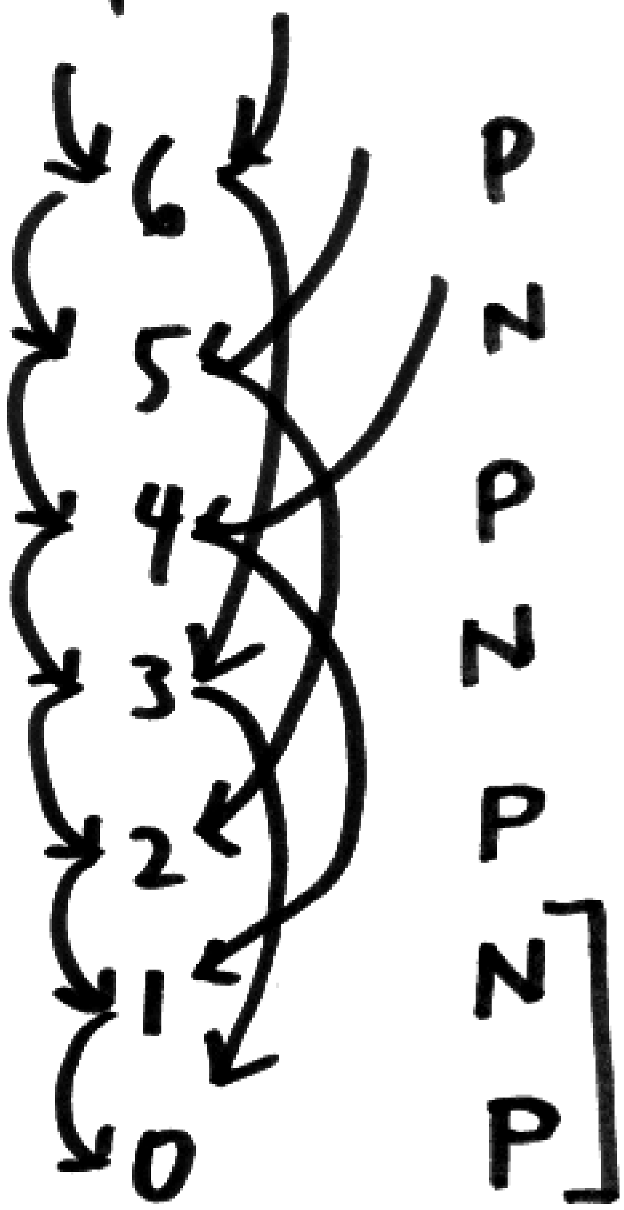
E.g. $S = \{1, 2\}$



n is P $\Updownarrow$ 3 divides n

pattern repeats

$S = \{1, 3\}$



n is P $\Leftrightarrow$ $2 \mid n$

Sum of games

Γ_1, Γ_2 (impartial games)

Define $\boxed{\Gamma = \Gamma_1 * \Gamma_2}$

In Γ a <u>move</u> is

either a move in Γ_1

<u>or</u> a move in Γ_2

E.g. Nim with k-piles

is $\underbrace{\Gamma * \cdots * \Gamma}_{k}$

Γ substraction game $S = \mathbb{N}$

Labellings on $\Gamma_1 * \Gamma_2$
are <u>not</u> determined by
the labellings in Γ_1 & Γ_2

For example

Γ_1 substraction game $S = \{1, 2\}$

Γ_2 " " $S = \{1, 3\}$

$\Gamma = \Gamma_1 * \Gamma_2$

positions in Γ are
pairs of positions (n_1, n_2)

$n_1 = 0, 1, 2 \ldots$
$n_2 = 0, 1, 2, \ldots$

Γ_2

$\{1,3\}$

1	7	N						
0	6	P						
1	5	N						
0	4	P	N	N	P			
1	3	N	P	N	N			
0	2	P	N	N	P	N		
1	1	N	P	N	N	P		
0	0	P	N	N	P	N	N	P
		0	1	2	3	4	5	6
		0	1	2	0	1	2	0

$\{1,2\}$

Γ_1

$P \rightarrow P$, $\uparrow P$

$N \rightarrow N$, $\uparrow P$

$N \rightarrow ?$, $\uparrow N$

1st row & column do not determine the square.

We need something more elaborate than just labellings.

Define a numerical value to any position in a game

<u>Nim value</u> , <u>Grundy function</u>

In case of NIM

$$G = n_1 * \cdots * n_K$$

$\uparrow \quad \uparrow \quad \nearrow \quad \uparrow$

number of objects in the piles

P label $\leftrightarrow$ $G = 0$

N label $\leftrightarrow$ $G \neq 0$

Grundy function

on vertices of Γ

Define recursively

$$G_\Gamma(v) := \text{mex}\{G_\Gamma(w) \mid v \to w\}$$

mex = minimum excludant

$S \subseteq \{0, 1, 2, 3, \dots\}$

$\text{mex}(S) :=$ smallest number NOT in S

Example

$$S = \{0, 1, 2, 4, 6, 7, 10, 15\}$$

$$\text{mex}(S) = 3$$

$$\text{mex}(\phi) = 0$$

KEY PROPERTY OF MEX

$$\text{mex}(S) = 0$$

$$\Updownarrow$$

$$0 \notin S$$

Γ_1 $S = \{1, 2\}$

$\vdots$		
6	P	
5	N	2
4	N	1
3	P	0
2	N	2
1	N	1
0	P	0
n		$G_\Gamma(n)$

$$G_r(1) = \operatorname{mex}\{0\} = 1$$

$$G_r(2) = \operatorname{mex}\{0,1\} = 2$$

$$G_r(3) = \operatorname{mex}\{1, 2\} = 0$$

$$G_r(v) = 0$$

$$\Longleftrightarrow \operatorname{mex}\{G_r(w) \mid v \mapsto w\} = 0$$

$$\Longleftrightarrow 0 \neq \underset{v \mapsto w}{G_r(w)}$$

$$P \longleftrightarrow G_r(v) = 0$$

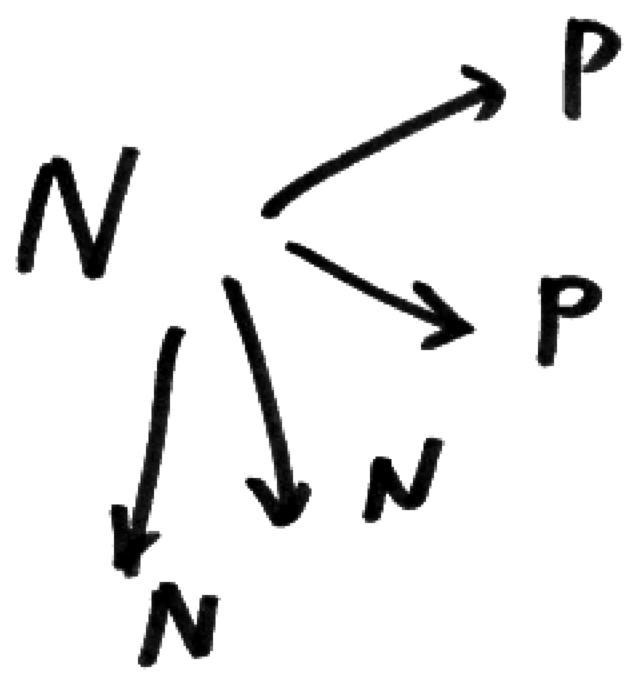
$$\begin{tikzcd} P \arrow[r] \arrow[dr] \arrow[ddr] \arrow[d] & N \\ N & N \\ & N \end{tikzcd}$$

$$G_\Gamma(v) > 0 \quad \longleftrightarrow \quad N$$

$$0 \neq \operatorname{mex}\{G_\Gamma(w) \mid v \mapsto w\}$$

$$\Updownarrow$$

at least one child of v has value 0.



$S = \{1, 3\}$

6	0
5	1
4	0
3	1
2	0
1	1
0	0

$$G_{\Gamma_2}(2) = \text{mex}\{1\} = 0$$

$1 * 2 = 3$

$$\begin{array}{r} 01 \\ 10 \\ \hline 11 \end{array}$$

THM (Sprague - Grundy)

$\Gamma_1, \Gamma_2, \dots, \Gamma_k$

$$G_{\Gamma} = G_{\Gamma_1} * \cdots * G_{\Gamma_k}$$

$$\Gamma = \Gamma_1 * \cdots * \Gamma_k$$

(vast generalization of Bouton)

For Nim

1 ~~column~~ pile $G(n) = n$

k columns $n_1, n_2, \dots, n_k$

$G = n_1 * \cdots * n_k$

We may think of

$$G_{\Gamma_1}(v_1) = n_1$$

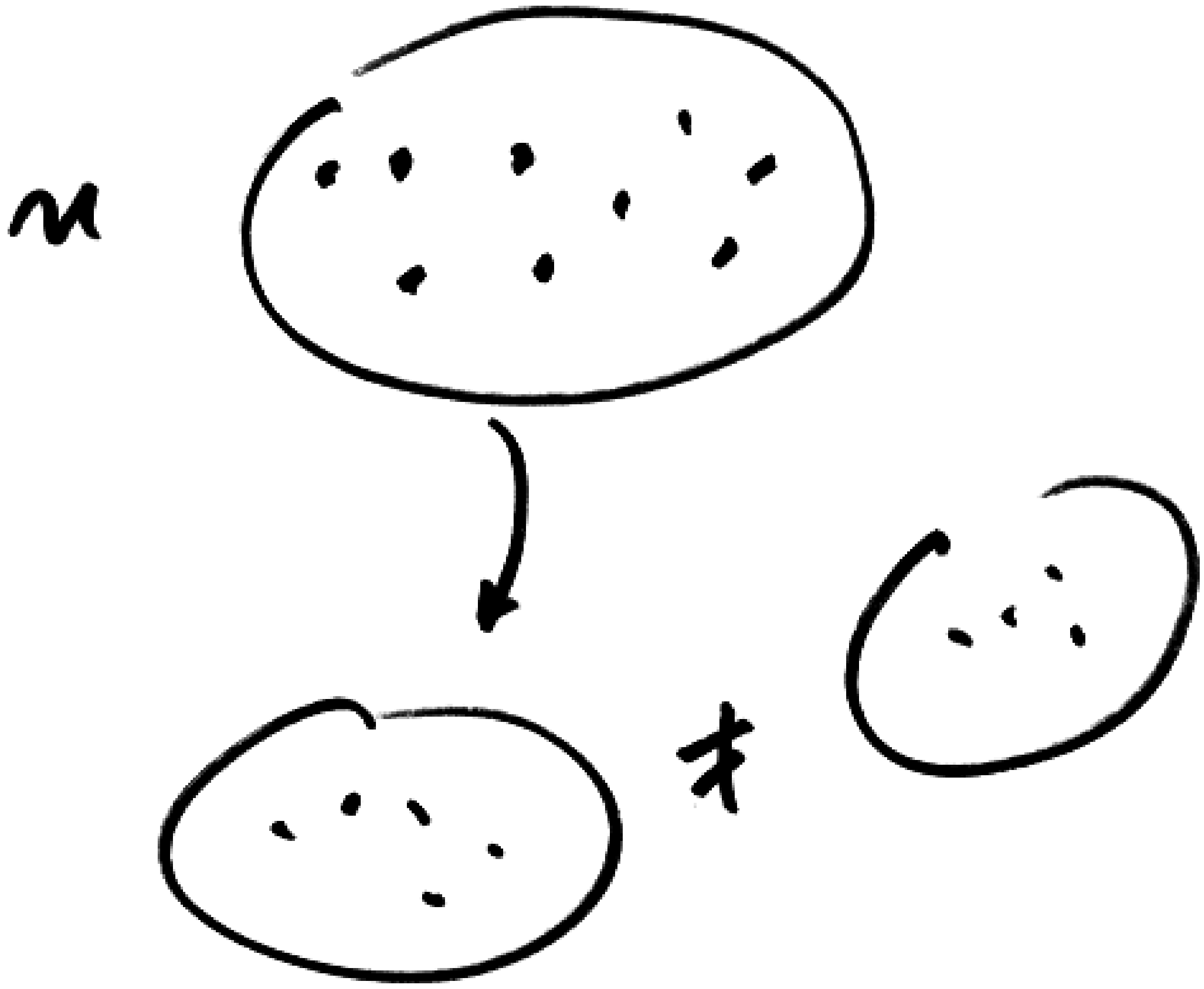
as the size of a virtual pile in Nim

$v_1, v_2, \ldots v_K$ in each game $\Gamma_1, \ldots \Gamma_K$

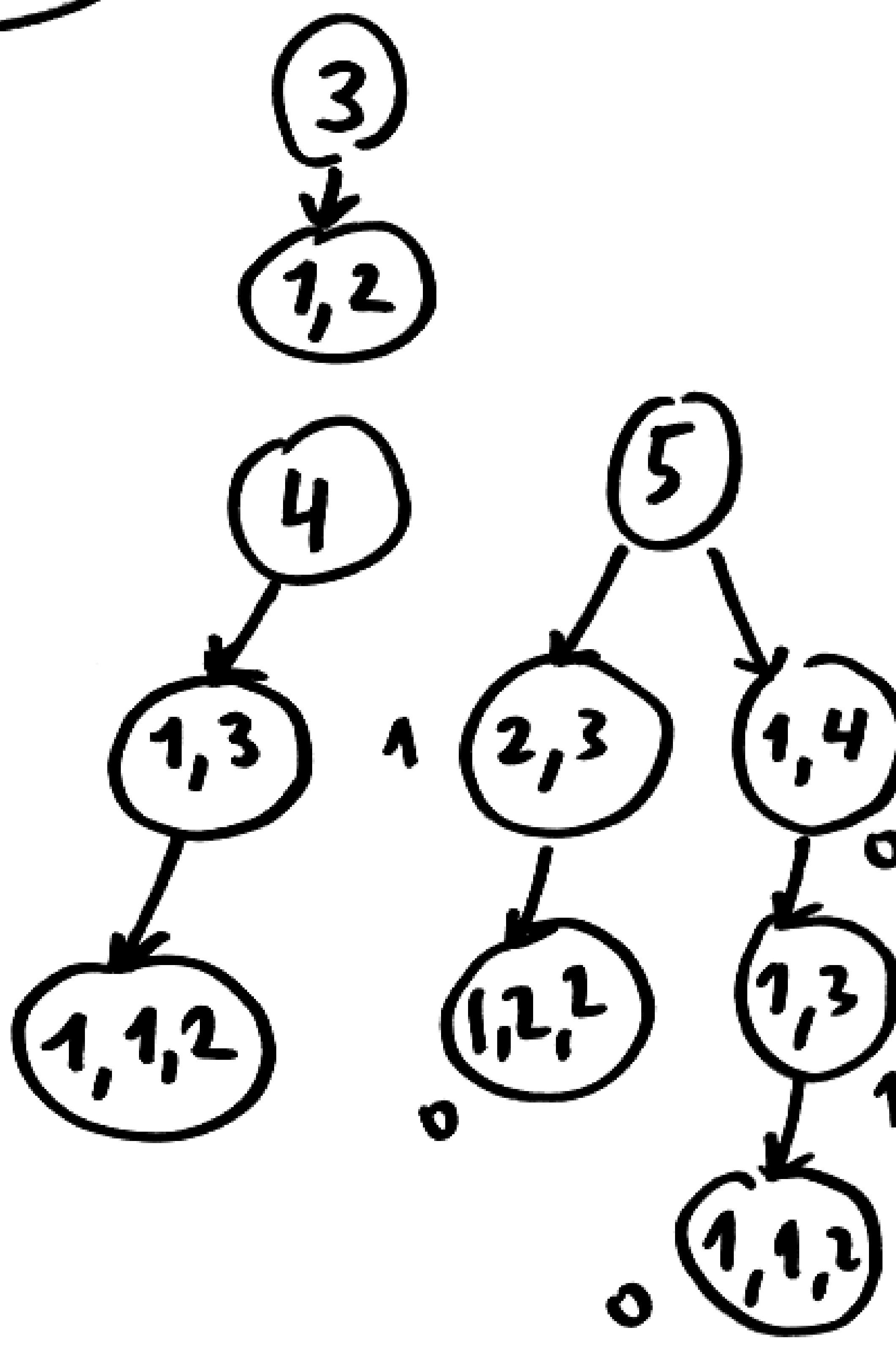
$$G_{\Gamma_2}(v_2) = n_2 \ldots$$

$$G_{\Gamma}(v) = n_1 * \ldots * n_K$$

Grundy game



m	G(n)
1	0
2	0
3	1
4	0
5	2



Turning Turtles

1 2 3 4 5 6 7 8 9 10
T H T T H T H T T T . . .

Rule Turn at most two coins. (H → T rightmost)

Positions $(n_1, n_2, \ldots, n_k)$

Positions of the H's.

If I play

$$\begin{array}{cc} 4 & 7 \\ T & H \\ \downarrow & \downarrow \\ H & T \end{array}$$

replaced ~~the~~ 7 → 4

$(2, 5, 7) \longrightarrow (2, 5, 4)$

If I play

$$\begin{array}{cc} 2 & 7 \\ H & H \\ \downarrow & \downarrow \\ T & T \end{array}$$

$$(2,5,7) \longmapsto (2,5,2)$$

$$G(n) = \operatorname{mex}\{G(0), G(1), \ldots, G(n-1)\}$$

$$G(0) = 0$$

$$G(n) = n$$

Mock Turtles

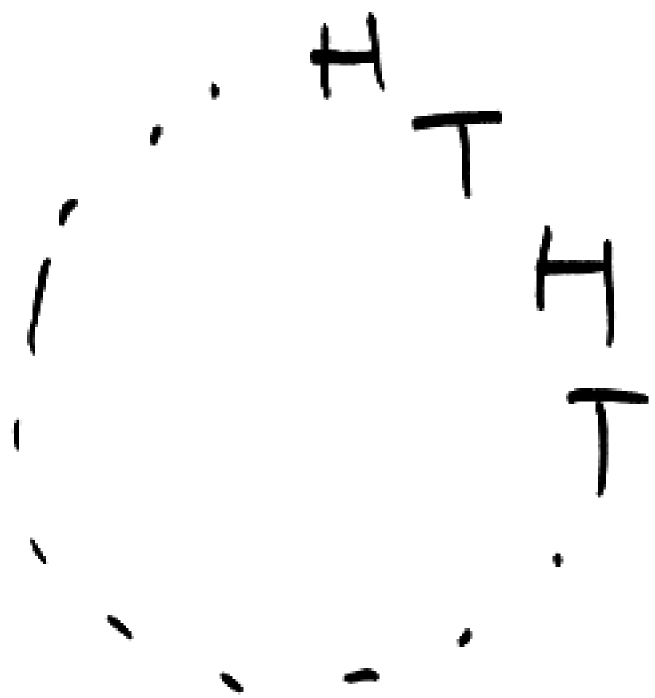
Turn up to 3 coins

(right most $H \to T$)

Positions $(n_1, n_2, \ldots, n_k)$

n_i = square with an H

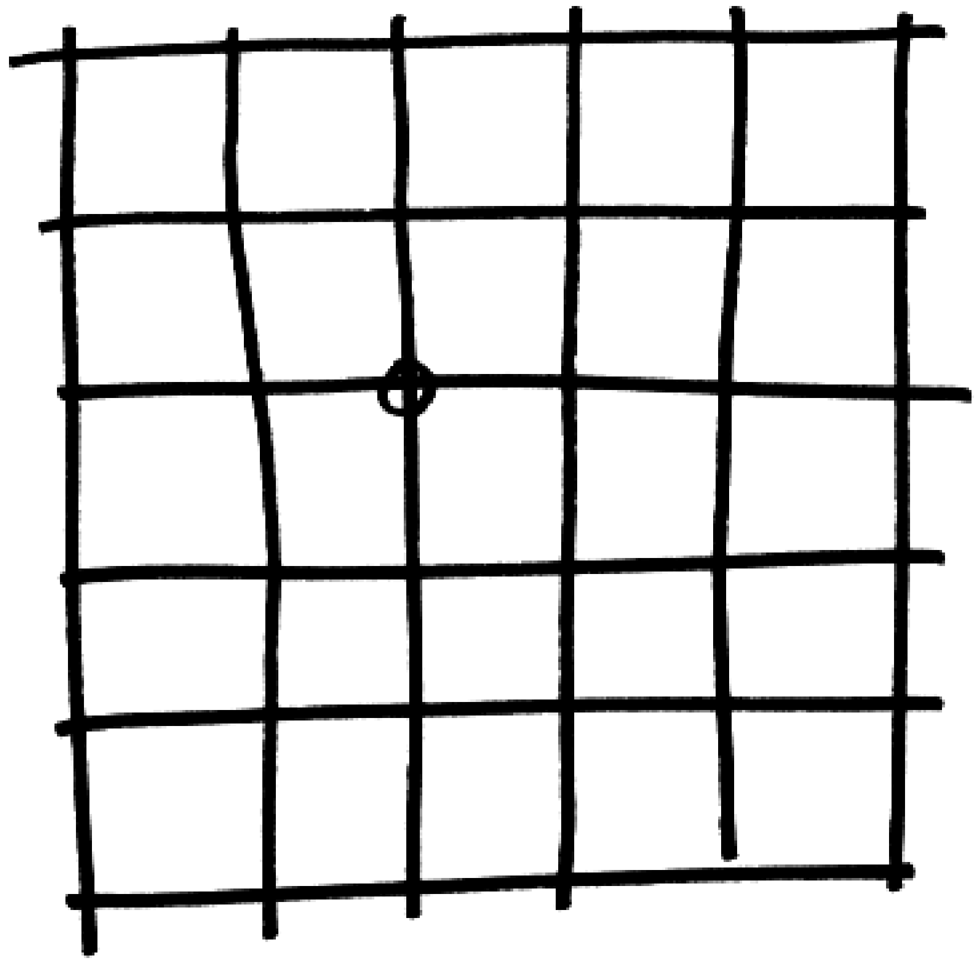
Blet



Rule: $\boxed{HTH} \longleftrightarrow \boxed{THT}$

Goal: Minimum # of T's
(maximum # of H's)

Moves available depend on the state puzzle.



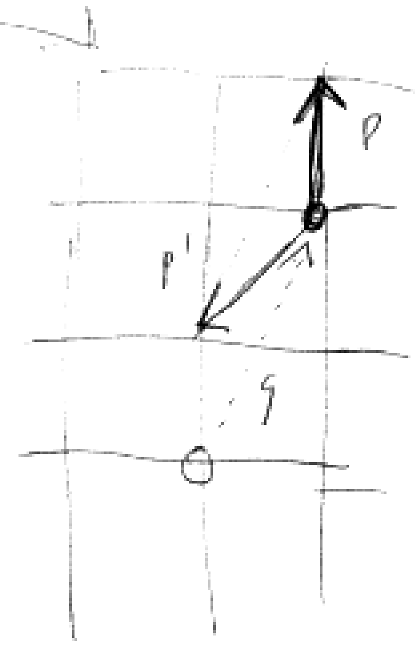
q = position
p = momentum

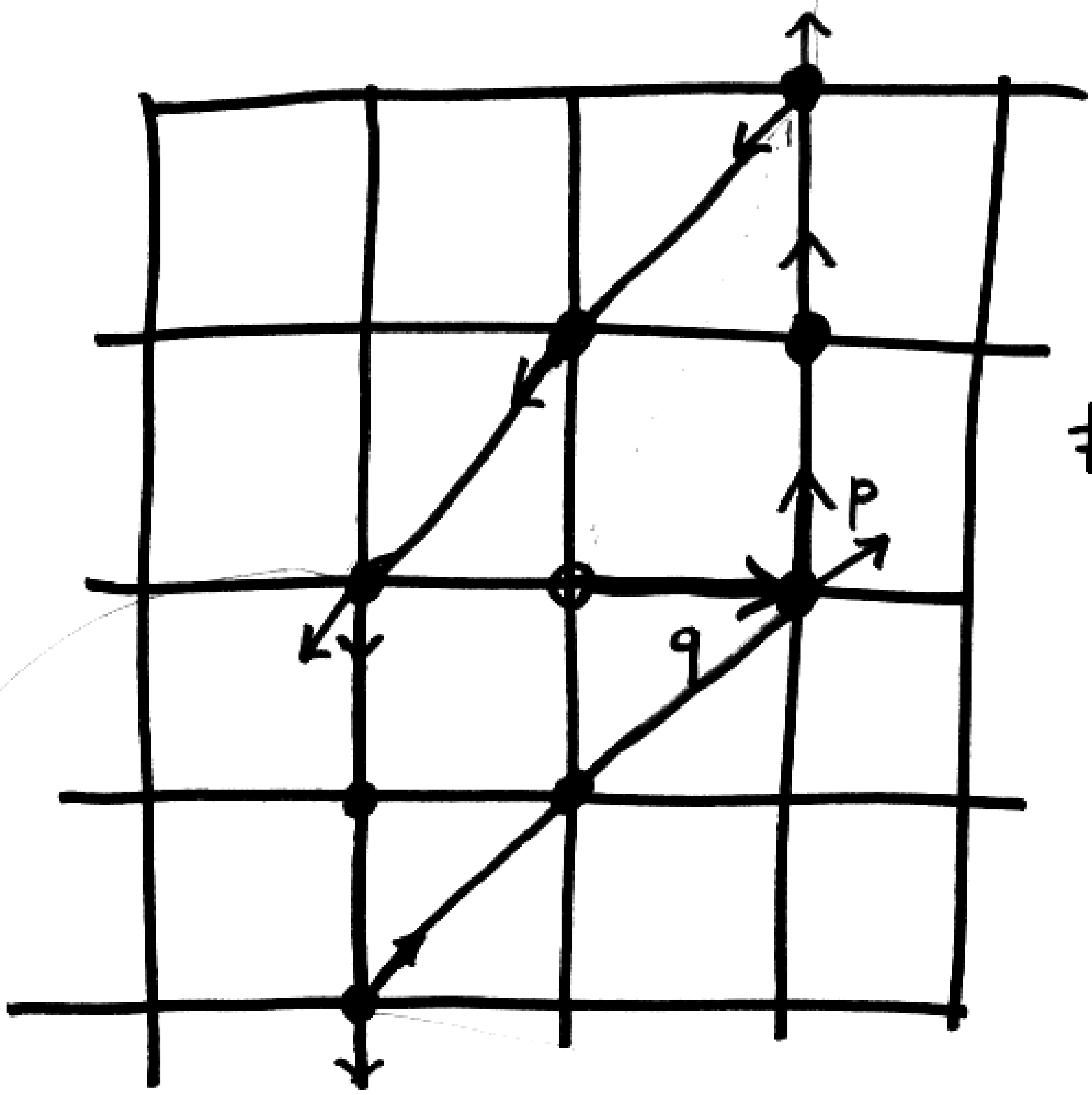
Associate a path in this lattice to any sequence of H's & T's.

$$H : \begin{cases} q \mapsto q + p \\ p \mapsto p \end{cases}$$

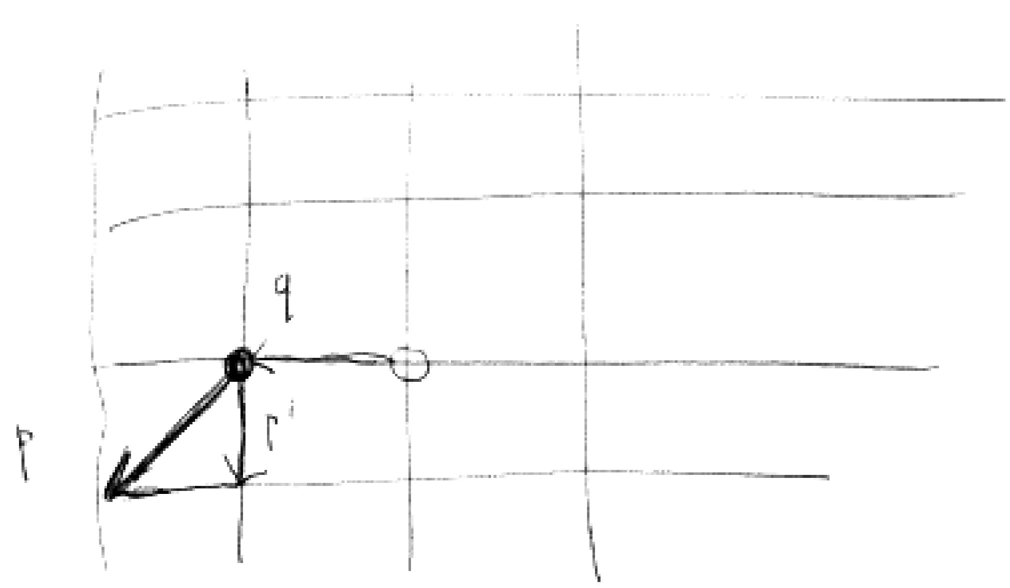
$$T : \begin{cases} q \mapsto q \\ p \mapsto p - q \end{cases}$$

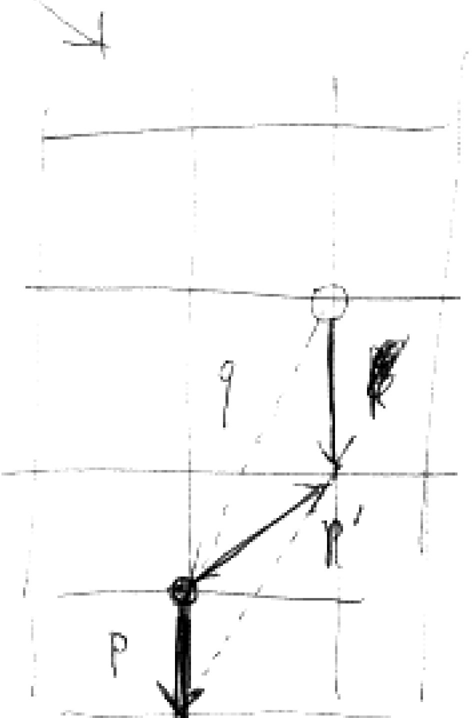
T H T H H T H H T H H

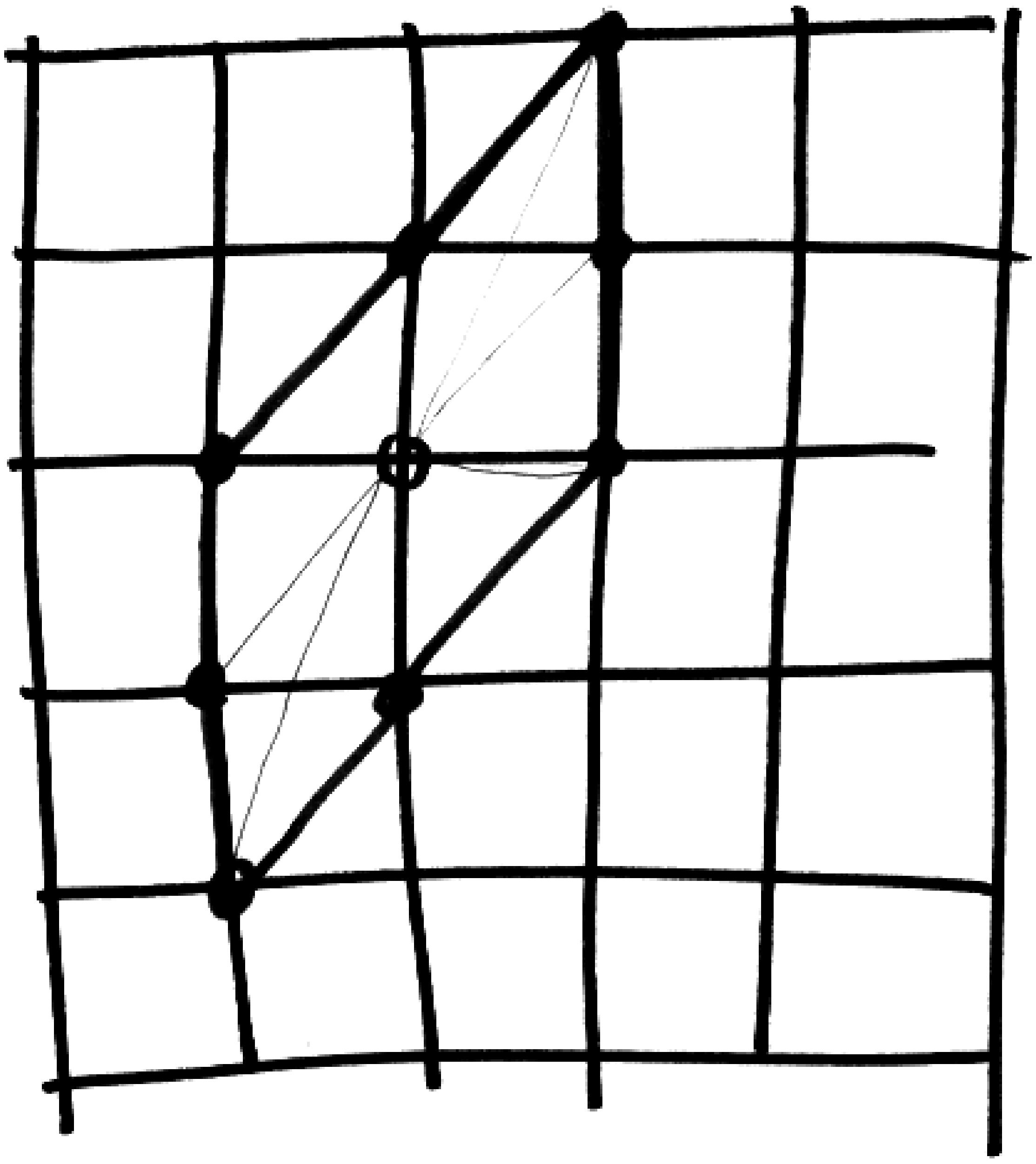




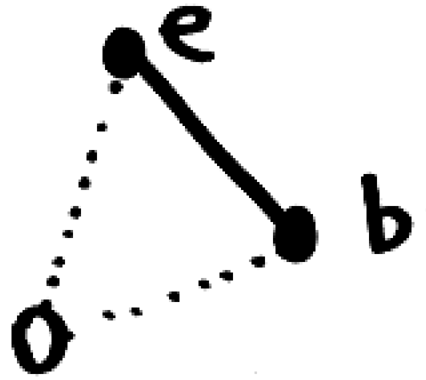
\# grid points on this path = 8







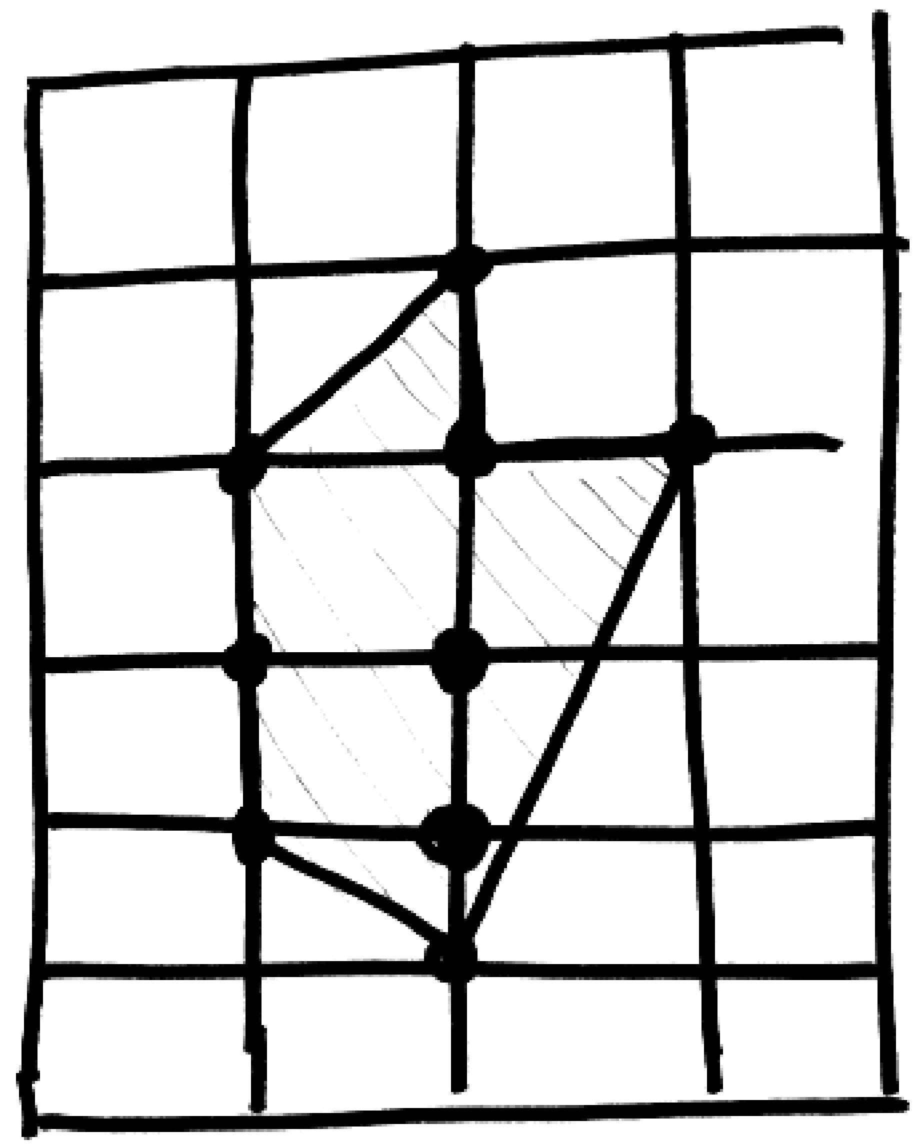
Paths that we obtain ~~by~~ from a sequence of H's & T's can be described as follows:



This triangle should contain only o, e, b as grid points

Equiv. Area of the triangle $= 1/2$

Pick's theorem

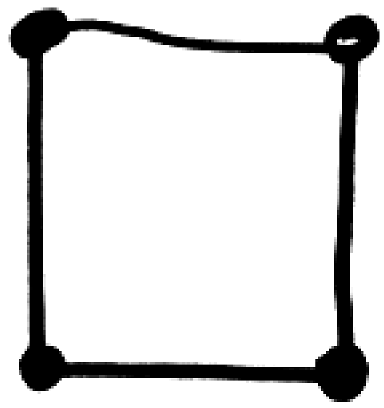


Area = # interior pts

$+ \frac{1}{2}$ # boundry points

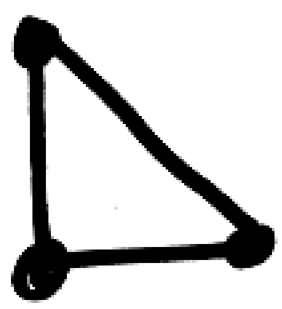
$- 1$

$$A = 2 + \frac{1}{2} 7 - 1 = \frac{9}{2}$$



$A = 1$

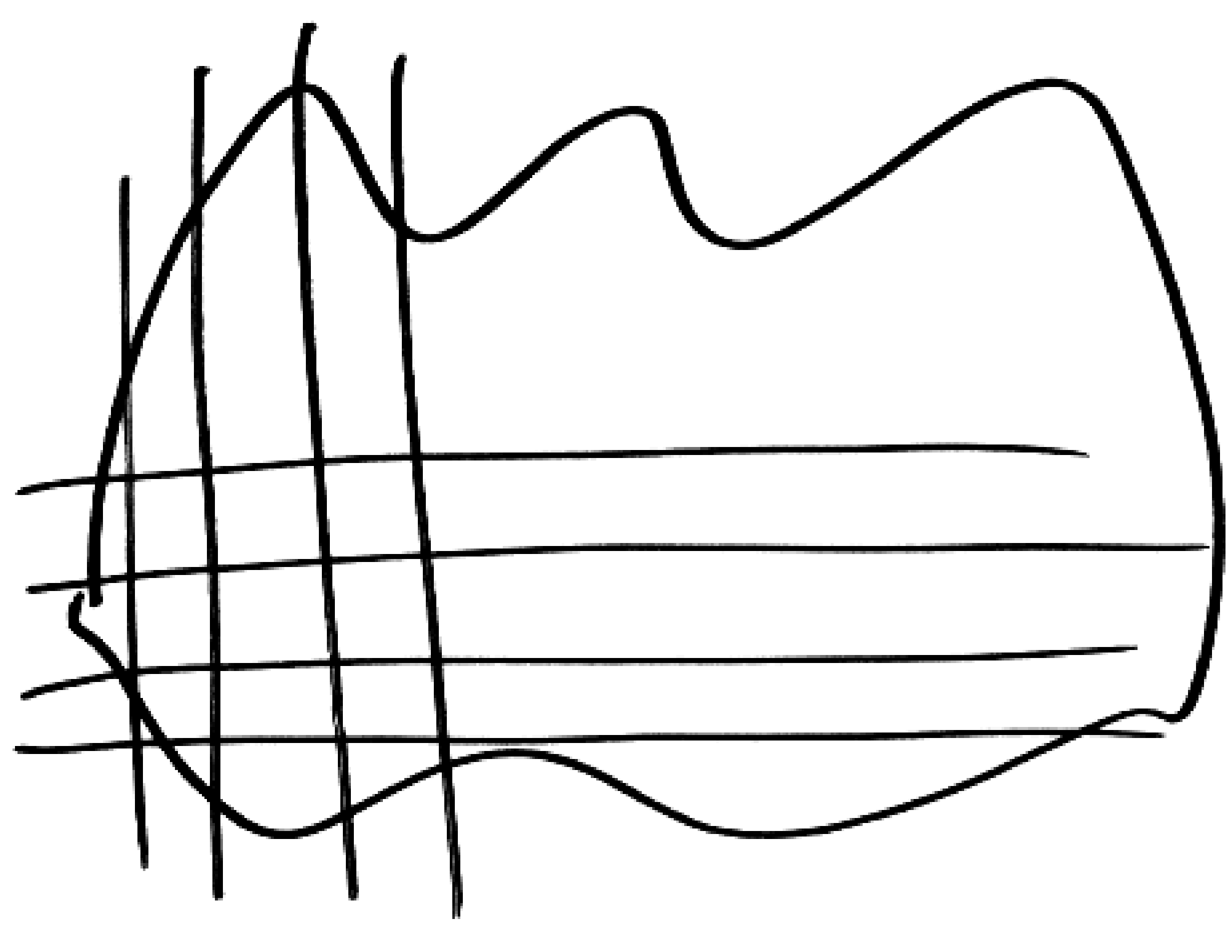
$$0 + \frac{1}{2}4 - 1 = 1$$



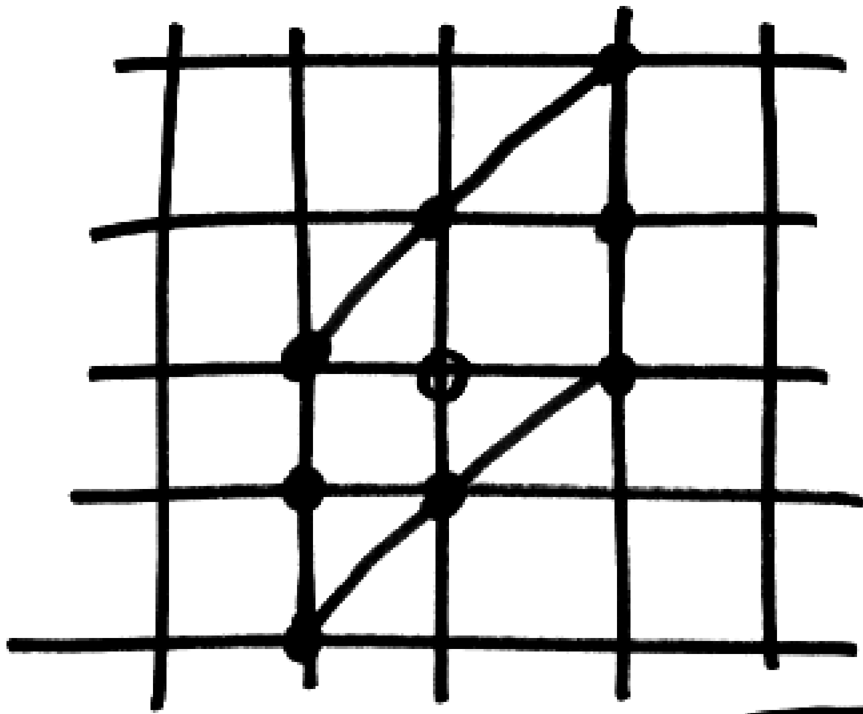
$A = \frac{1}{2}$

$$0 + \frac{3}{2} - 1 = \frac{1}{2}$$

$$\vdots$$



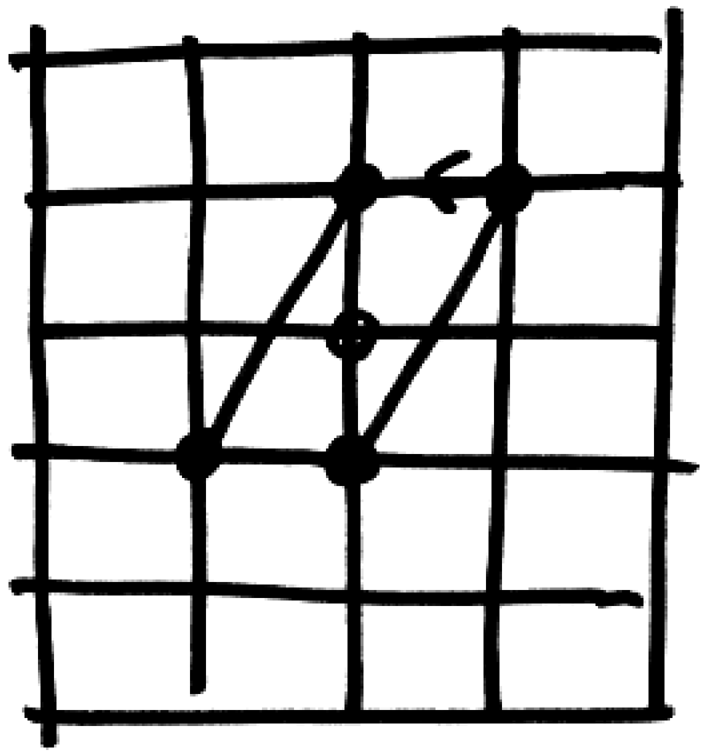
dots
q's



8

..... THH

dots
p's



4

$\boxed{8+4=12}$

TTH

THH THH THH THH

In original path $\mathcal{L}$

# of boundary points is

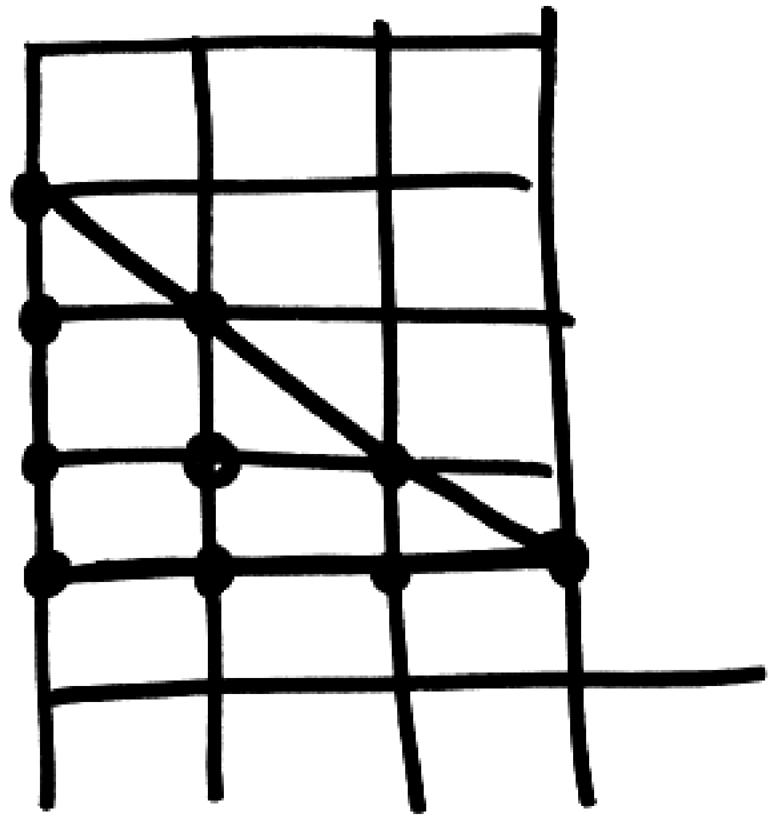
= # H's in the sequence

In dual path $\hat{\mathcal{L}}$

boundary points is

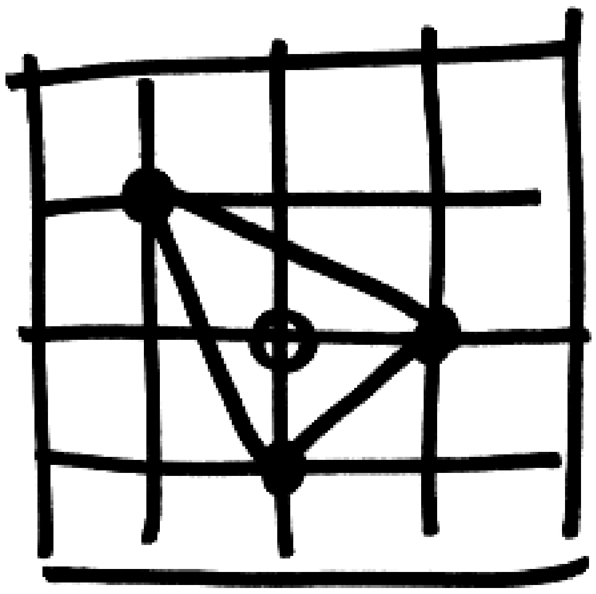
= # T's in the sequence

$\mathcal{L}$



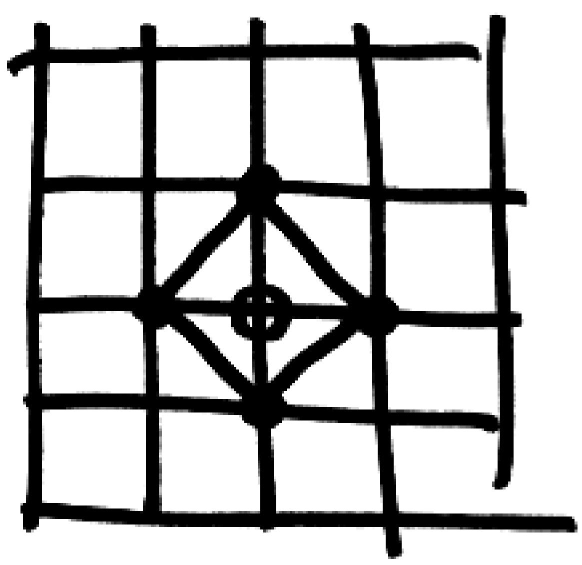
9 pts

$\hat{\mathcal{L}}$



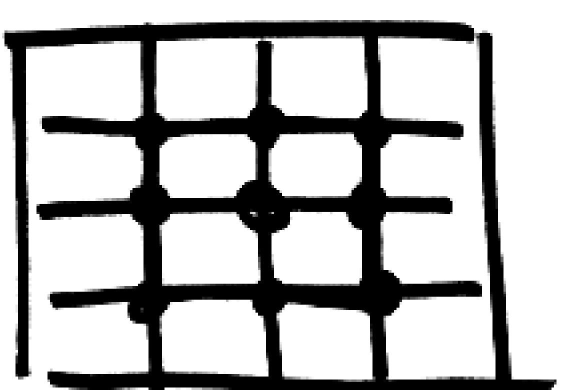
3 pts

$\mathcal{L}$



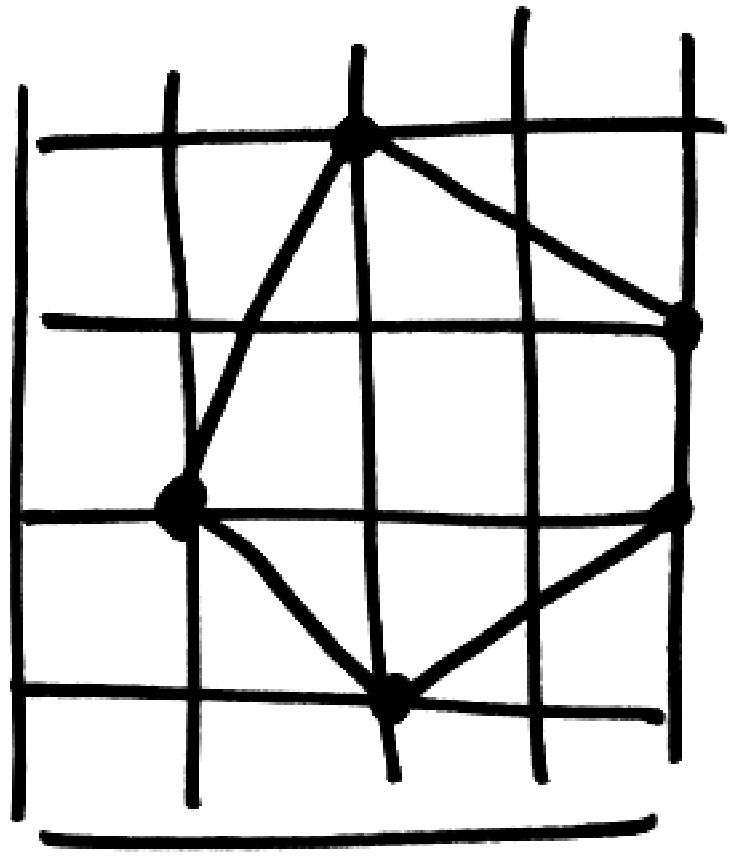
4 pts

$\hat{\mathcal{L}}$



8 pts

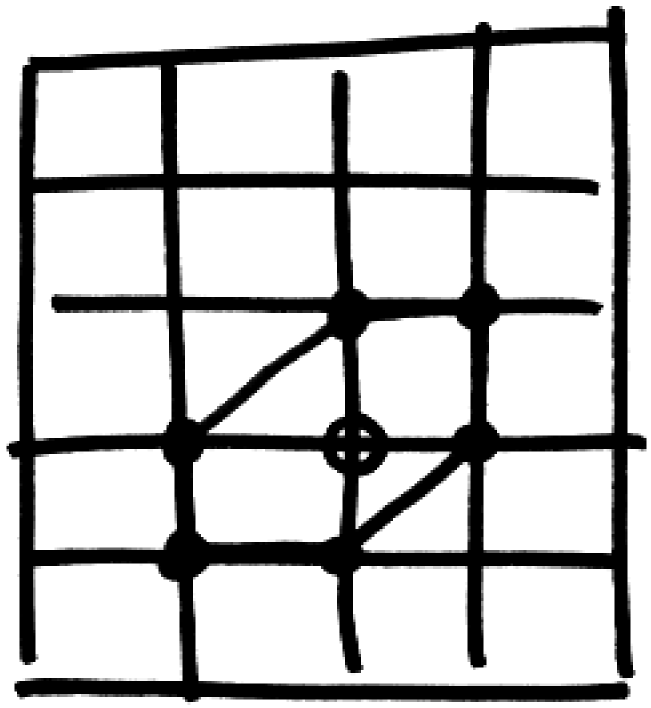
Theorem (Scott).



Consider convex polygons with k interior points $k>0$. Then up to change of basis there are only finitely many.

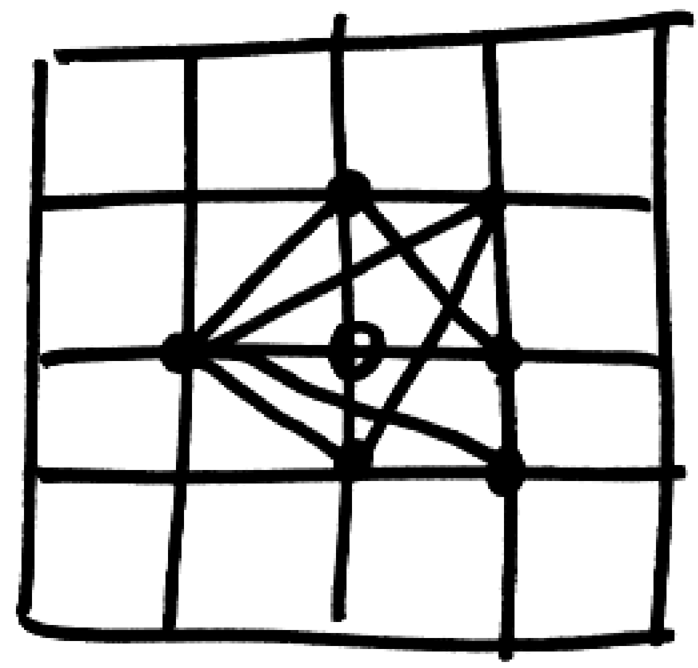
Sequence of H's, T's

$\rightsquigarrow$ path



T H
T H

E.g.
total 24



winding
number
of path
= 2

\# H's = # of boundary pts

THM

ℓ_H = # H's

ℓ = # H's & T's
(total length)

$$\frac{1}{6} < \frac{\ell_H}{\ell} < \frac{5}{6}$$

→ gives an upper bound for how many H's we can possibly have.

$$\ell_H < \frac{5}{6}\ell$$

E.g. $\ell = 12 \implies \ell_H < 10$

$\ell = 24 \implies \ell_H < 20$

In fact: this upper bound is achieved

Hence gives the largest number of H's that we <u>can</u> achieve.

on the web $\ell = 28$

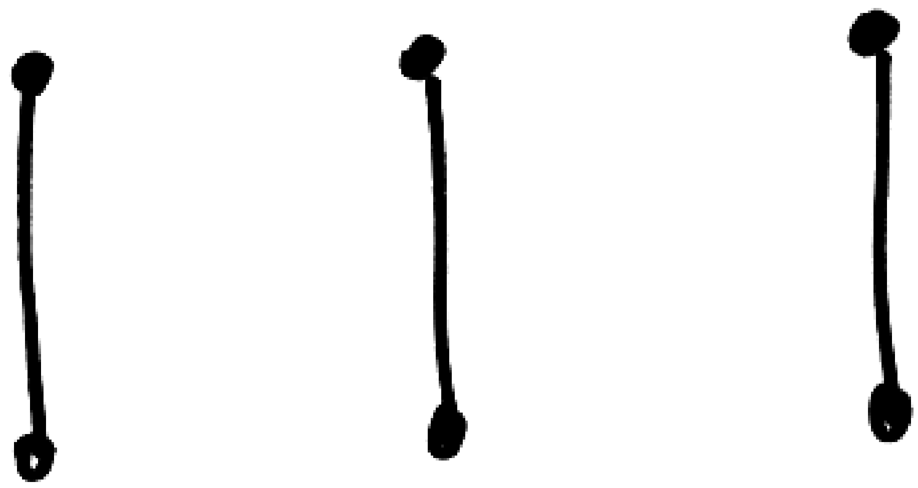
$$\ell_H < \frac{5}{6} \cdot 28 = ?$$

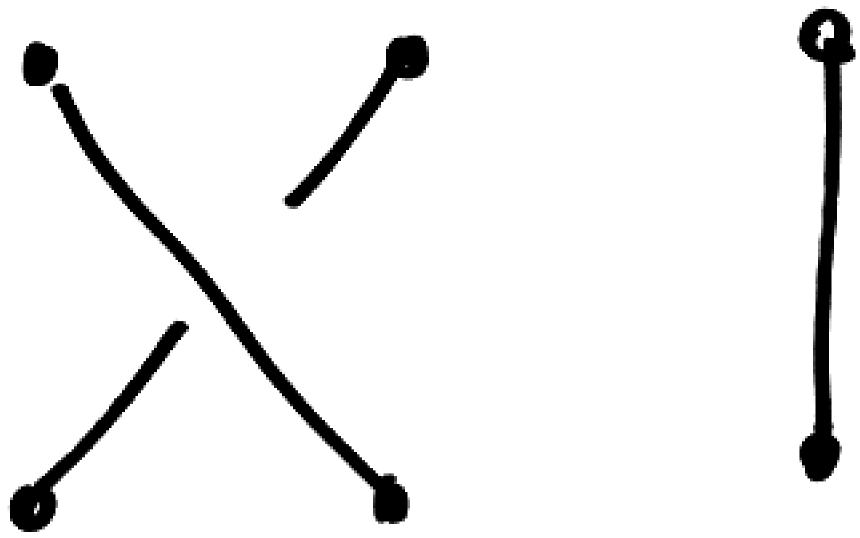
$$* \quad \ell_H \leq 23$$

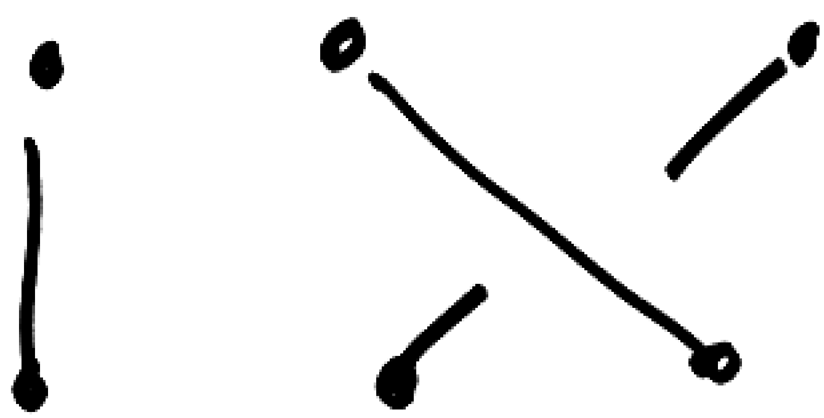
greedy algorithm will only get 21 H's.

H, T

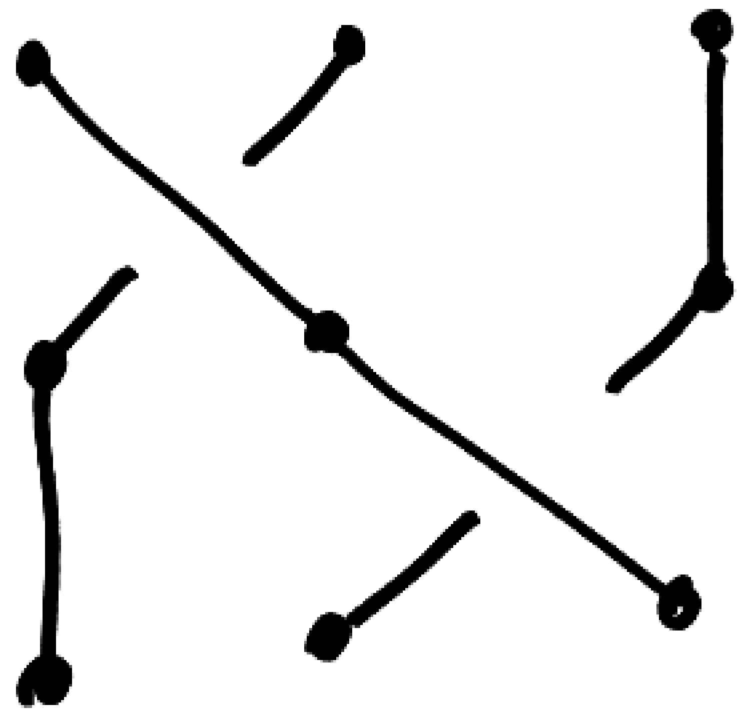
Braid group



H 

T 

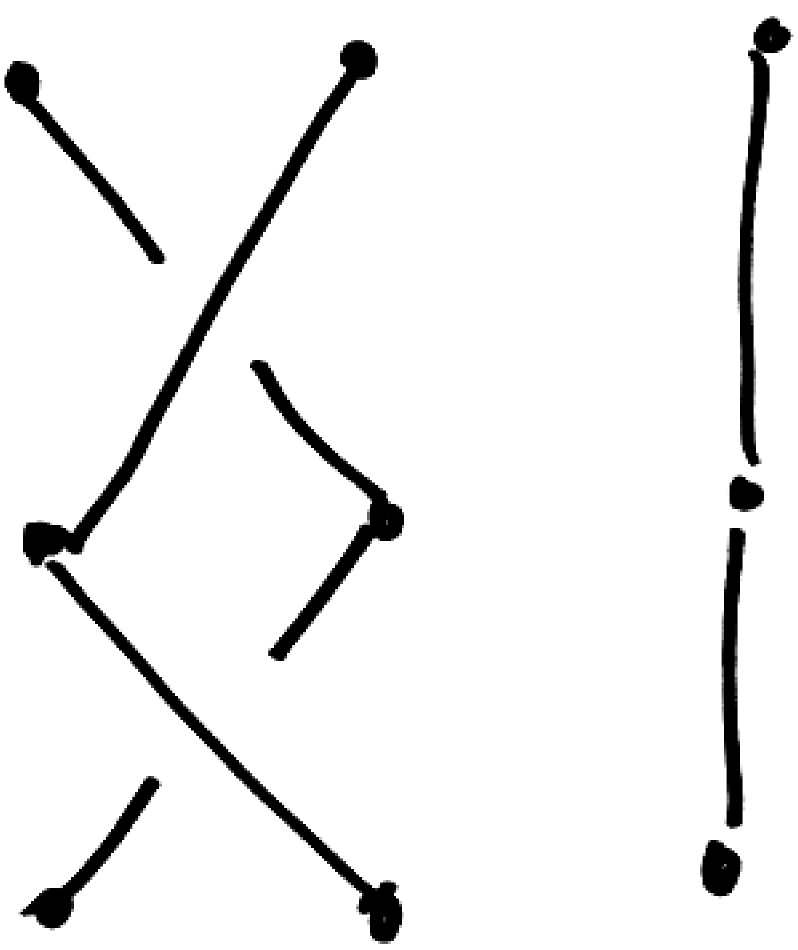
T H



H^{-1}

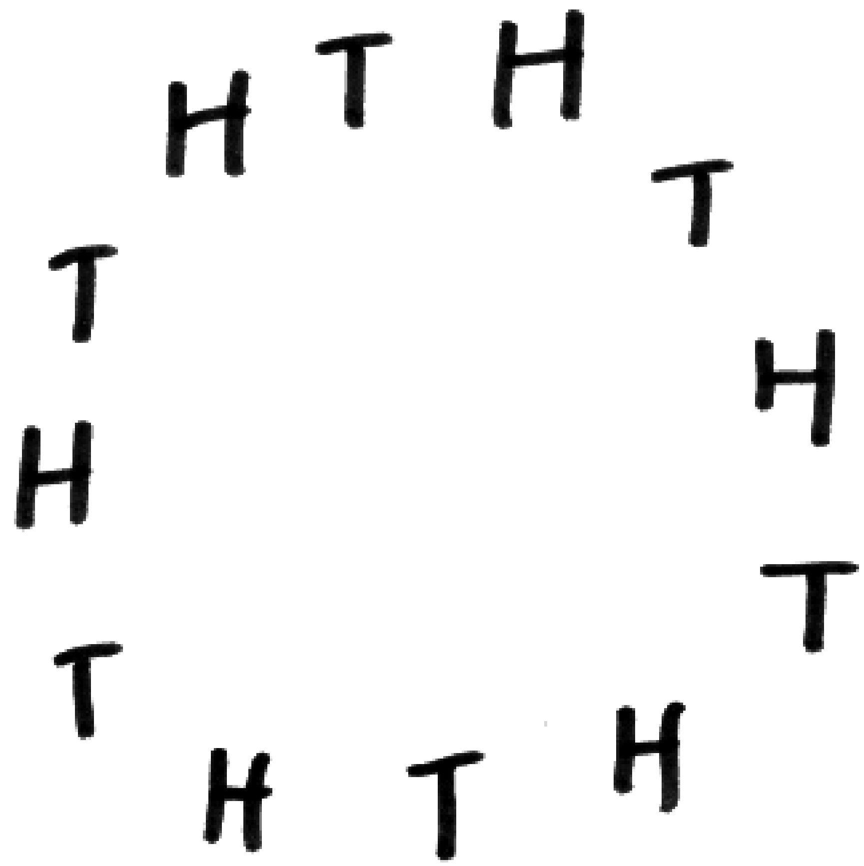
H^{-1}

H



HTH = THT

Blet



Rule: $\boxed{HTH} \longleftrightarrow \boxed{THT}$

Goal: Maximize the number of H's.
(Minimize # of T's)

(F. Voloch, L. Sadun)

Simulated annealing.

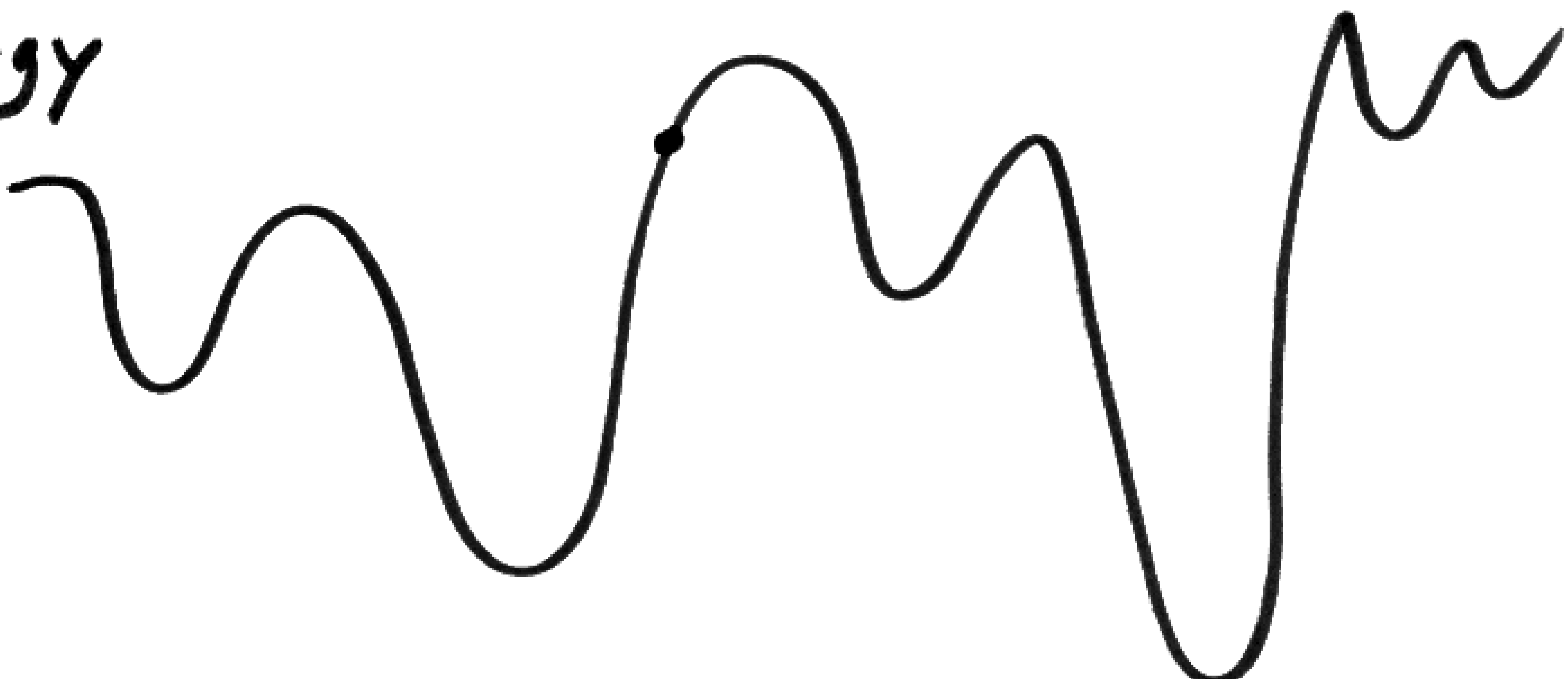
N. Metropolis algorithm

travelling sales man pblm.

Minimize a function on

a discrete huge space.

Energy



Start at random state E

Pick a neighbor state E′

If $E' < E$

then move to that state

If $E' > E$ then

compute

$$p = e^{-(E'-E)/KT}$$

k = Boltzmann

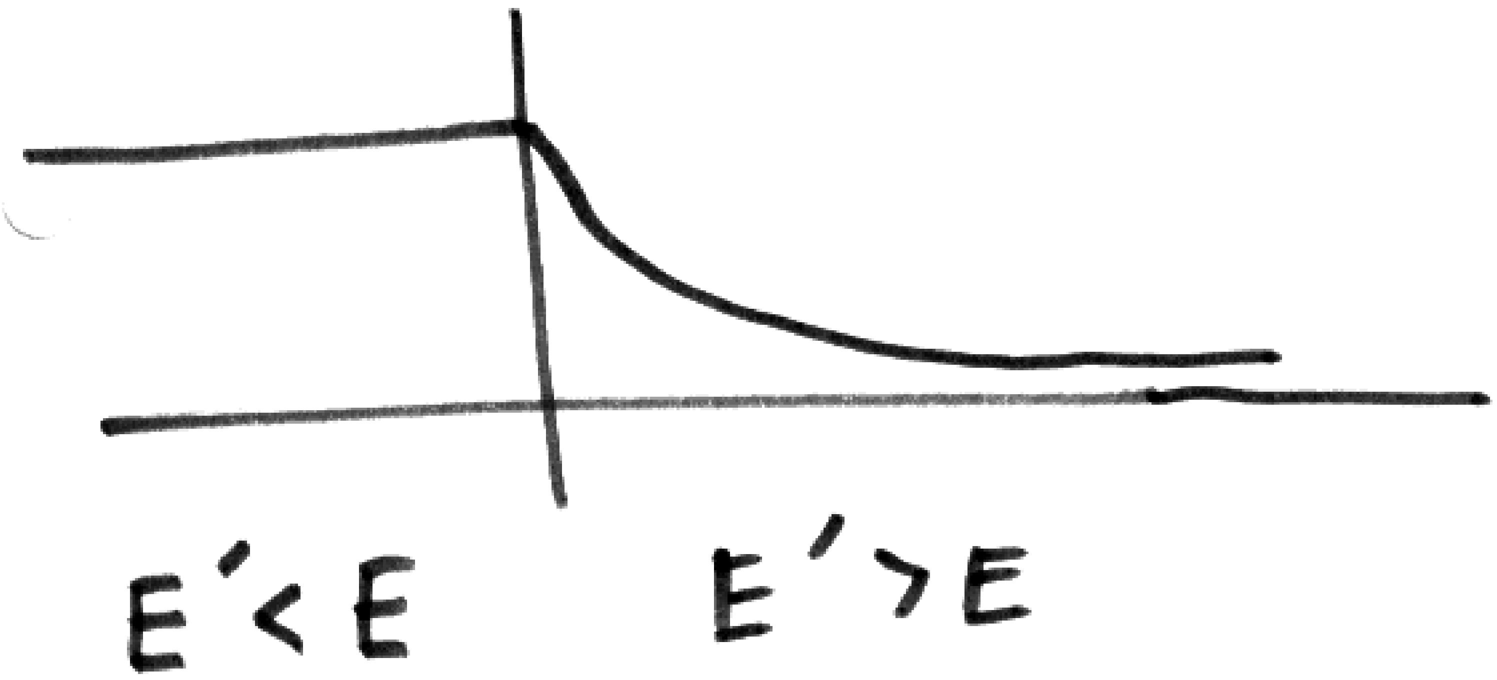
T = temperature

compare with a random sample $[0, 1]$

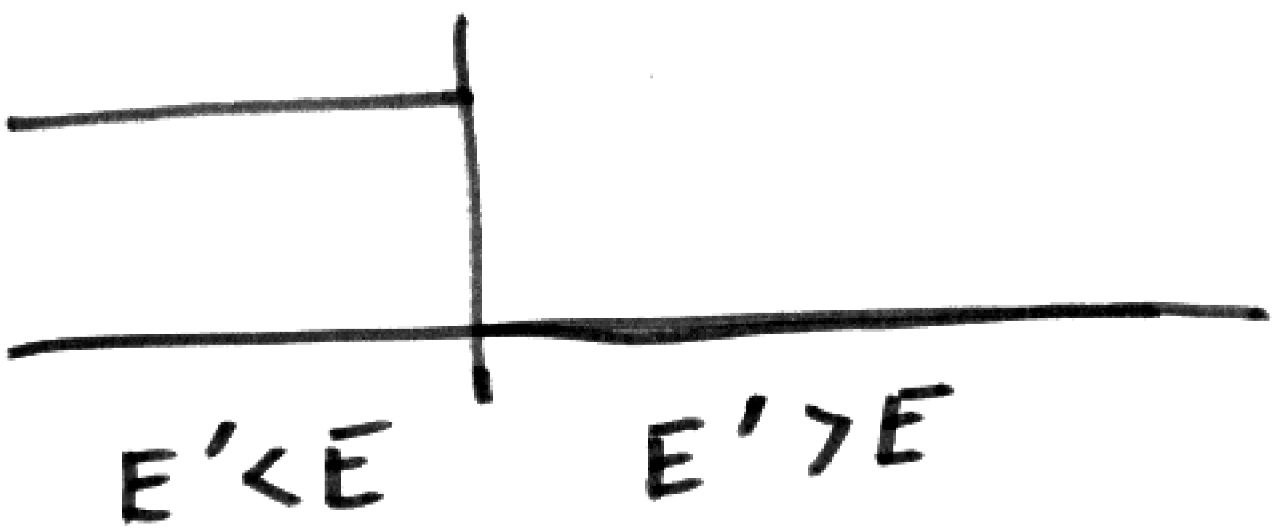
$0 \leq q \leq 1$

If $q \leq p$ then

move to new state.



Greedy algorithm



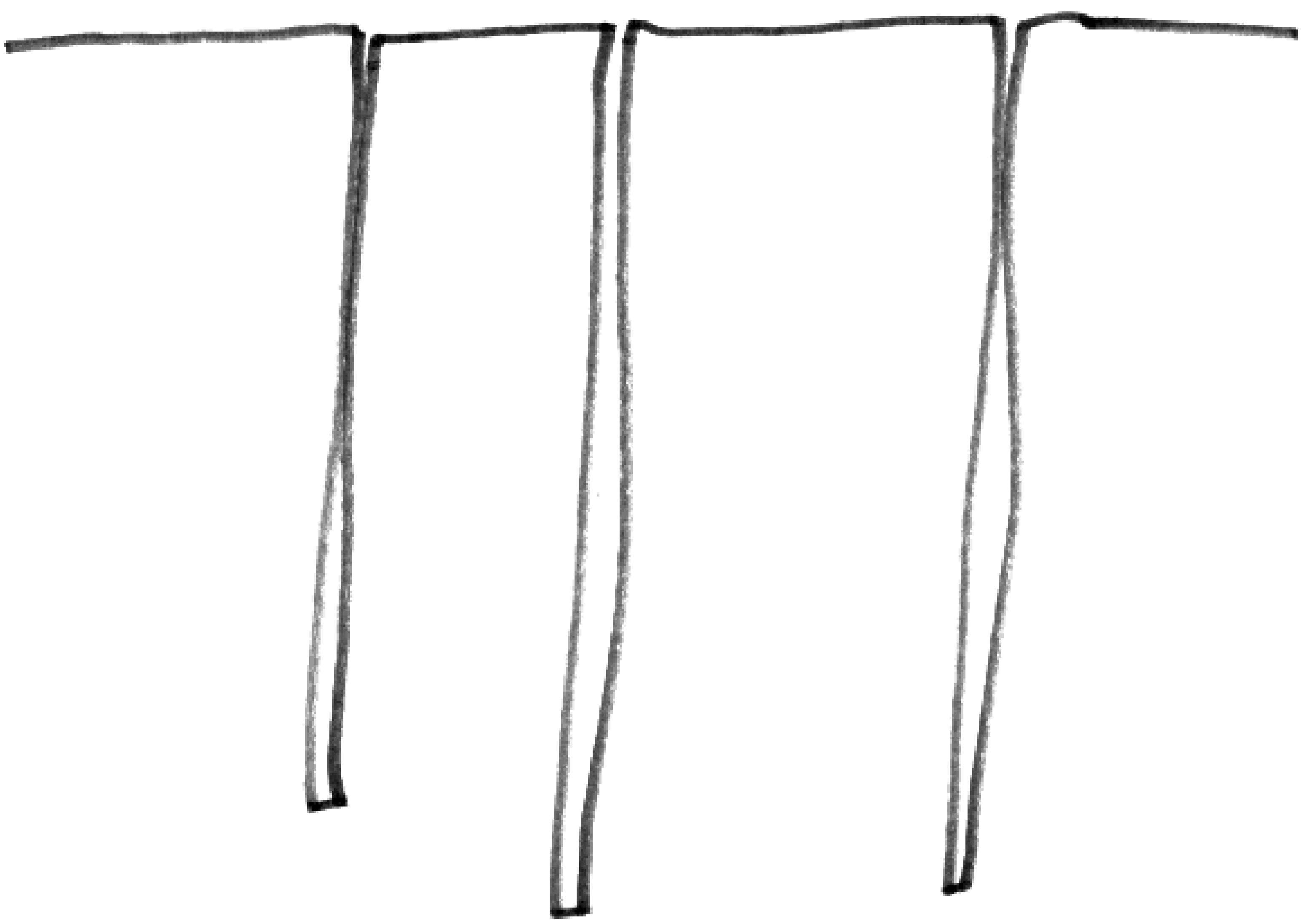
T = 0

High T gives a basically random behaviour.

Typically take small T

T decreases with time.

Algorithm fails if graph (5)
of Energy golf course



Scientific American
Algorithm of the gods.

Associate to

$$H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

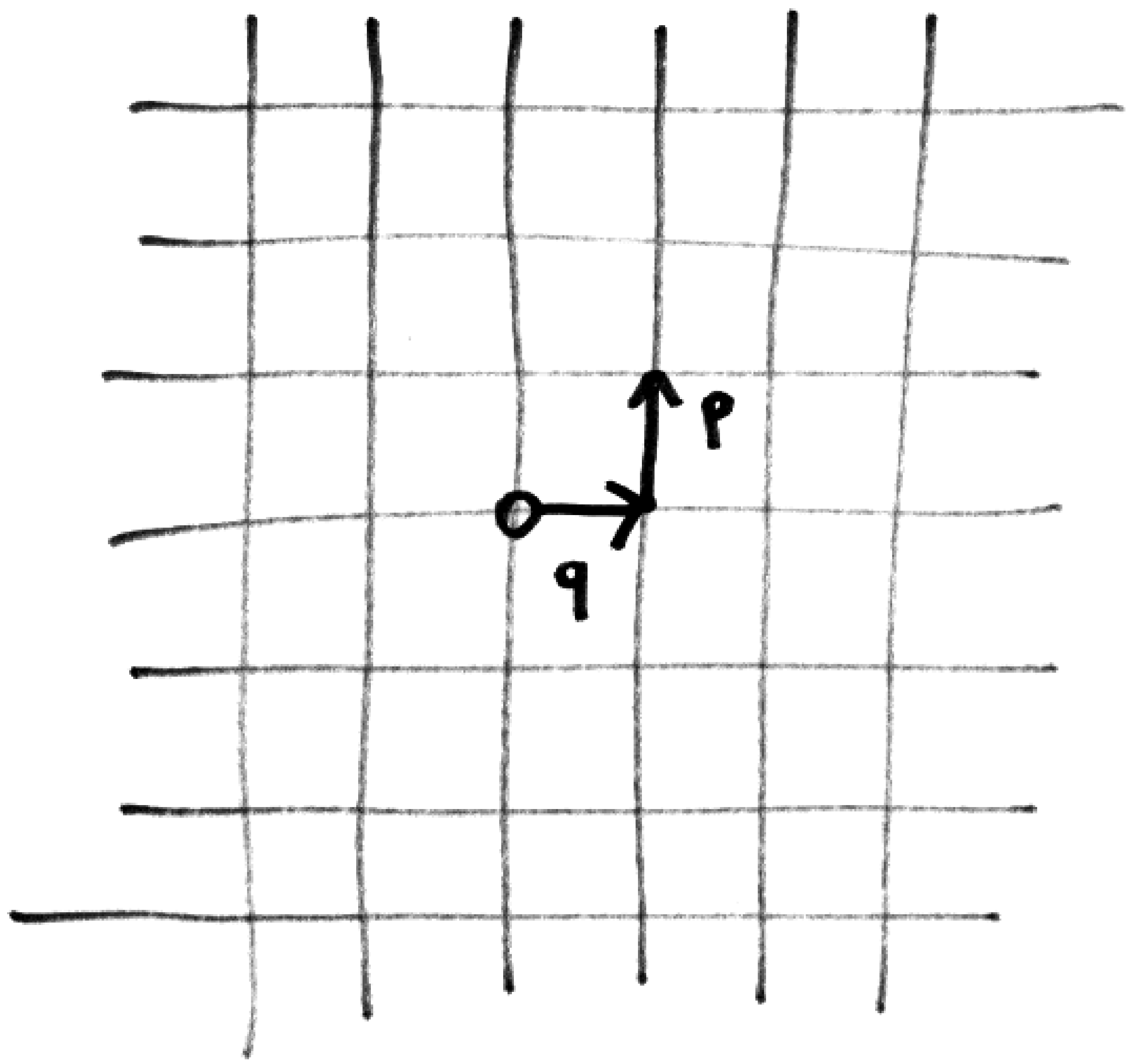
<u>Key</u> <u>property</u>

$$\boxed{HTH = TH \; [\text{redacted}] \; T}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Associate to a sequence of H's & T's a path in the lattice $\mathbb{Z}^2$:



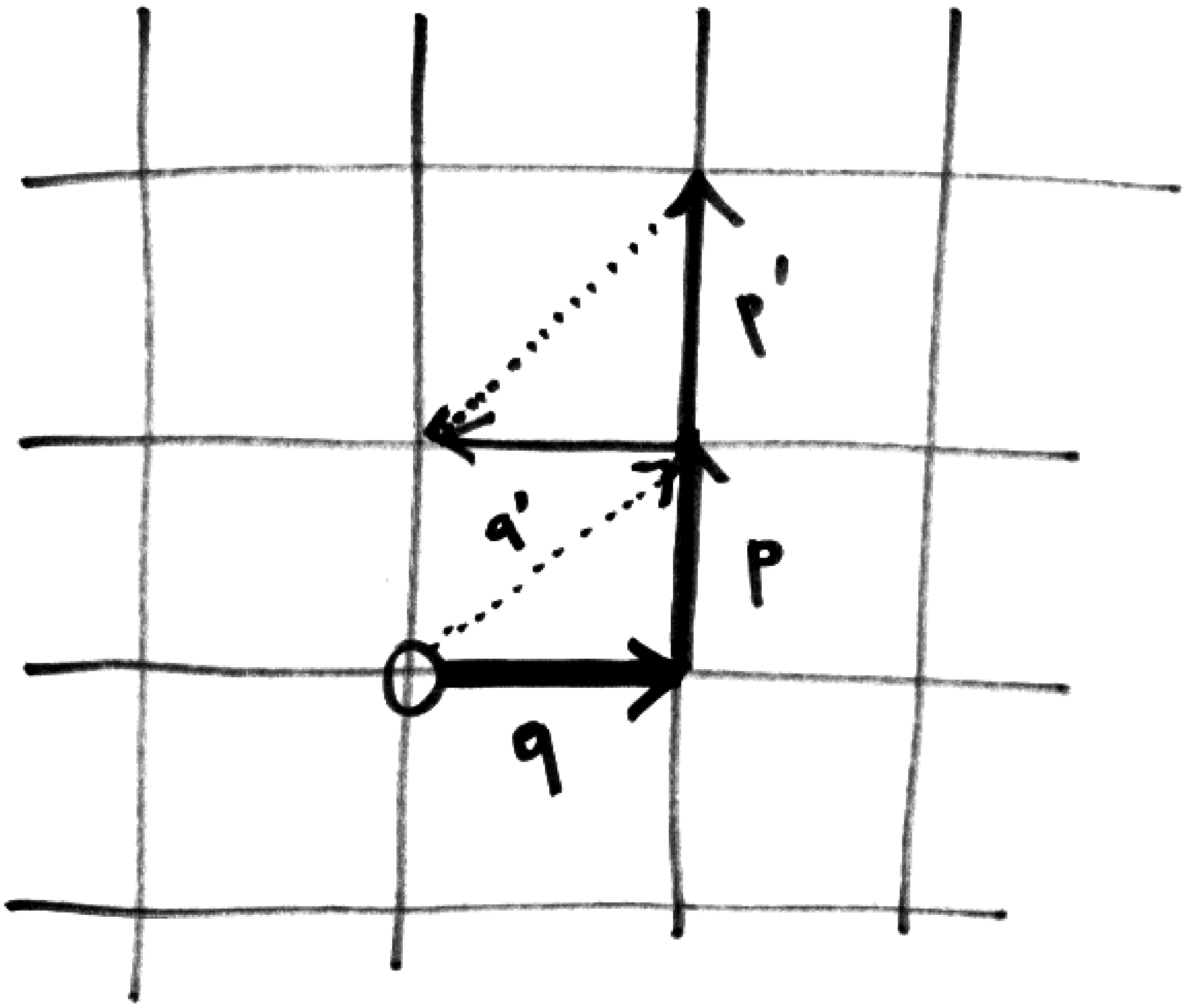
T H T H T H

$$H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} q \longmapsto q + p \\ p \longmapsto p \end{cases}$$

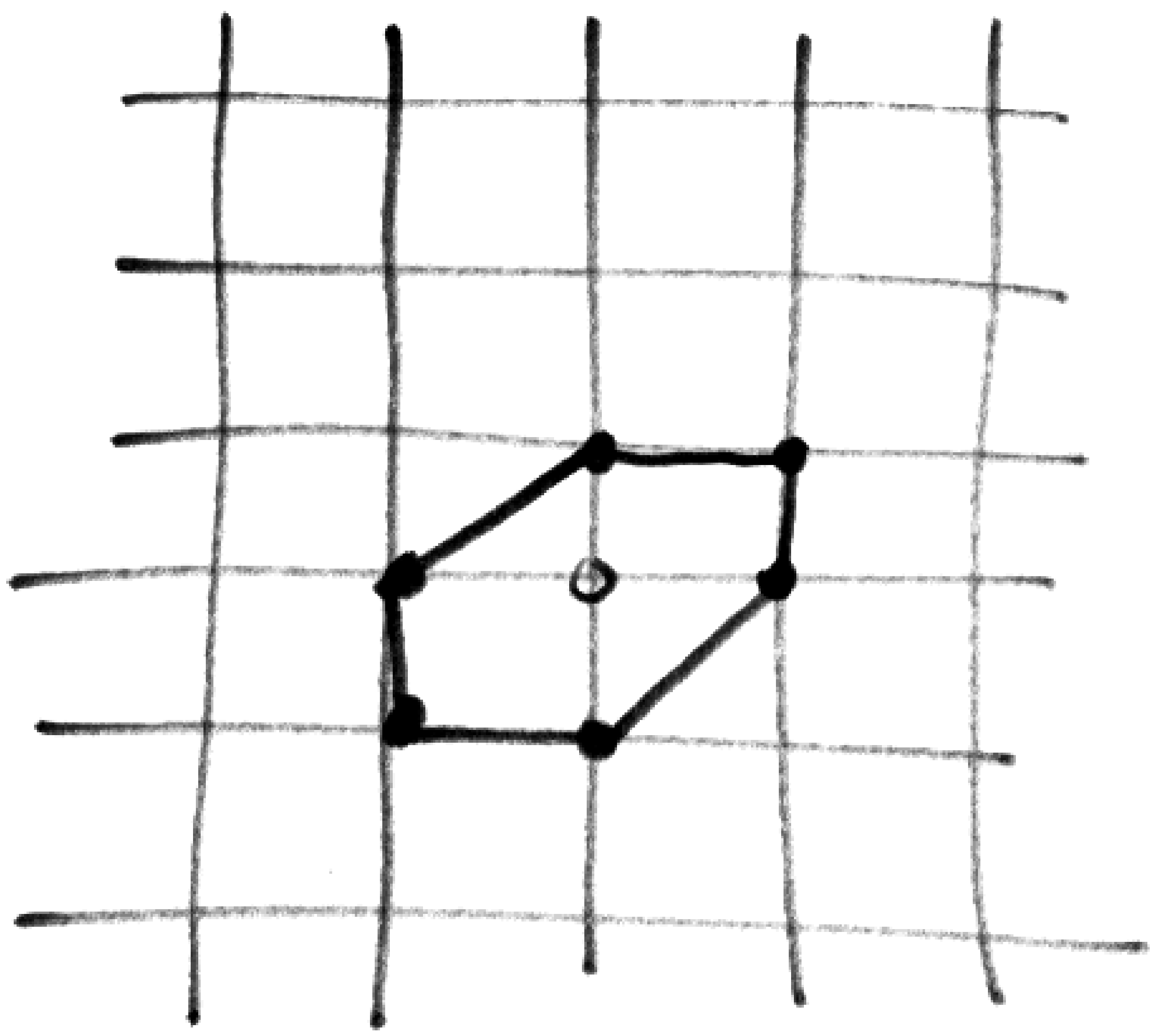
$$T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} q \longmapsto q \\ p \longmapsto p - q \end{cases}$$



$\cdots$ T H T H

$\leftarrow$



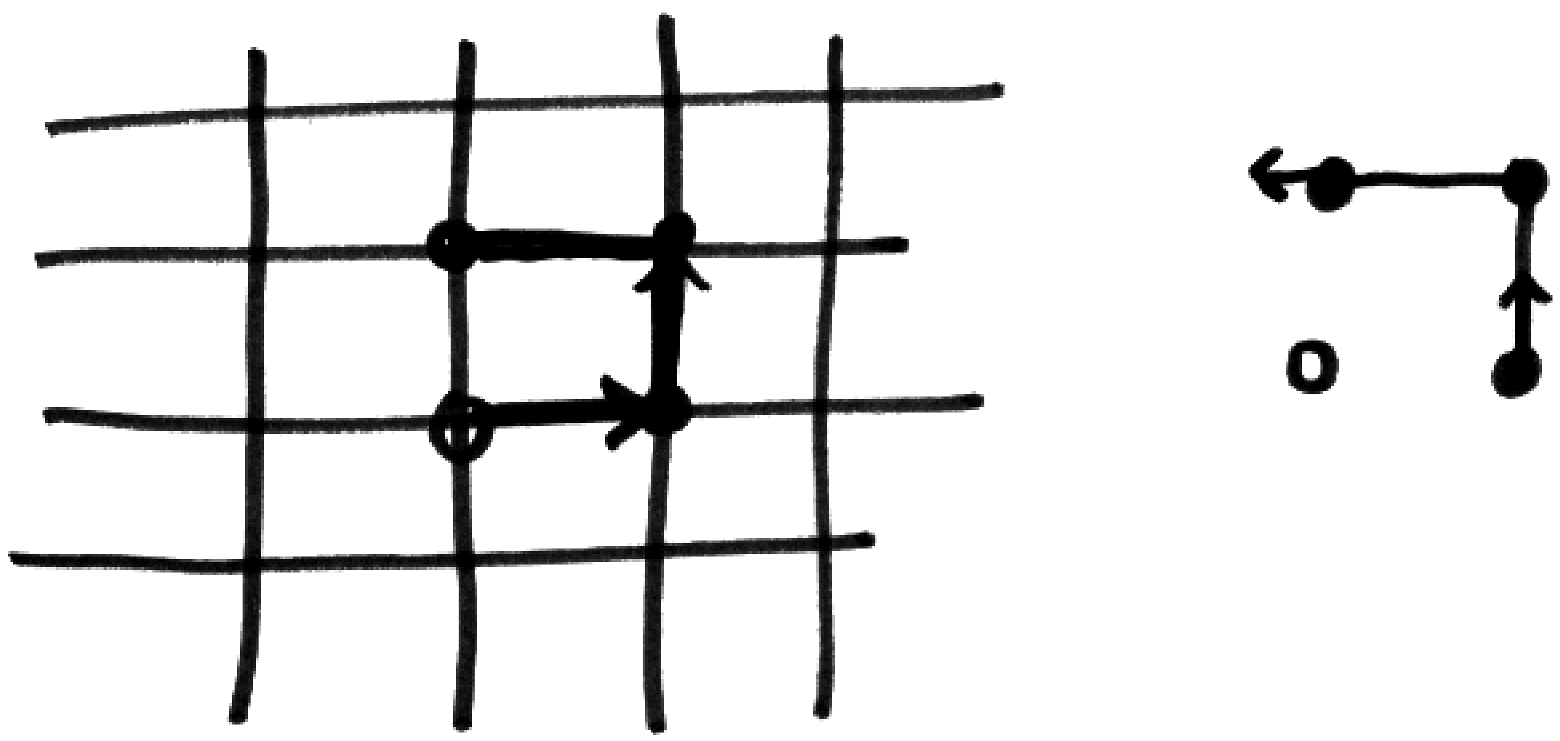
THTH THHT H T H T H

Come back to the exact same q. and p.

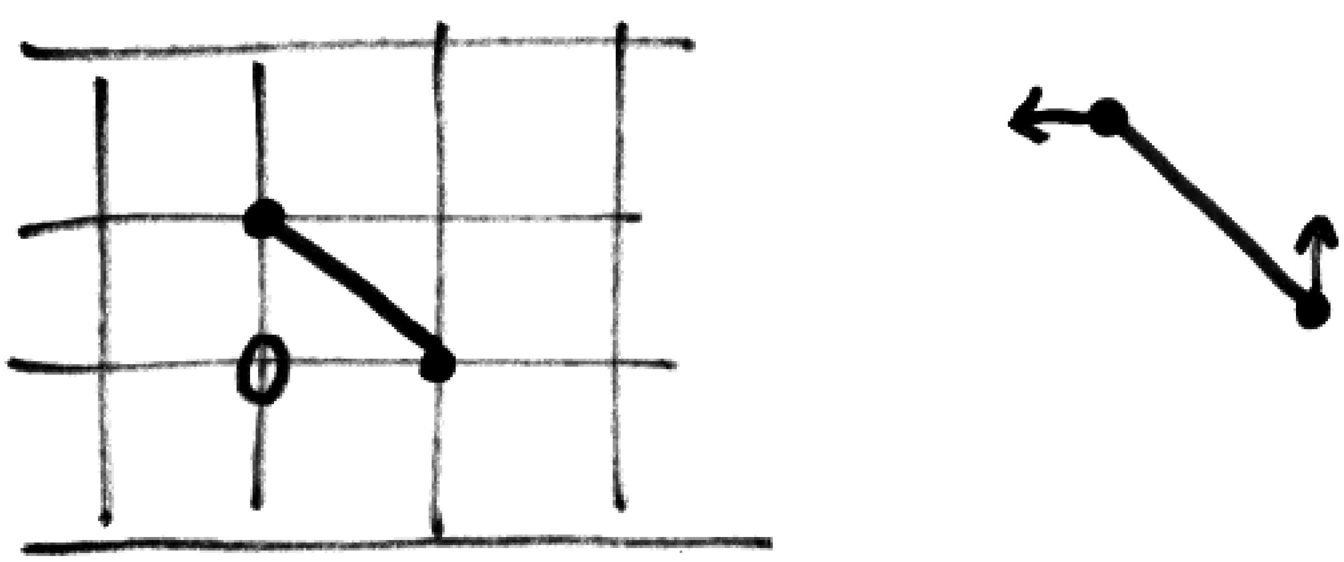
Notice that a sequence of T's in the sequence corresponds to a vertex on this path.

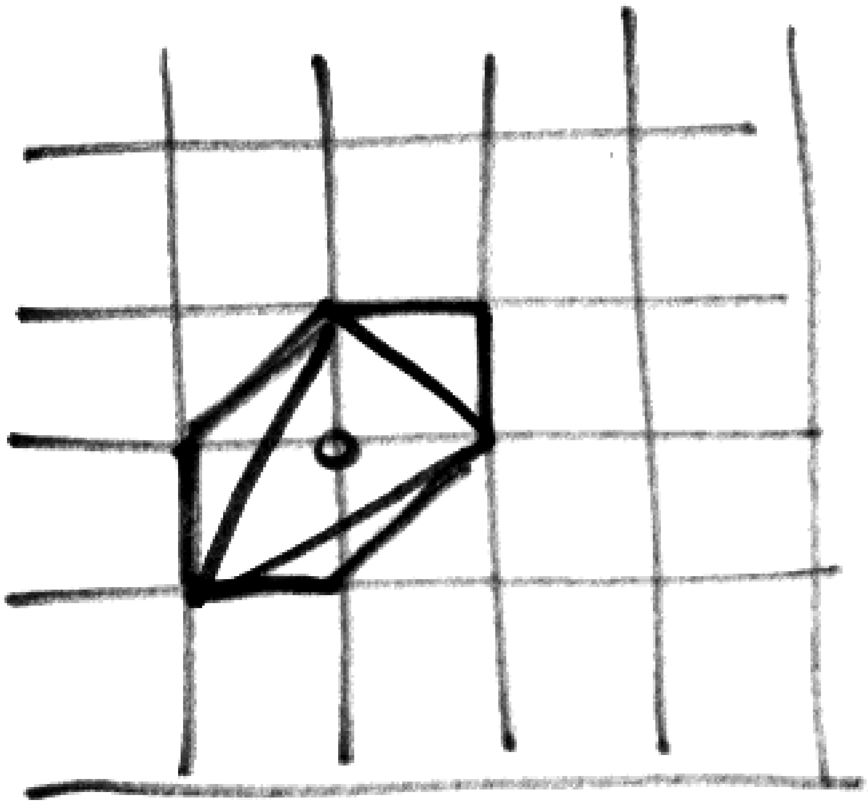
What happens to this picture when we apply the rule of the game?

H T H

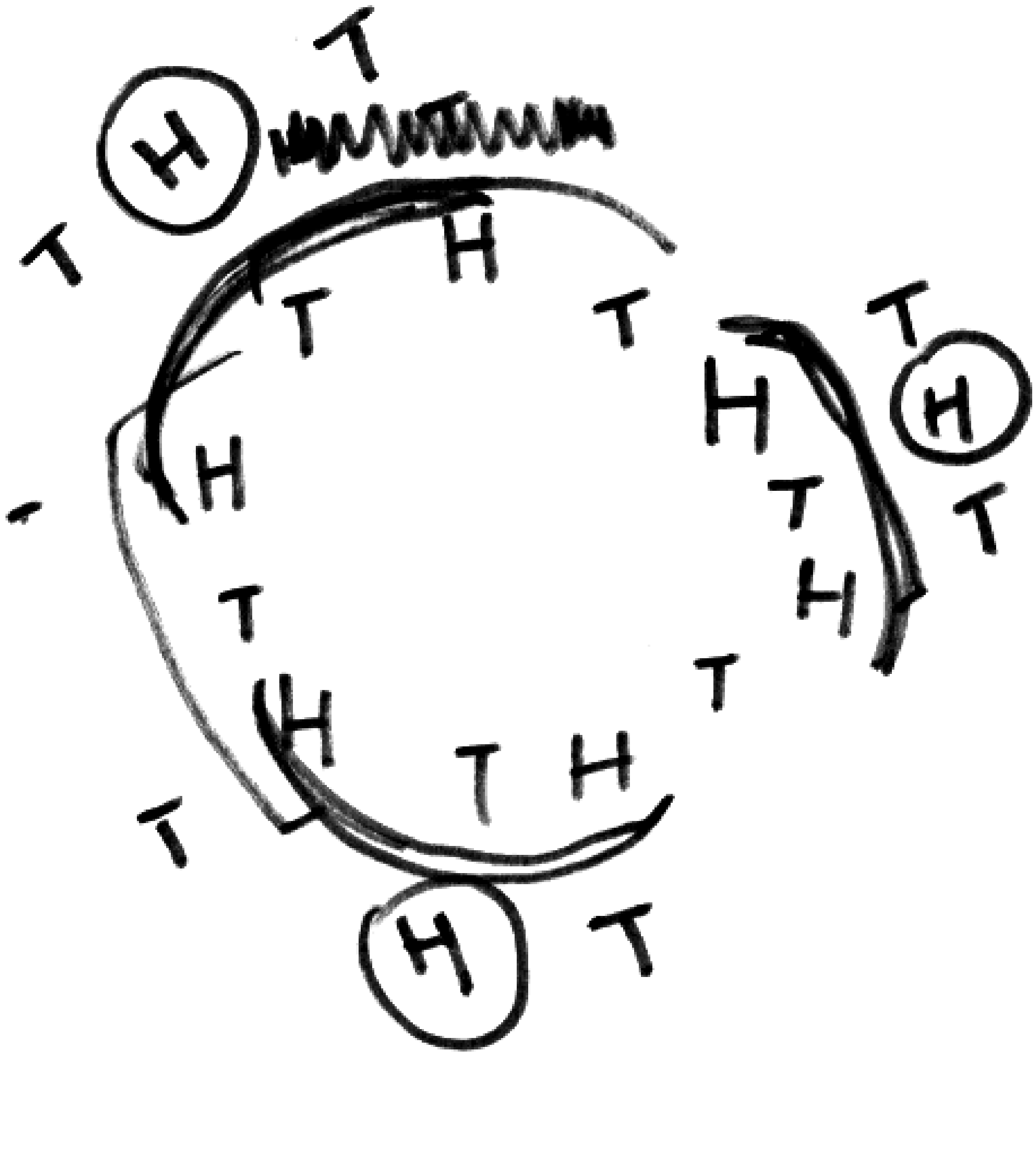


T H T



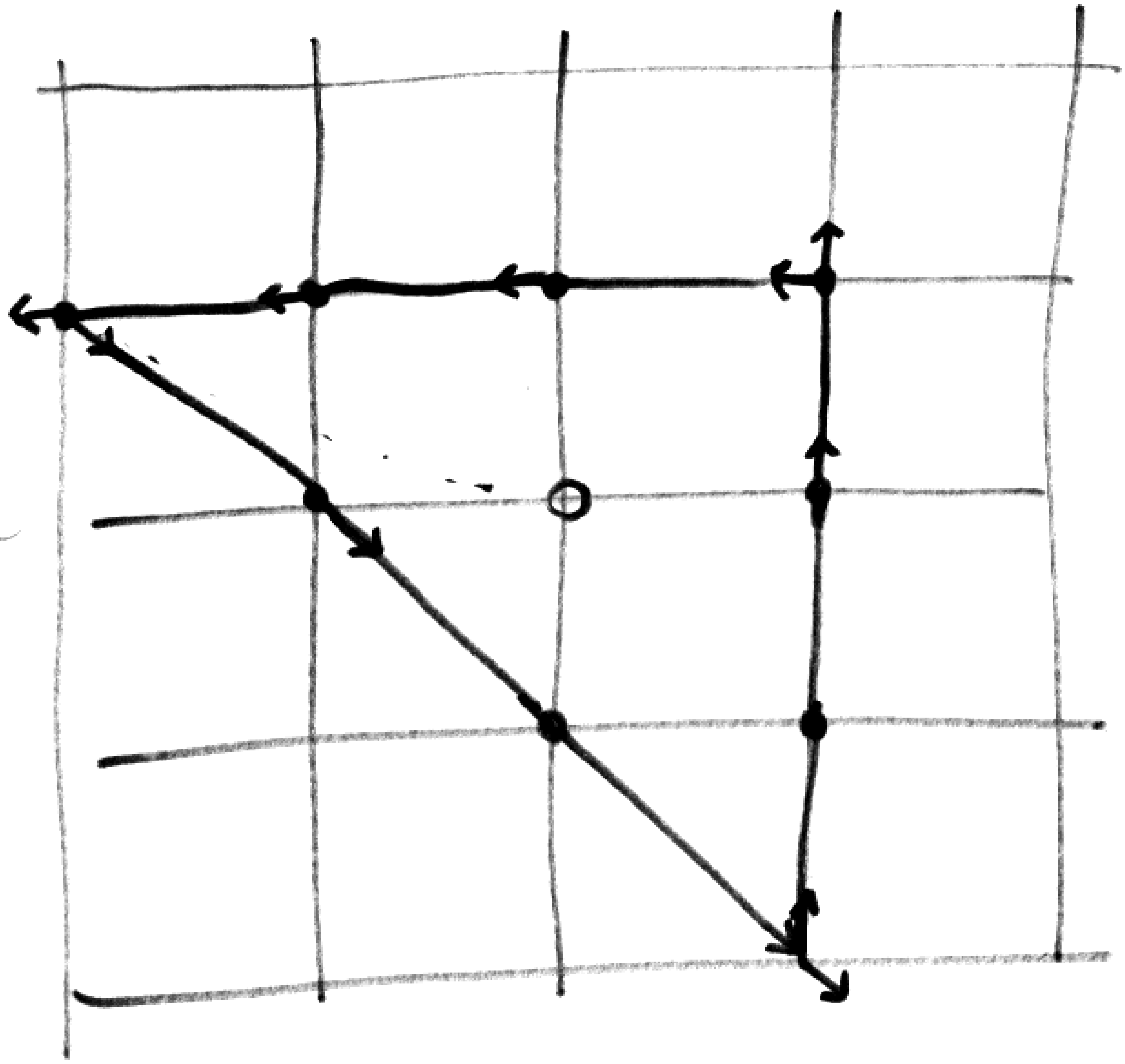


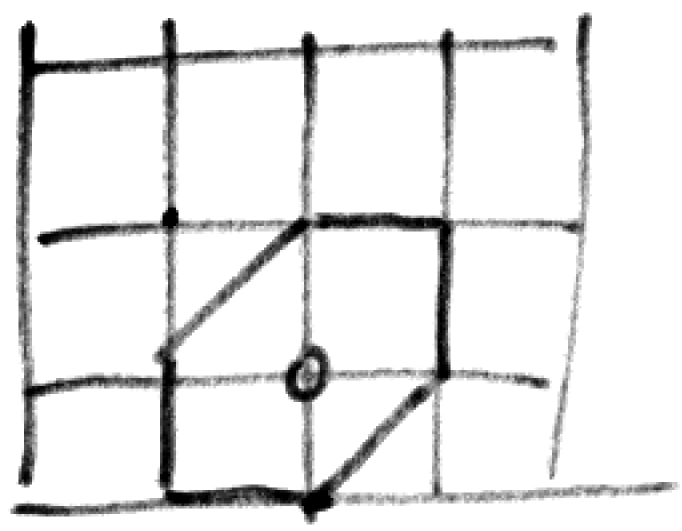
minimum possible H's!

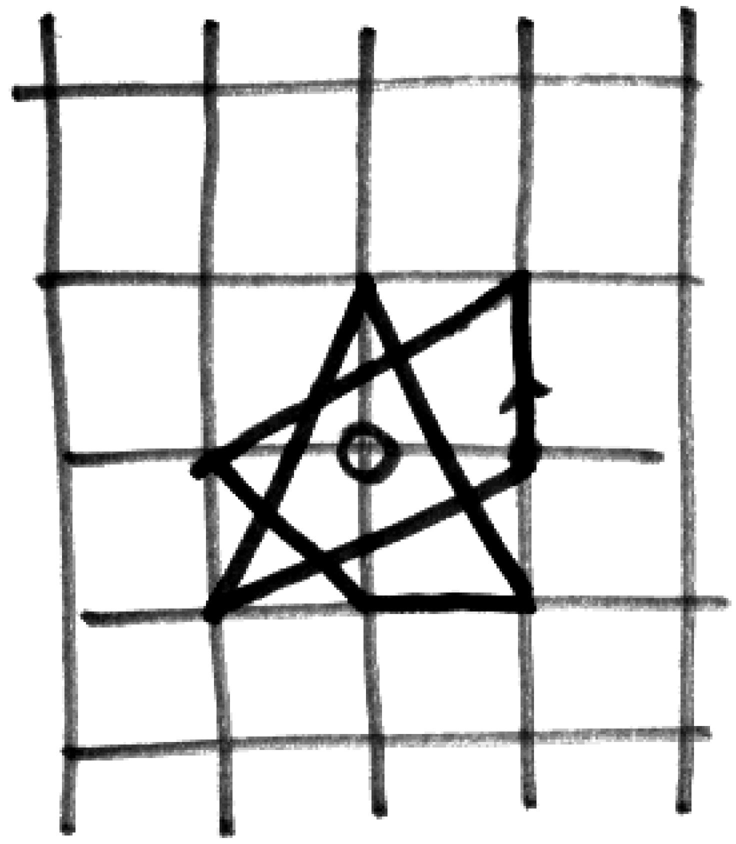


13

H H T H H H T H H H T H





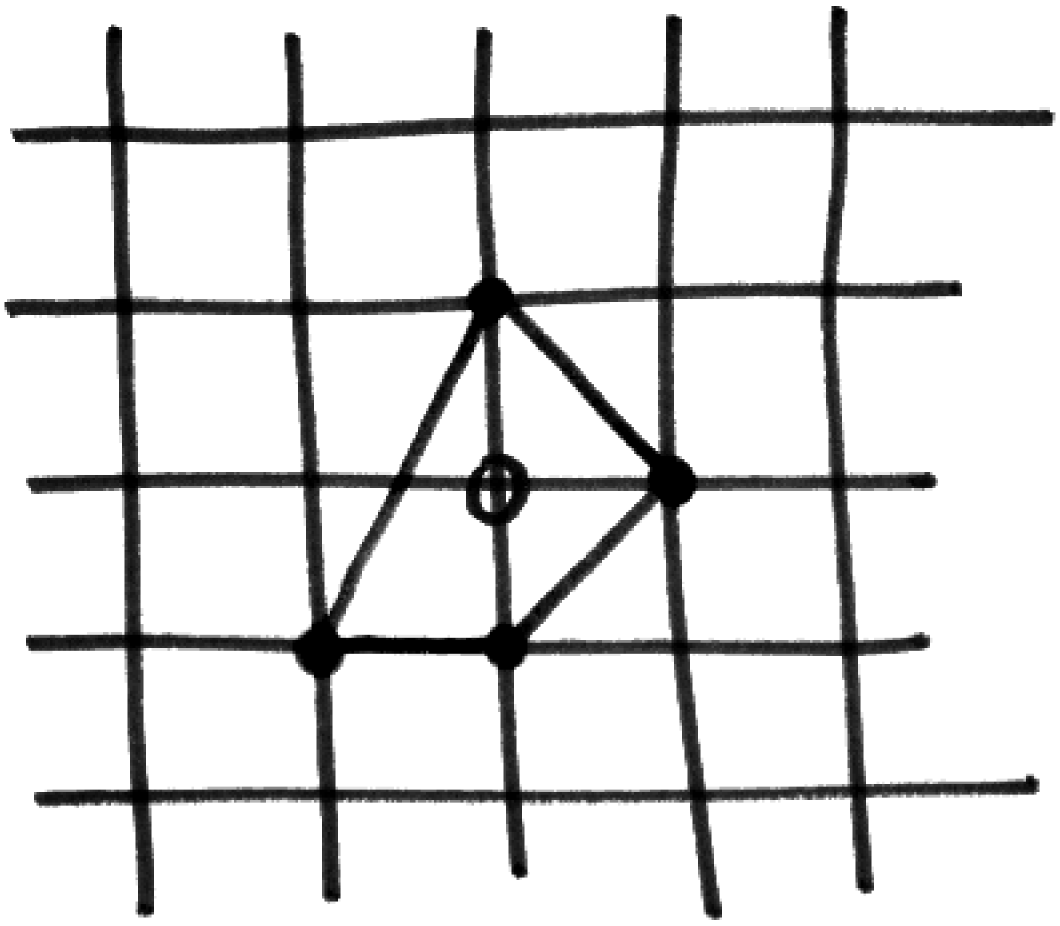


w = 2

<u>Fact</u>. Suppose our path closes up. Then the number of H's and T's in our word is 12 × the number of turns of the path.

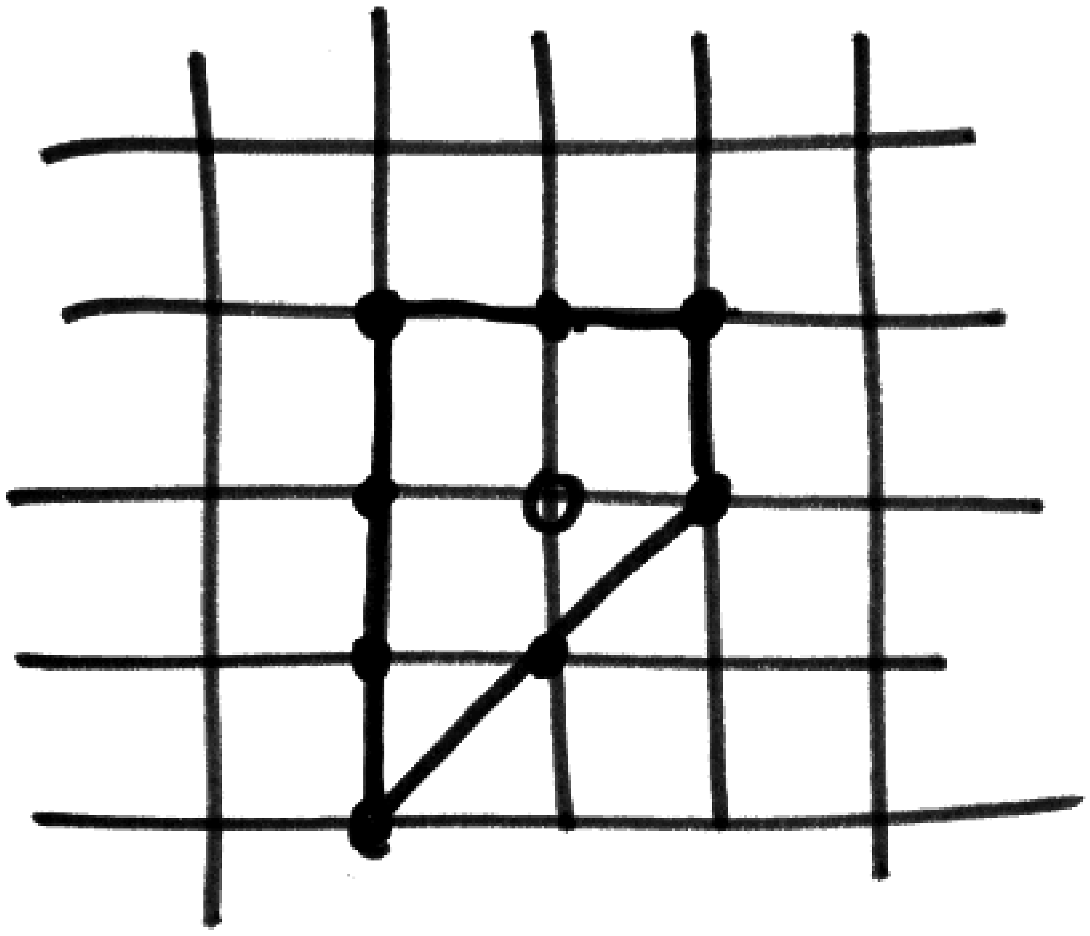
winding number = w

4



no other lattice points.

8



$8 + 4 = 12$

<u>goal</u> : maximize H's

$\mathcal{L}$: largest number of dots possible

$\hat{\mathcal{L}}$: smallest number of dots possible

<u>THEOREM</u>

ℓ = length of sequence

ℓ_H = # of H's

ℓ_T = # of T's

$\ell = \ell_H + \ell_T$

$$\frac{1}{6} < \frac{\ell_H}{\ell} < \frac{5}{6}$$

$$\frac{1}{6} < \frac{\ell_T}{\ell} < \frac{5}{6}$$

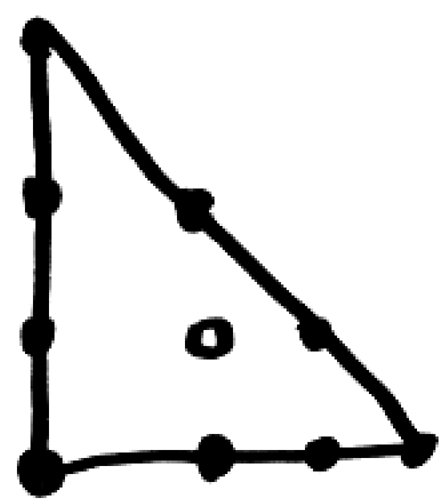
Conclusion

$$\max \# H\text{'s} < \frac{5}{6}\ell$$

e.g. $\ell = 12$

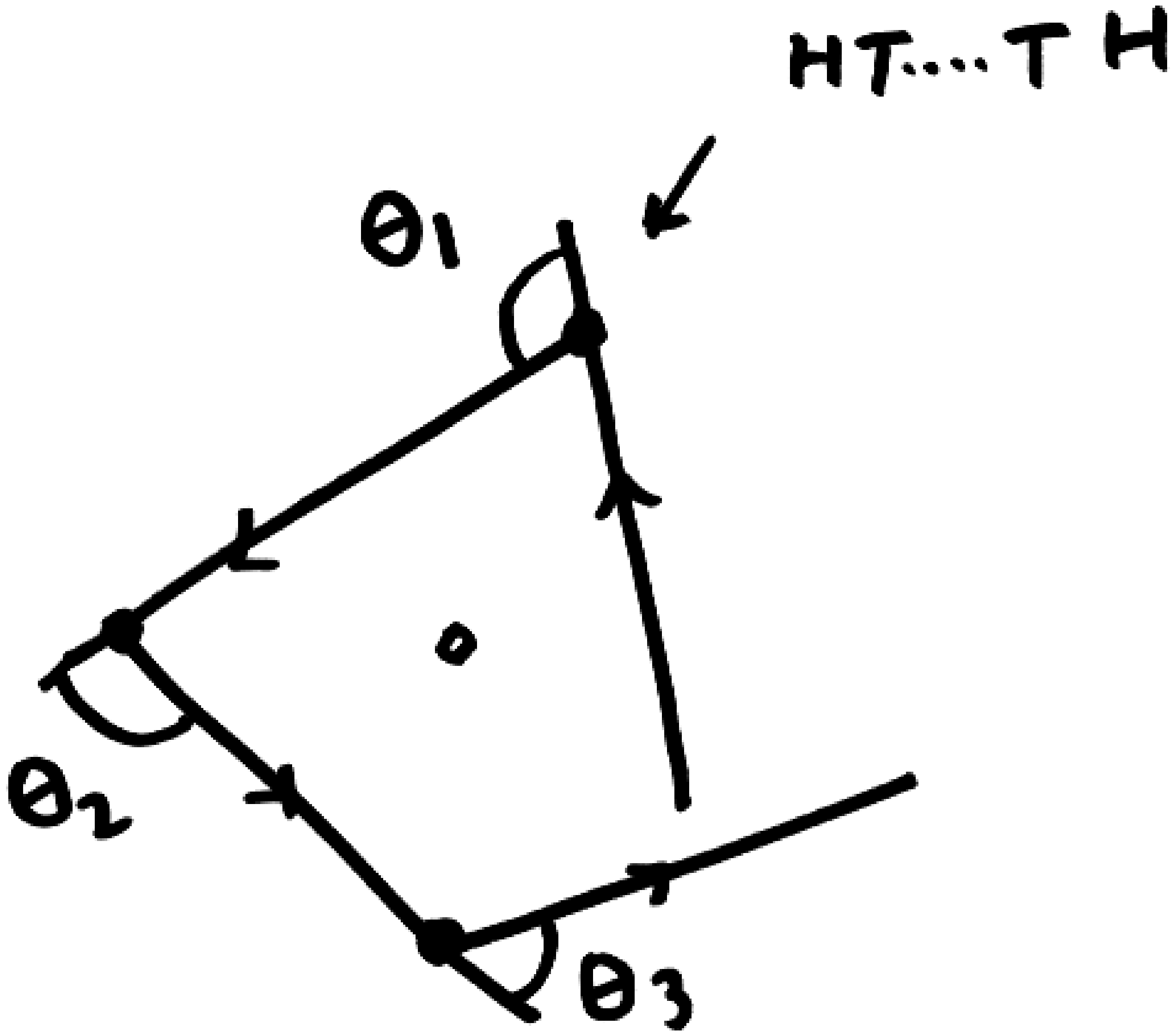
$$\frac{5}{6} \cdot 12 = 10$$

$$\max \# H\text{'s} = 9$$



$$T H^3 T H^3 T H^3$$

$\mathcal{L}$



say r corners in $\mathcal{L}$

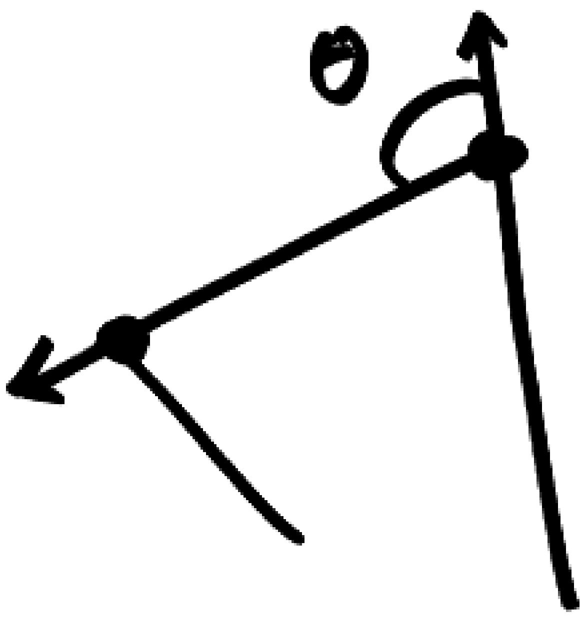
$$H\underbrace{\boxed{T \cdots T}}_{\theta_1} HH \underbrace{\boxed{T \cdots T}}_{\theta_2} HH \underbrace{\boxed{T \cdots T}}_{\theta_3} H \cdots \quad \cdots$$

$m :=$ winding number of $\mathcal{L}$

$\theta_1, \theta_2, \ldots, \theta_r$

$$\boxed{\theta_1 + \cdots + \theta_r = 2\pi m}$$

$$\boxed{\ell = 12\ m}$$



$$0 < \theta < \pi$$

$$r\pi > \theta_1 + \cdots + \theta_r = 2\pi m$$
$$= \pi \frac{\ell}{6}$$

$$r > \frac{\ell}{6}$$

$$\ell_T \geqslant r$$

$$\ell_T > \frac{\ell}{6}$$

$$\boxed{\frac{\ell_T}{\ell} > \frac{1}{6}}$$

7

$$\frac{\ell_T}{\ell} > \frac{1}{6}$$

$$\Rightarrow \quad \frac{\ell_H}{\ell} > \frac{1}{6}$$

$$\frac{\ell_T}{\ell} + \frac{\ell_H}{\ell} = 1$$

$$\Rightarrow \quad \frac{\ell_T}{\ell} < \frac{5}{6}$$

$$\frac{\ell_H}{\ell} < \frac{5}{6} \qquad \square$$

Can this theoretical bound be obtained?

Yes.

Winning strategy

$b(n)$ = max # of H's achievable with an open string

H T H T H ..... H T H T

Thm $b(n+6) \geq b(n)+5$

Pf

$$[\overbrace{\bullet\!-\!-\!-\!-\!-\!-\!-\!-\!\bullet}^{n}]\ \boxed{\text{HTHTHT}}$$

Play on the first n to get $b(n)$ H's

$$\underbrace{[\qquad\qquad\qquad]}_{b(n)\text{ H's}}\ \boxed{HTHTHT}$$

<u>Case 1</u>

$$(\cdots\cdots\ T)\ \boxed{HTHTHT}$$

$$\cdots\ \underbrace{THT}\ H\ \underbrace{THT}$$

$$\underbrace{\cdots}_{b(n)}\ \underset{1}{H}\ T\ \underset{2}{H}\ \underset{3}{H}\ \underset{4}{H}\ T\ \underset{5}{H}$$

total $= b(n) + 5$
H's

Case 2

(...... H) $\boxed{\text{H T H T H T}}$

? ..... H) H T H $\underline{\text{T H T}}$

... H) H T H H T H
1 2 3 4

? .... H) H $\underline{\text{T H T}}$ H T

... H) H H T H H T
1 2 3 4

Fact

slide * $\boxed{\text{H T H T H T}}$

* = H

H $\underline{\text{H T H}}$ T H T

H T H T $\underline{\text{T H T}}$

$\boxed{HTHTHT}$ H

⊛ = T

T H T H T H T

H T H H T H T

$\boxed{HTHTHT}$ T

Case 2

$$\underbrace{(\cdots\cdots H}_{b(n)})\ \boxed{HTHTHT}$$

$$\cdots\cdots \circledast)\ \boxed{HTHTHT}\ H$$

if $\circledast = T$ we are back to case 1

if $\circledast = H$ slide again

at some point we'll get a T and we are done. $\square$

$$b(n+6) \geq b(n) + 5$$

Largest number H's possible is $\left[\frac{5n-1}{6}\right]$

n = total length, which is the theoretical bound.

$$H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

sequence H's, T's

. . . . . H H T T T H

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

H

TH

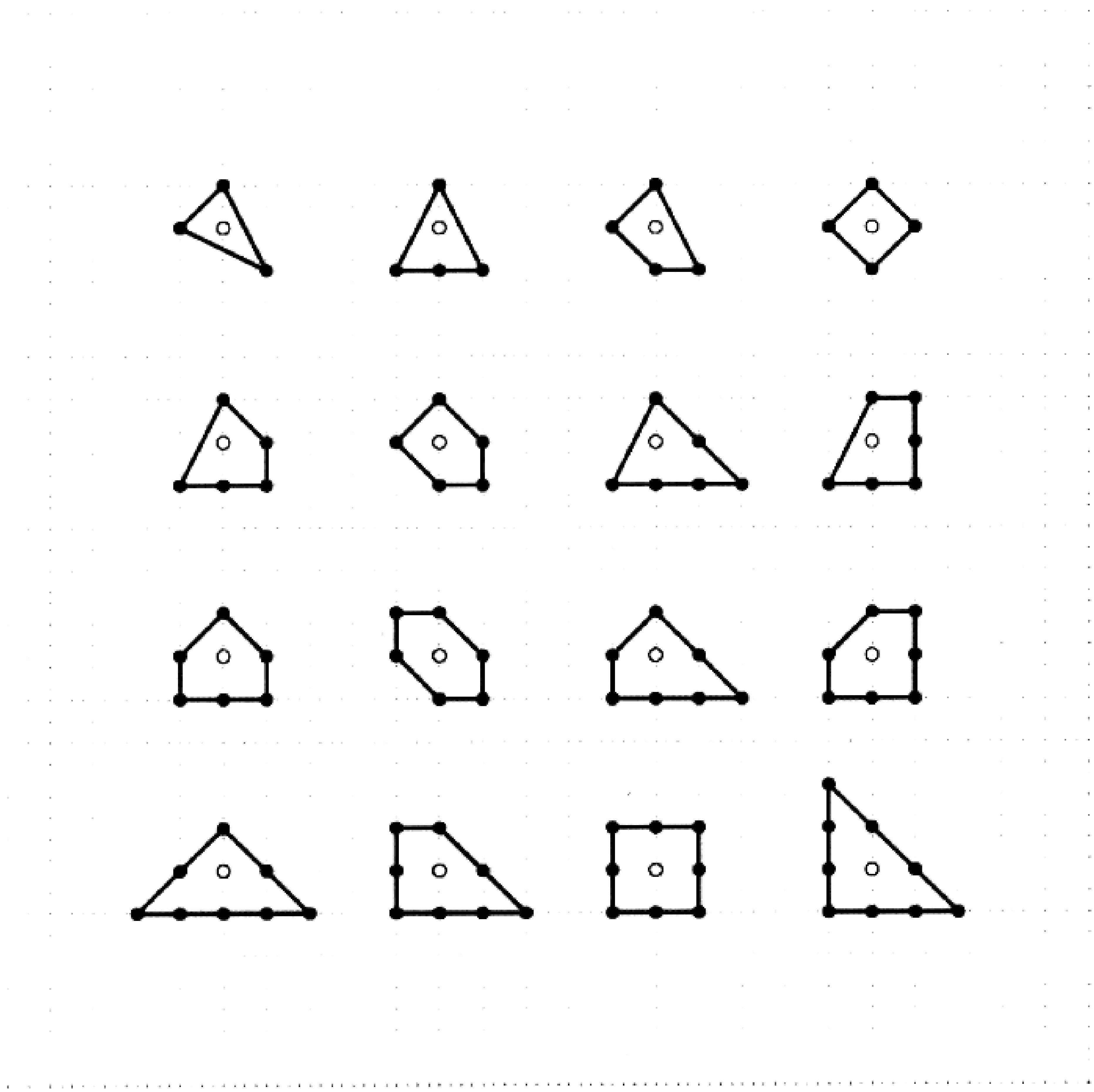
TTH

⋮

sequence of matrices

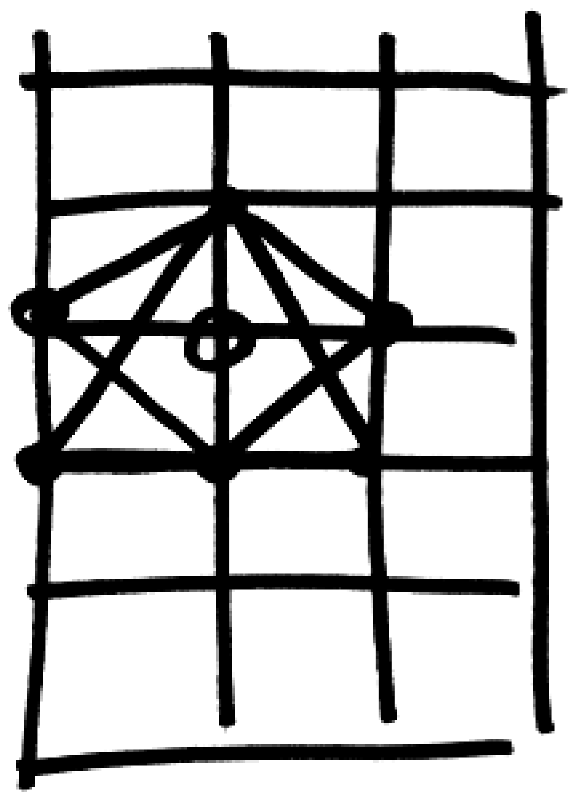
det = +1

integer coefficients



In fact path $\mathcal{L}$
and a dual path

$\mathcal{L}$



dot $\longleftrightarrow$ H

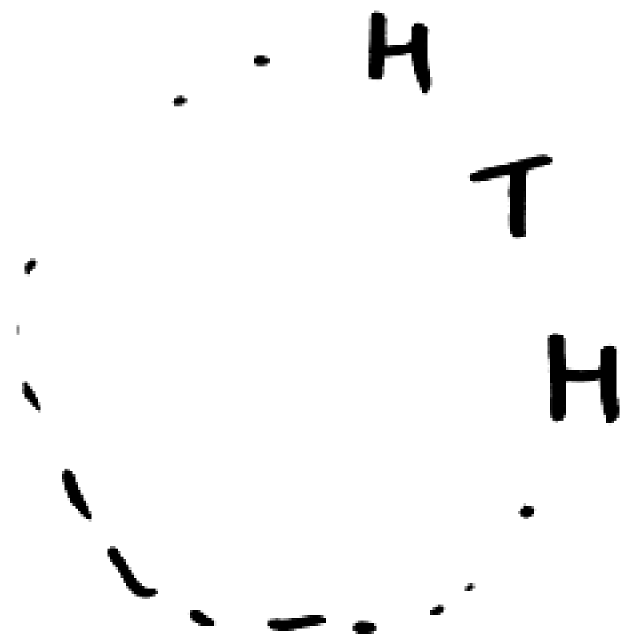
corner $\longleftrightarrow$ $H \underbrace{T \cdots T} H$

$\hat{\mathcal{L}}$ reverses role of H, T

dot $\longleftrightarrow$ T

corner $\longleftrightarrow$ $T \underbrace{H \cdots H} T$

Blet



$\boxed{HTH} \longleftrightarrow \boxed{THT}$

- Any position of puzzle sequence of H's, T's in a circle.
- position $\rightsquigarrow$ path in the plane

This is a group

$SL_2(\mathbb{Z})$

sequence $\rightsquigarrow$ a path in $SL_2(\mathbb{Z})$

encodes both $\mathcal{L}$, $\hat{\mathcal{L}}$ together

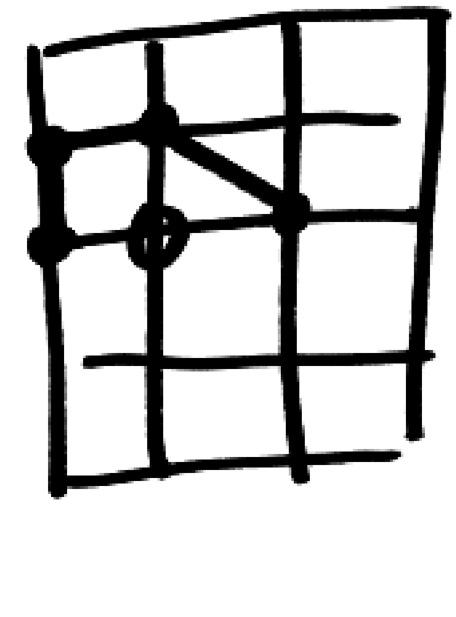
M

first column ↙ point of $\mathcal{L}$

↘ second column point of $\hat{\mathcal{L}}$

$\boxed{\text{HTHTHT}}$

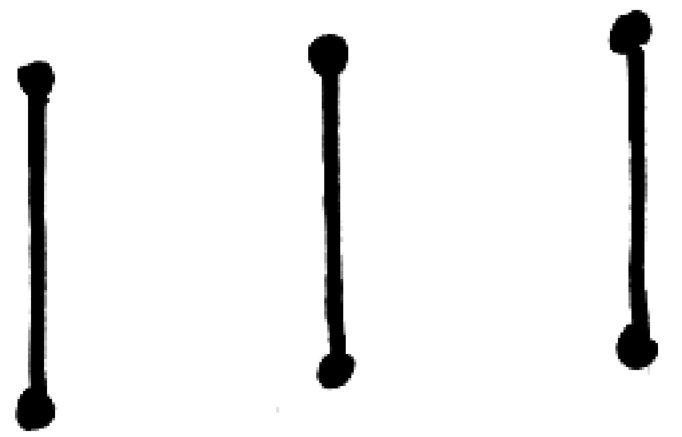
$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



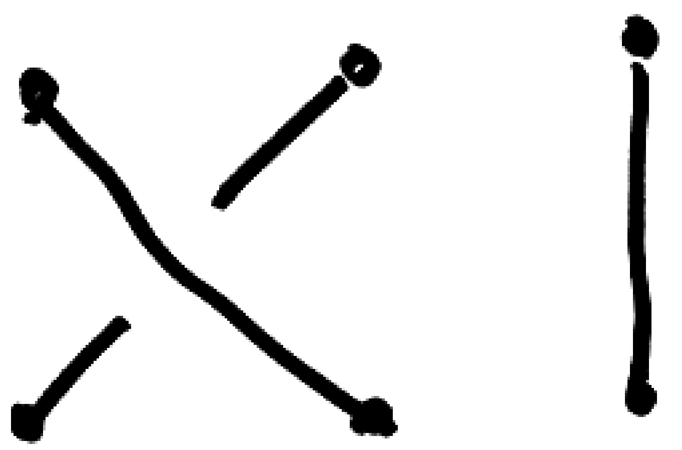
Braid gp on 3 strands

(Artin 20's)

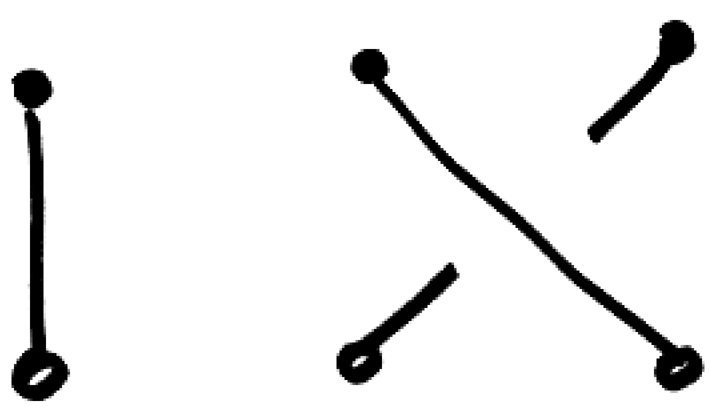
identity



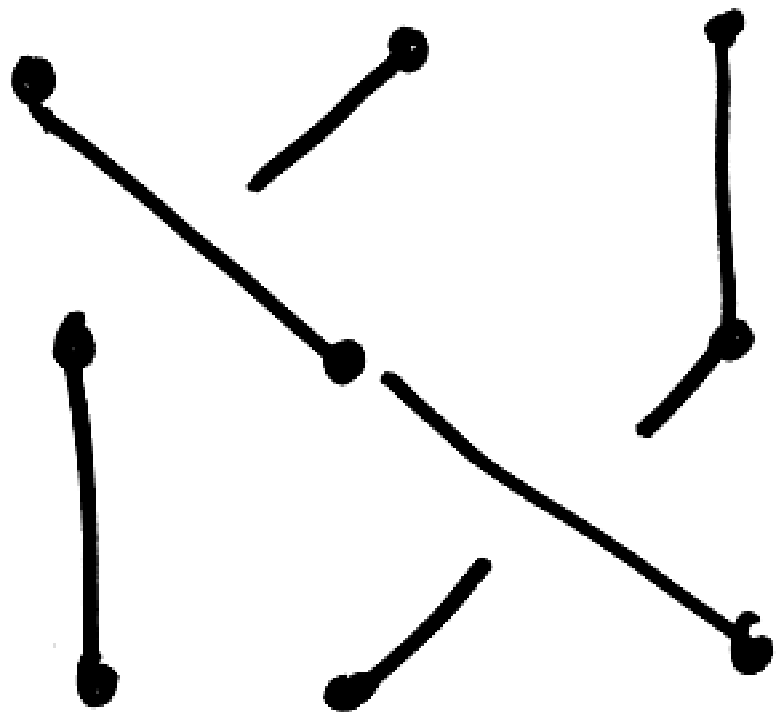
H



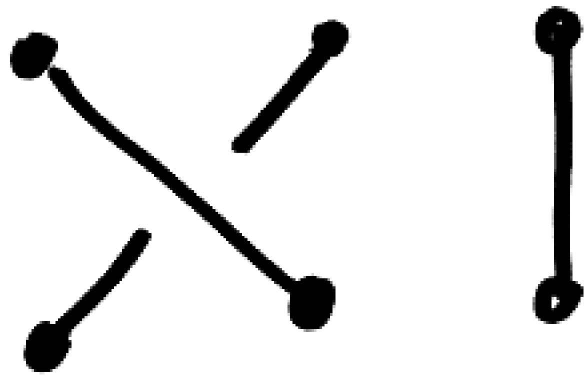
T

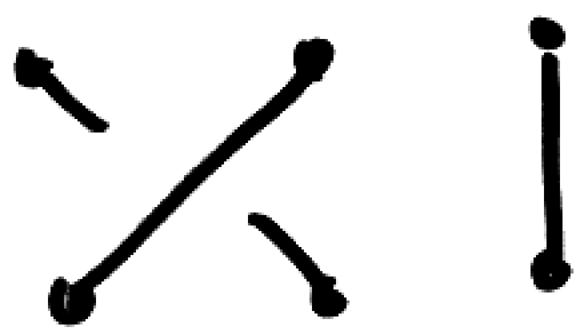


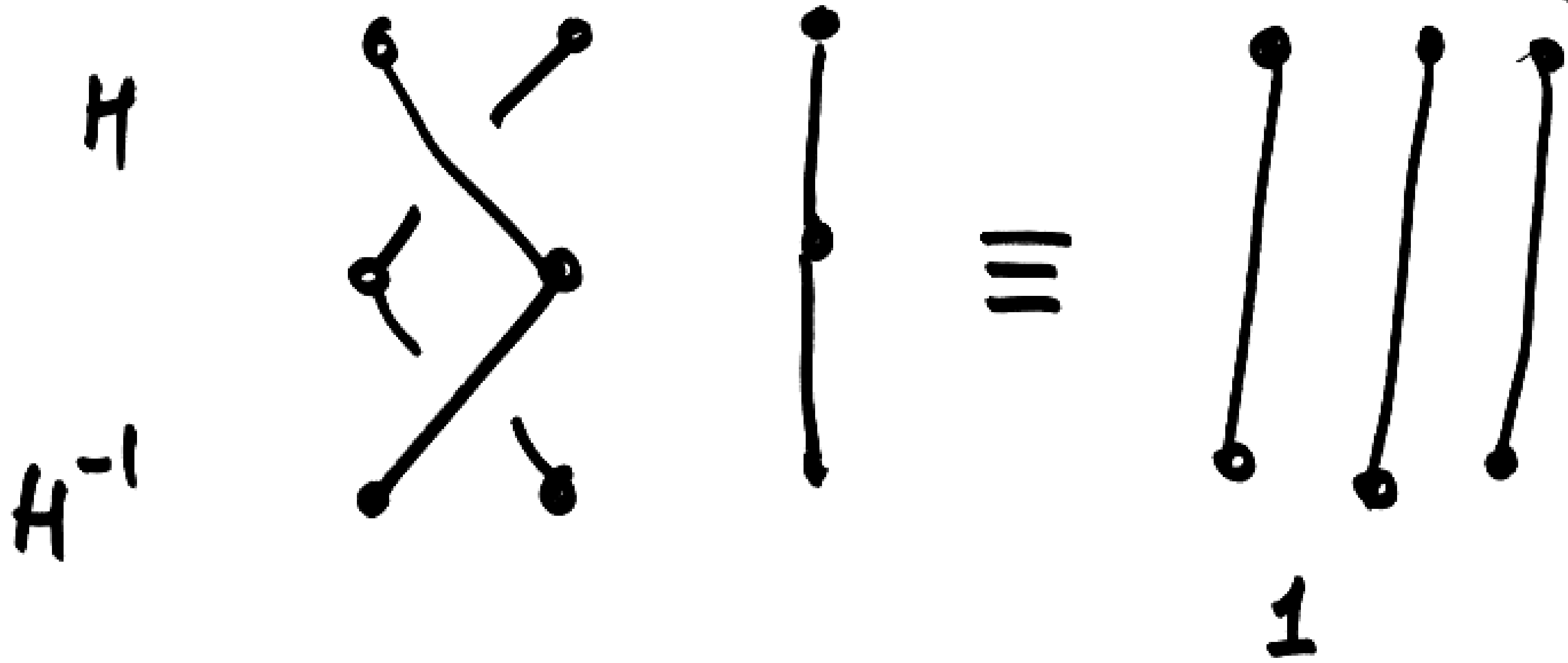
Multiply braids by concatenating



For example

H 

H^{-1} 



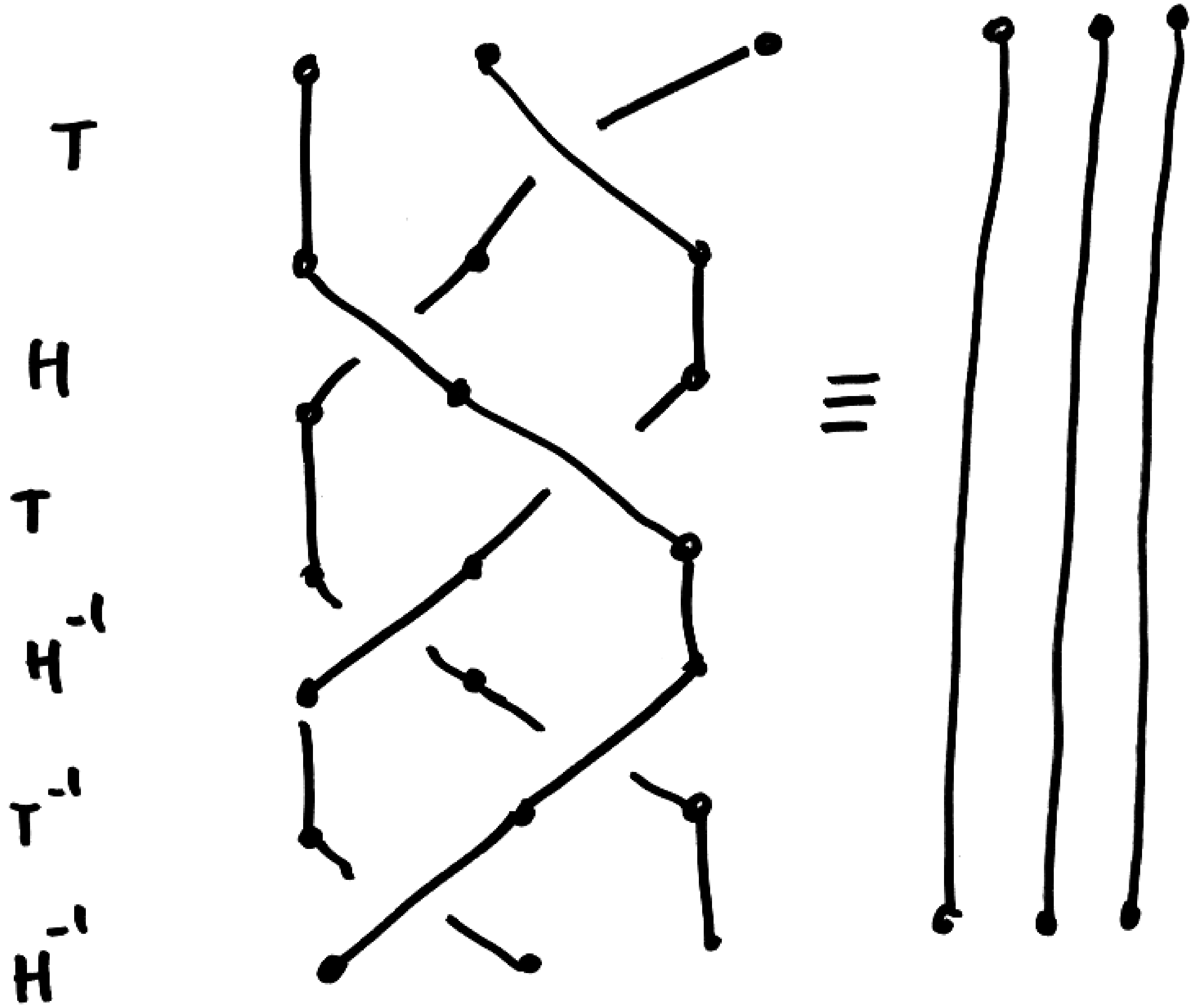
$$H \cdot H^{-1} = 1$$

$$\boxed{HTH = THT}$$

Braid gp is generated by H_0 & T.

$HTH = THT$?

$H^{-1}T^{-1}H^{-1}THT = 1$?



Every other relation among H & T arises from this basic one.

$$HTH = THT$$

$$U = \boxed{\underline{HTHTHT}}$$

<u>Claim</u>: generates the center of the braid group B_3

i.e. any braid which commutes w/every other is of the form

$$U^m$$

$$B_3 \longrightarrow SL_2(\mathbb{Z})$$

$$H \longmapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$T \longmapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

multiplicative

well defined.

$$HTH = THT$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

Table 2. A Nim-Addition Table.

(Berlekamp)

Theorem

Dots + Double crosses

= turns.

Theorem implies chain rule.

Suppose Dots is odd

turns opposite parity to double crosses

double crosses = long chains − 1

Player 1 → odd number of long chains

Player 0 → even number of chains

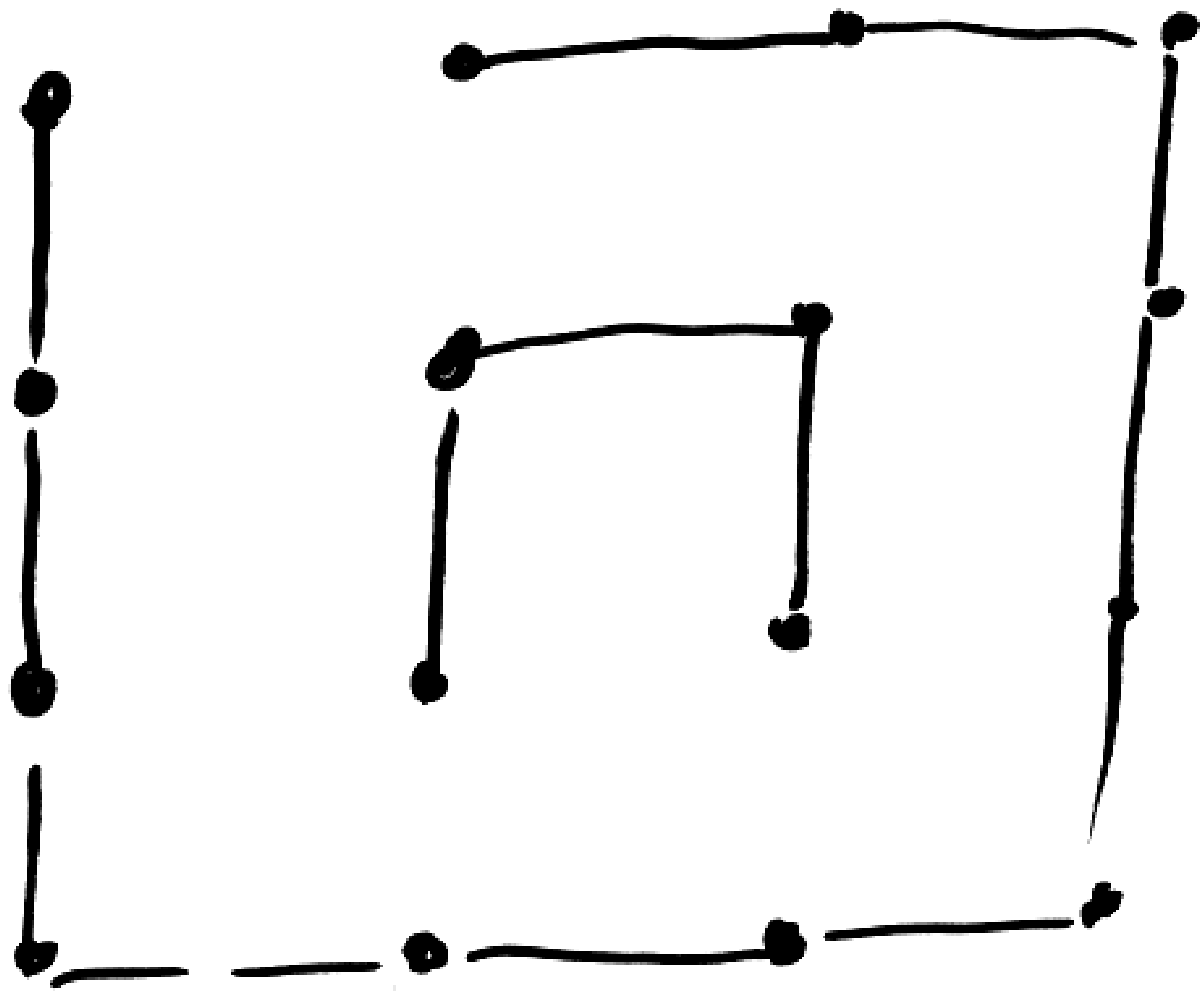
Dots even

Double crosses ≡ turns mod 2

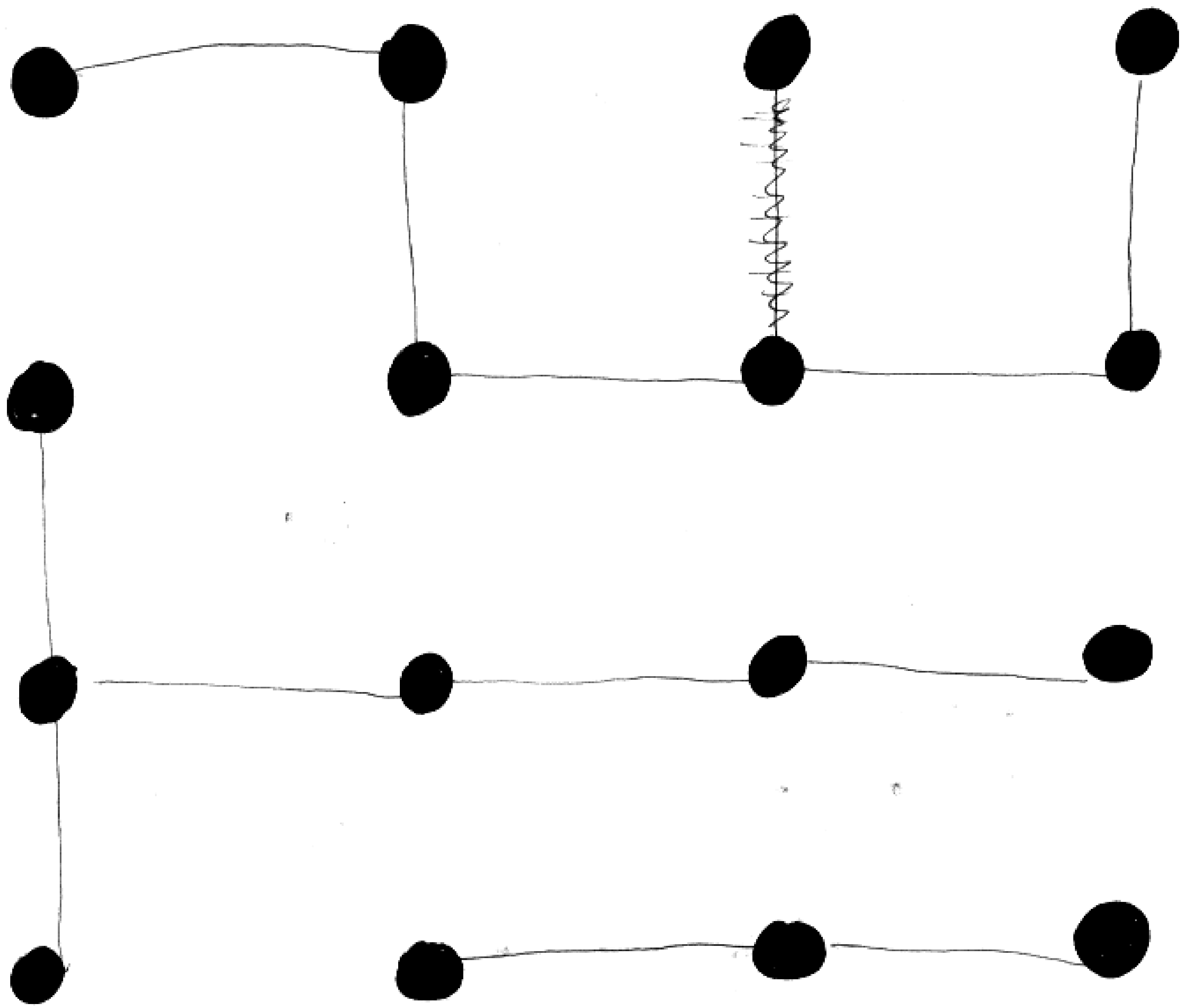
⟹ turns opposite parity long chains

Player 1 → even number long chains

Player 0 → odd number of chains



Q: What do we do with cycles?



Double dealing

Summary

long chains (≥ 3)

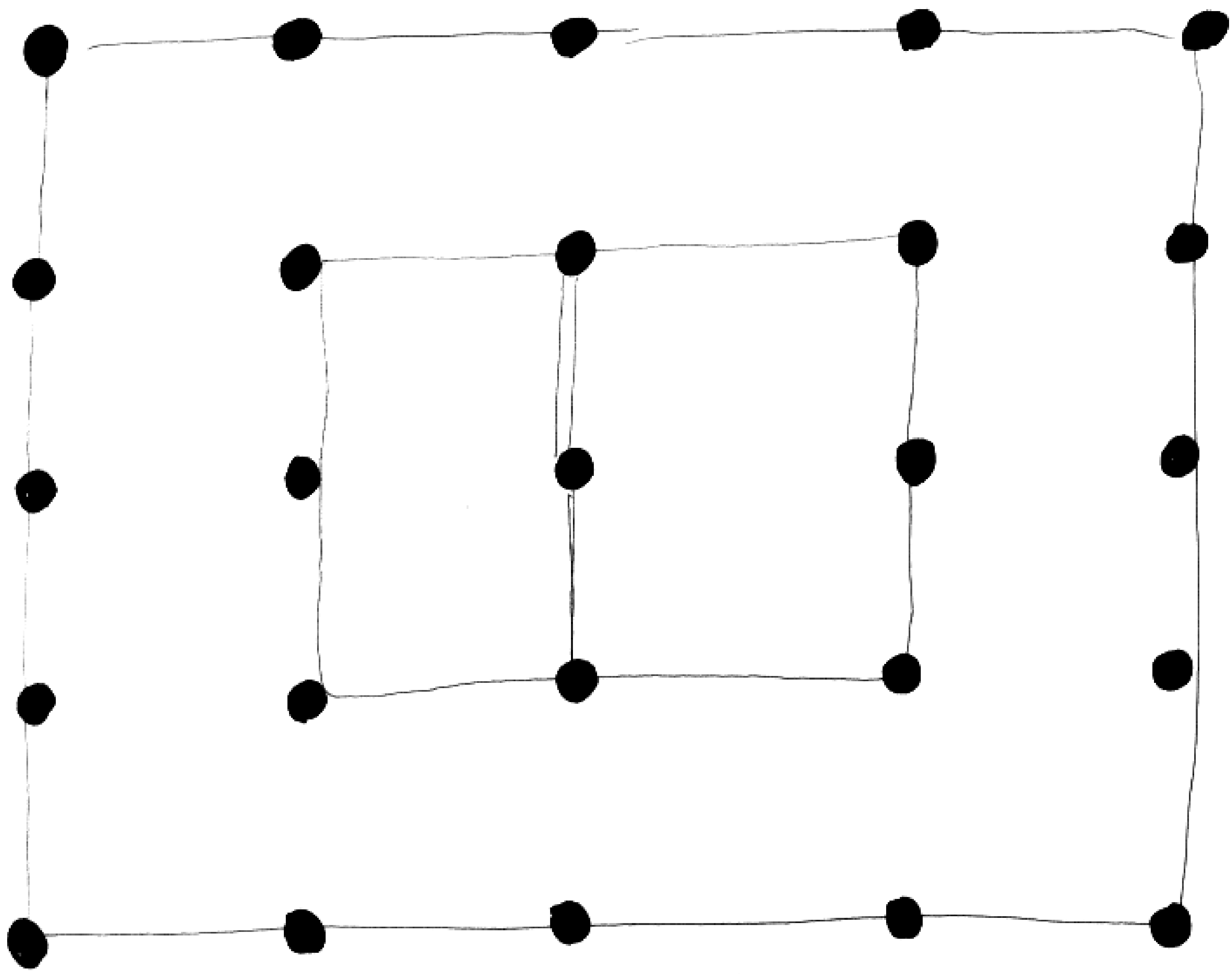
use one double-cross

cycles (≥ 4)

use two double-crosses

Chain Rule determines player forced into long chain or cycle

18 player 1.

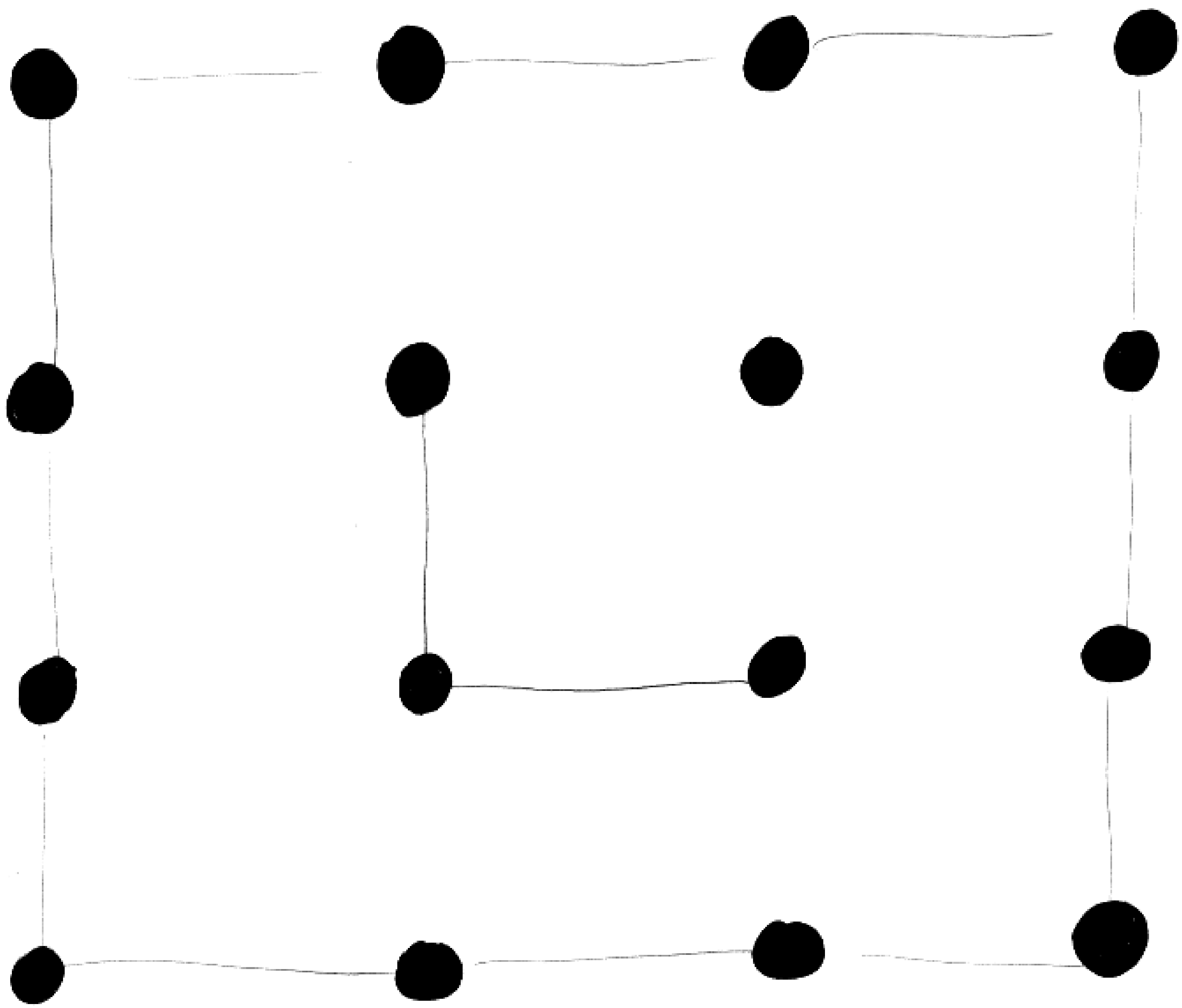


chain rule

Dots + l. chains = player

player 1 is forced into the cycle

Player 1's turn



no long chain

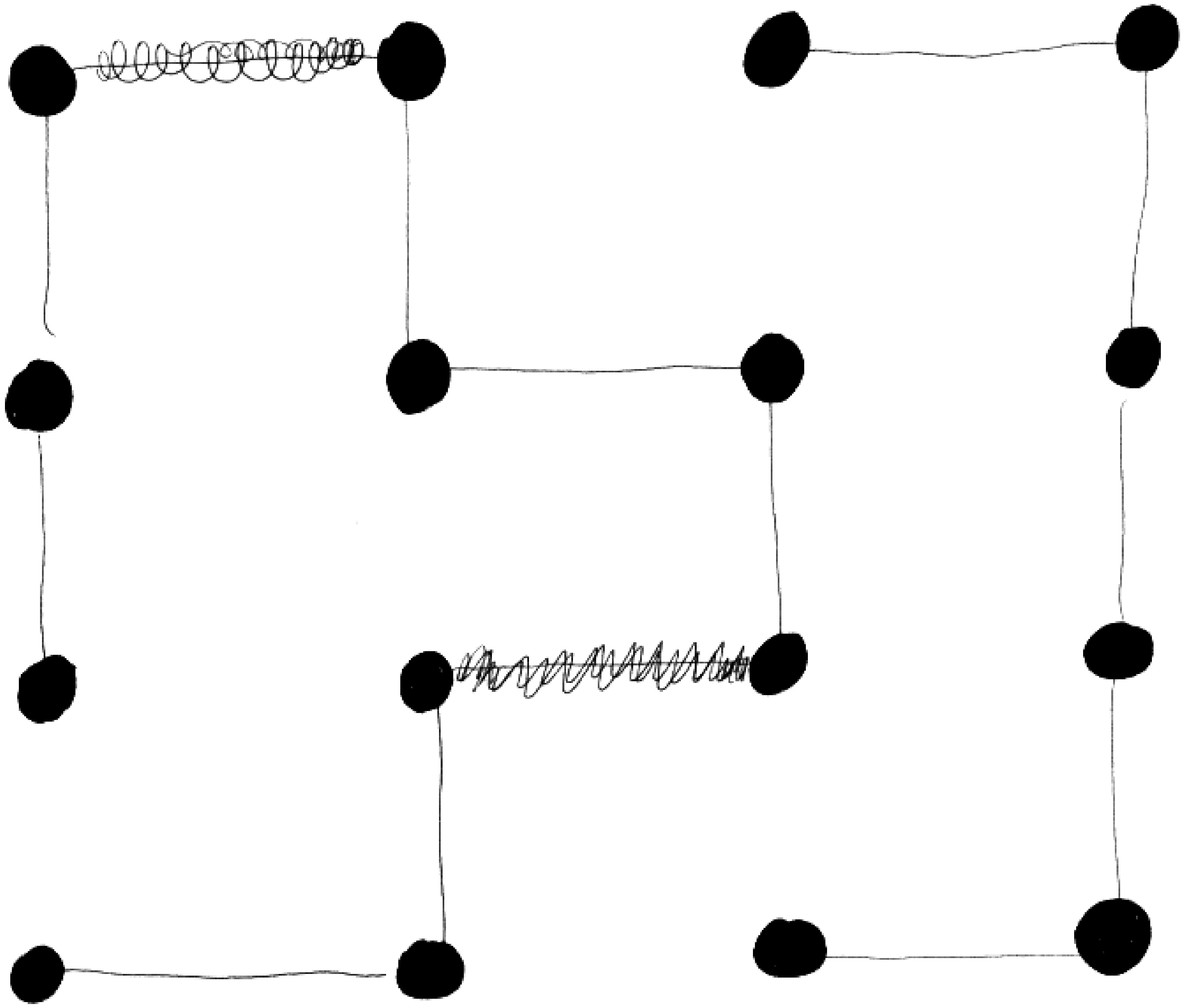
→ player 0

Chain Rule

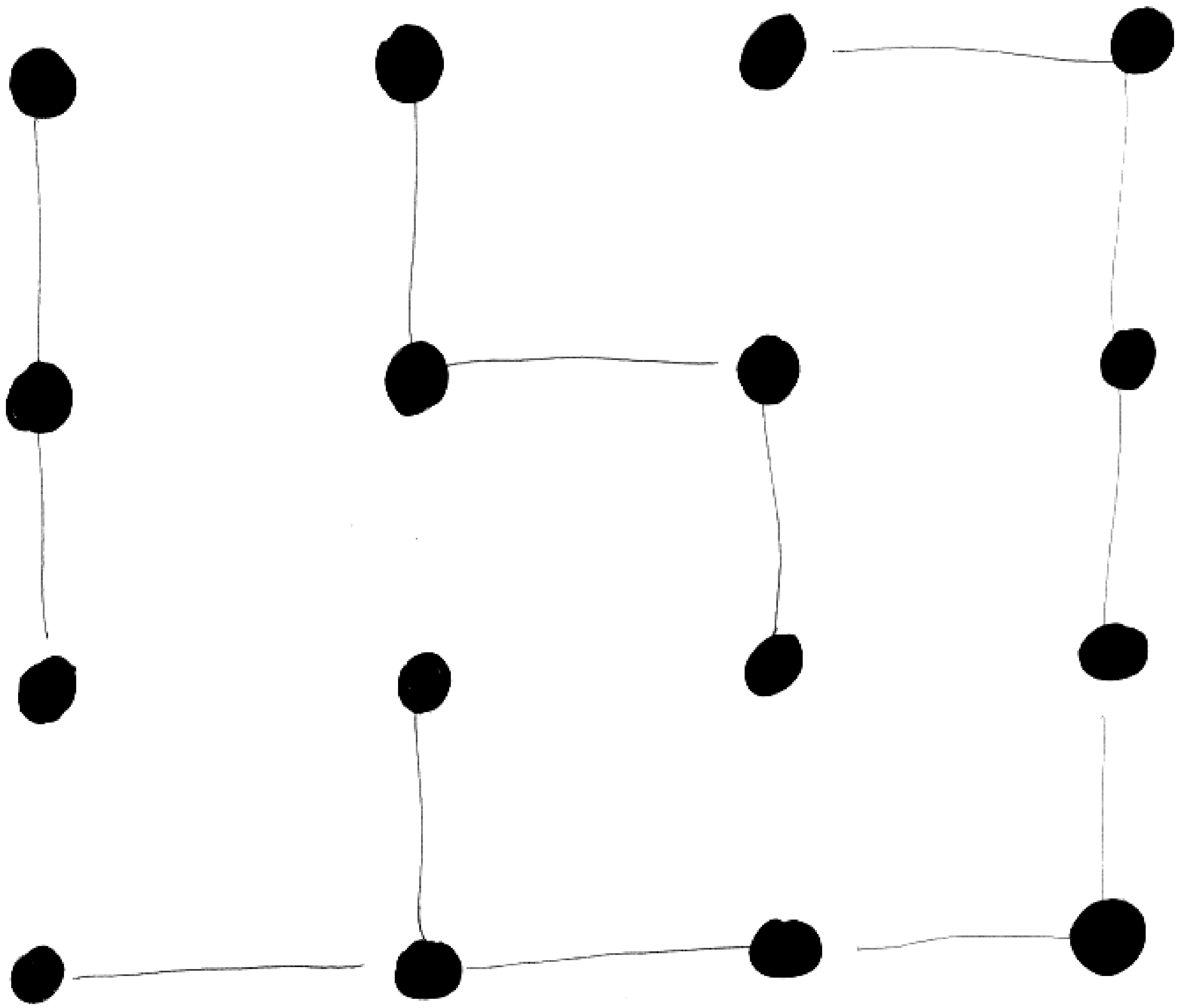
$$\text{total number of dots} + \text{total number of long chains}$$

$$\equiv 0 \quad \text{or} \quad 1$$

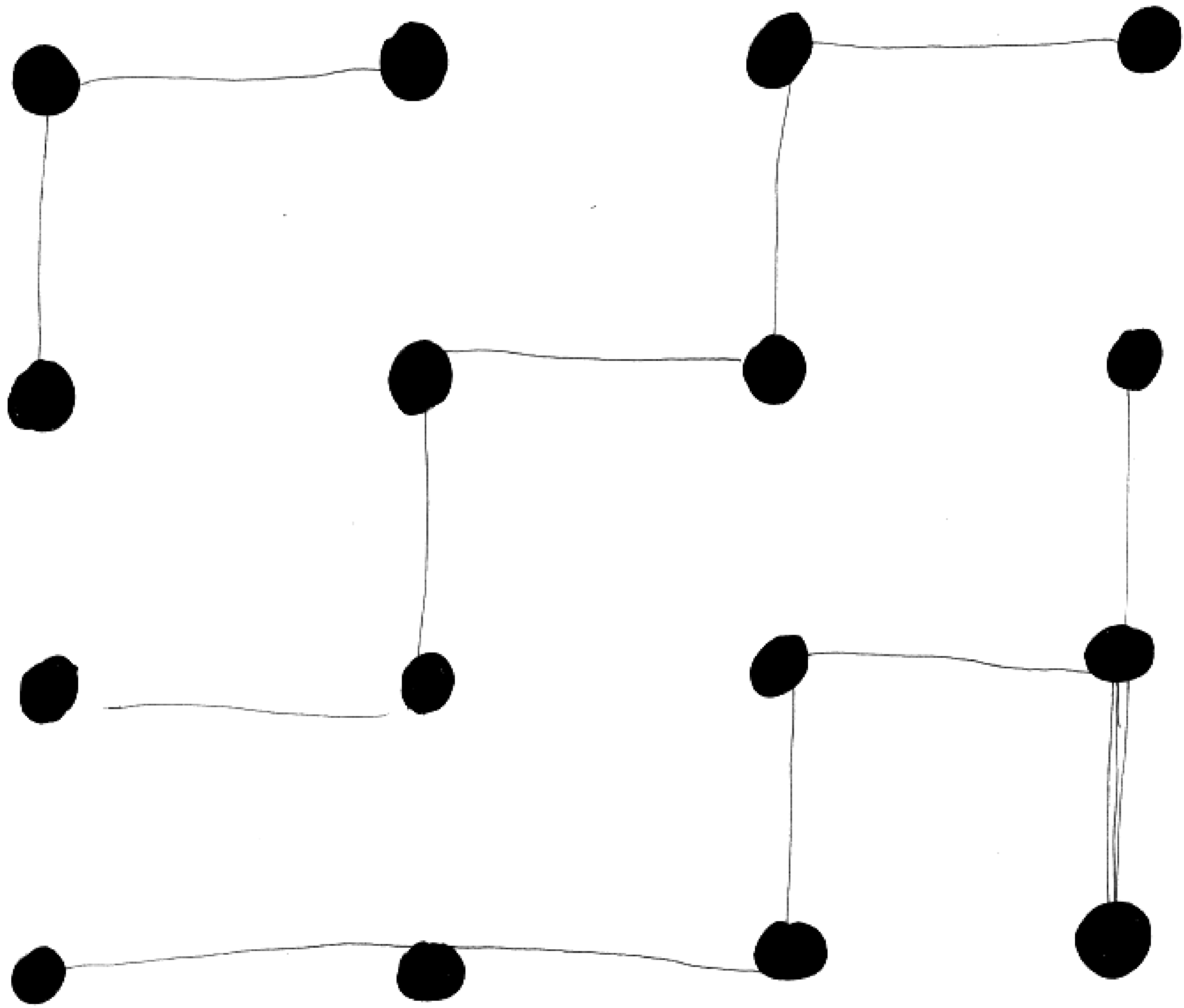
player forced into chains



1 chains



Dots = 16 Turn 1

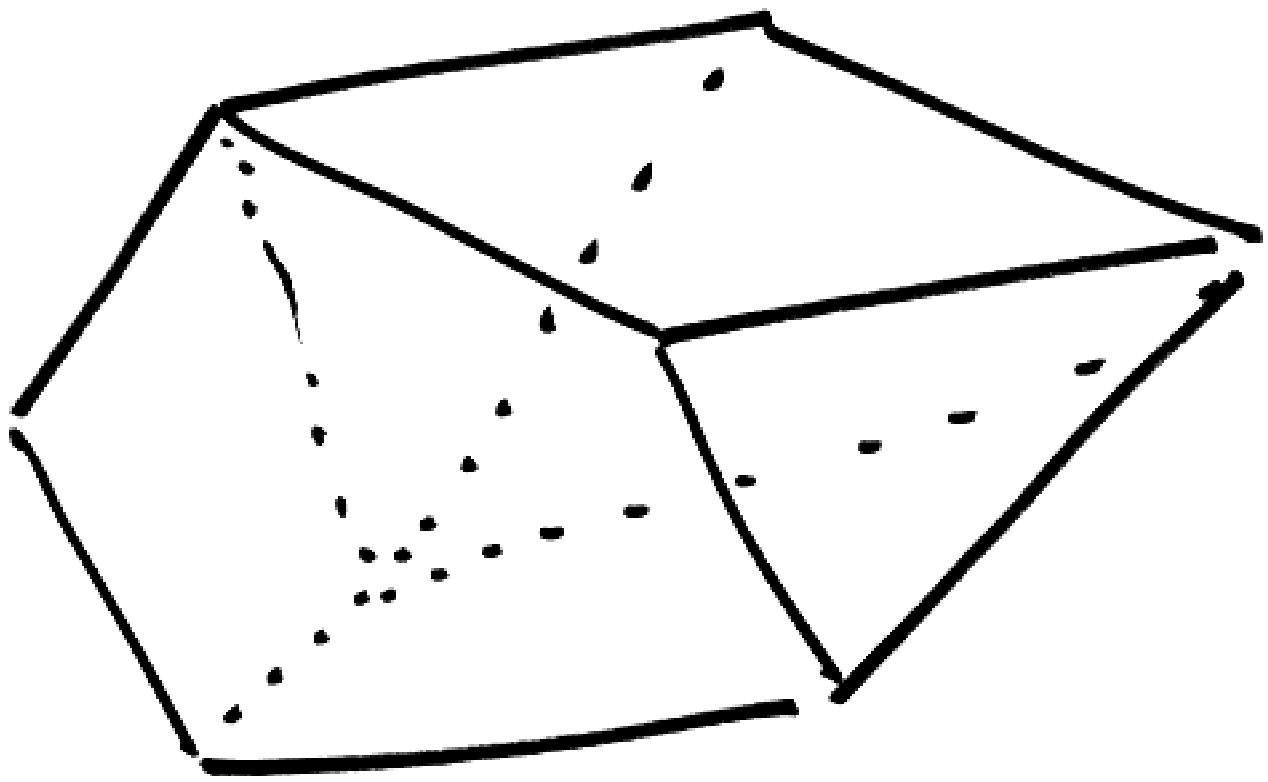


2 long chains

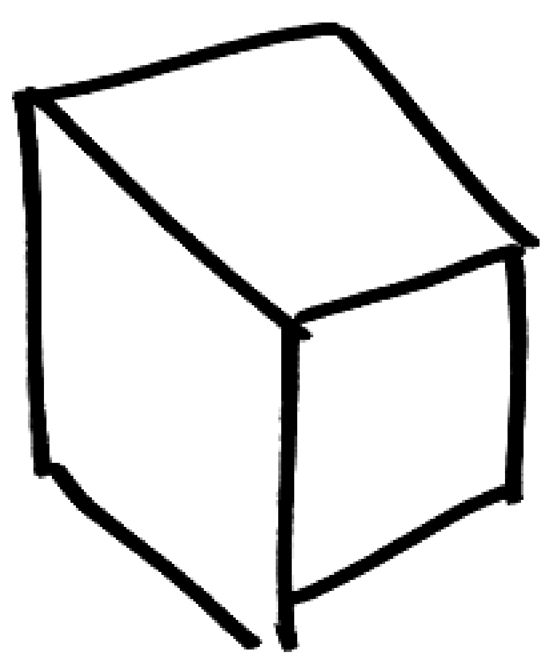
Dots + chains = loosing player

= 0

Euler's formula



$$V - E + F = 2$$



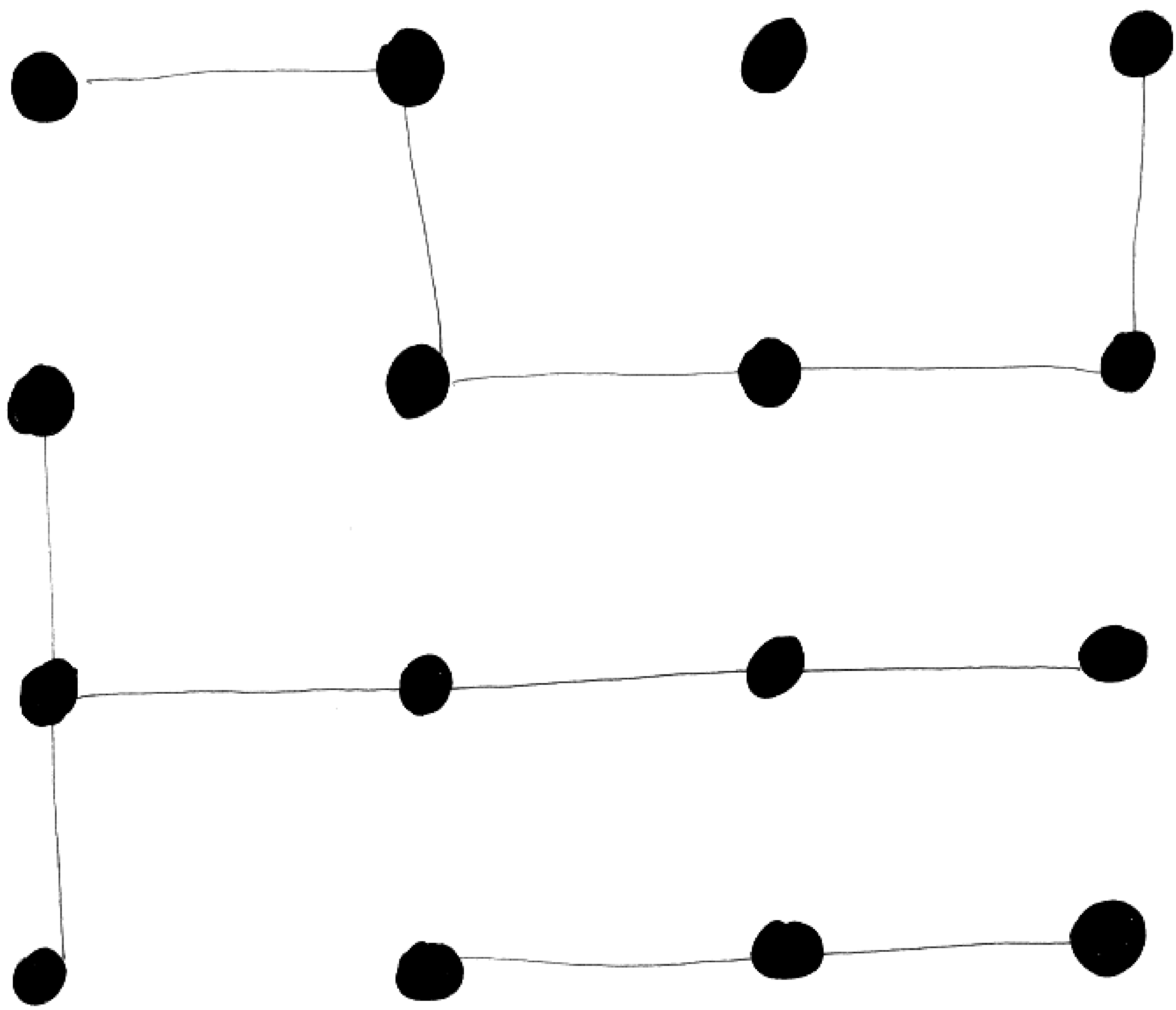
$V = 8$

$E = 12$

$F = 6$

$8 + 6 - 12 = 2$

Player 1's turn



2 long chains
16 dots
player 0 forced into chains

Assuming normal play
left with chains

Finally

$$\boxed{D + DC = T}$$ $\square$.

~1700's.

Euler characteristic

webpage
geometry junkyard

17 proofs of Euler's theorem.

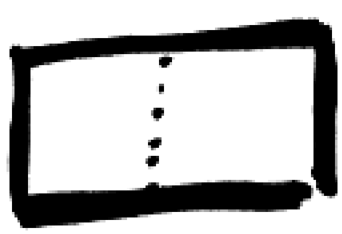
Planar graph

$$E = B + T - 1$$

$$E = B + D - 1$$

$$\Rightarrow \quad T = D$$

If we had DC we can do same kind of count



let's not count middle edge or these two boxes

$$E - DC = B - 2DC + T - 1$$

$$\rightarrow \quad E = B - DC + T - 1$$

Proof of Key Fact

No double crosses

B = # boxes

E = # edges

T = # turns

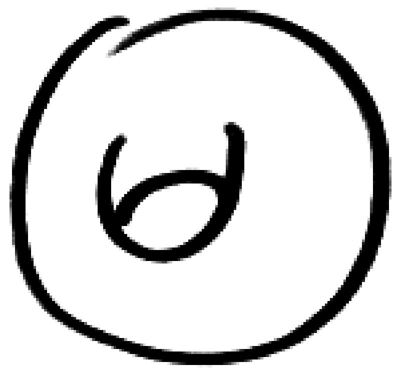
$$\boxed{E = B + T - 1}$$

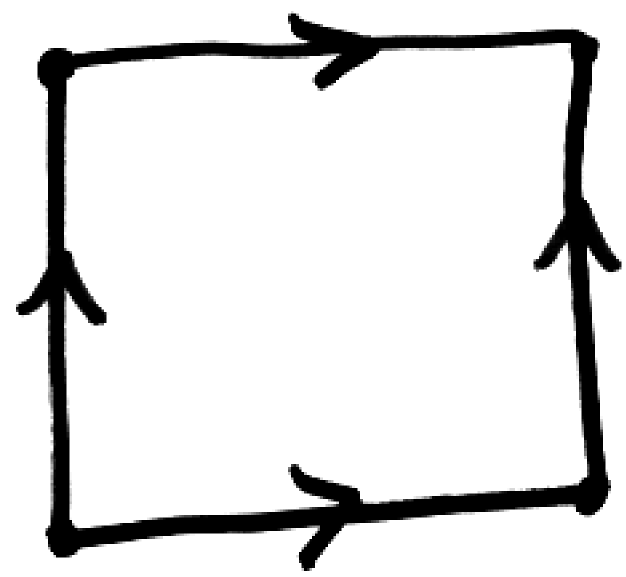
$$\underbrace{BB\cdots\cdots B*}_{\text{a turn}}$$

last turn: $\underbrace{BB\cdots\cdots B}$

Torus $g = 1$

genus = "the number of holes"





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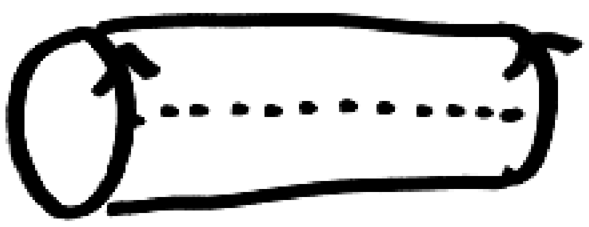
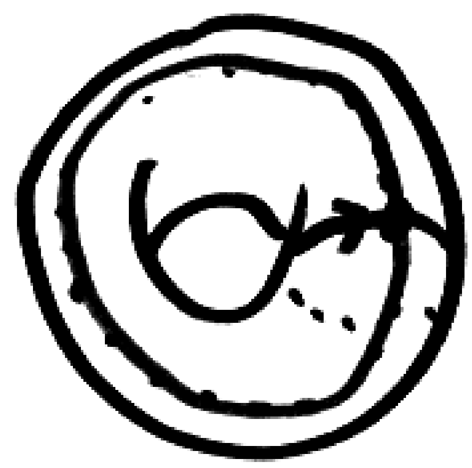


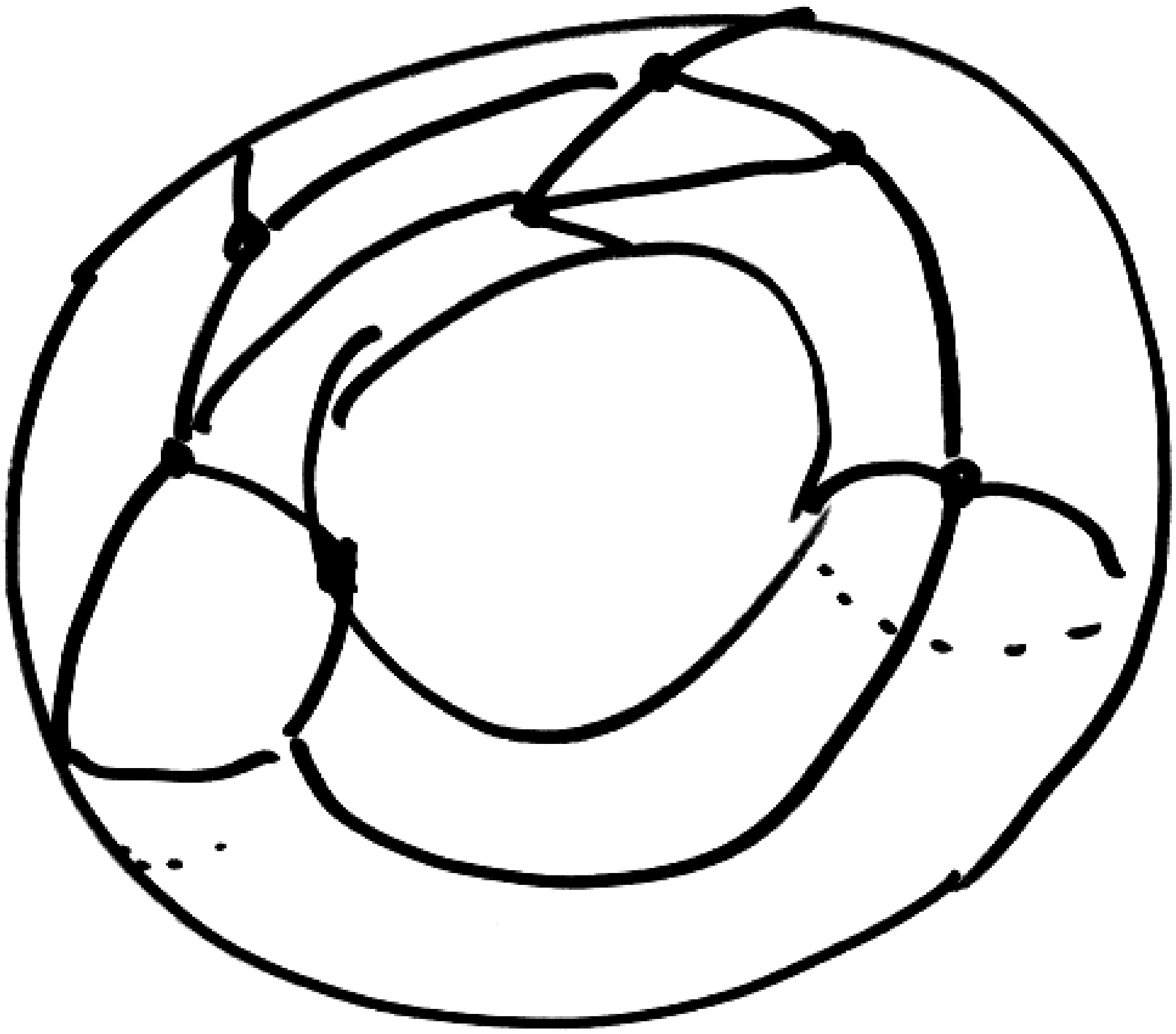


$V = 1$
$F = 1$
$E = 2$

$V - E + F = 0$

g holes

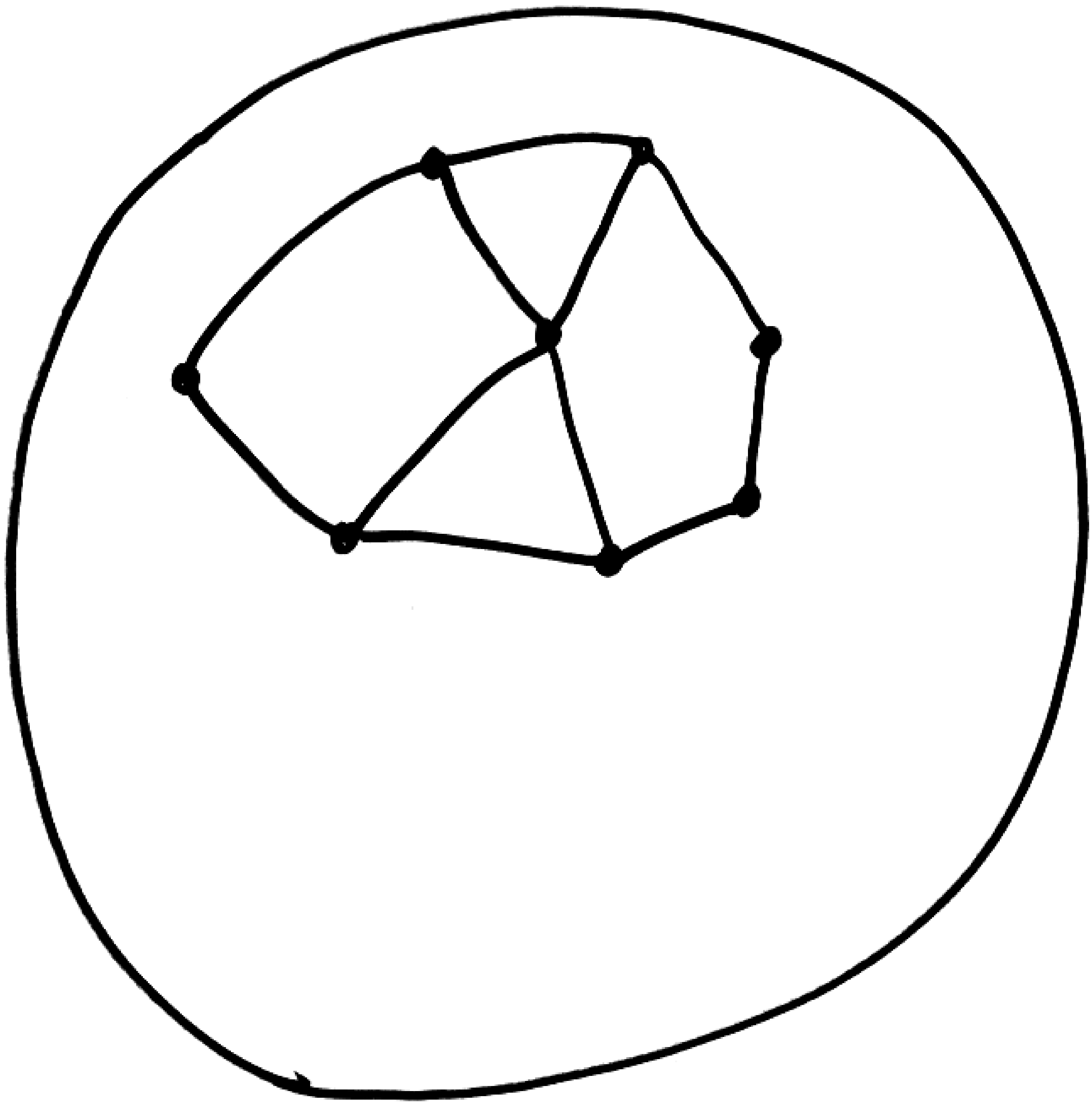
Euler char
$2 - 2g$



$$V - E + F = 0$$

Euler characteristic

$$V - E + F = -2$$

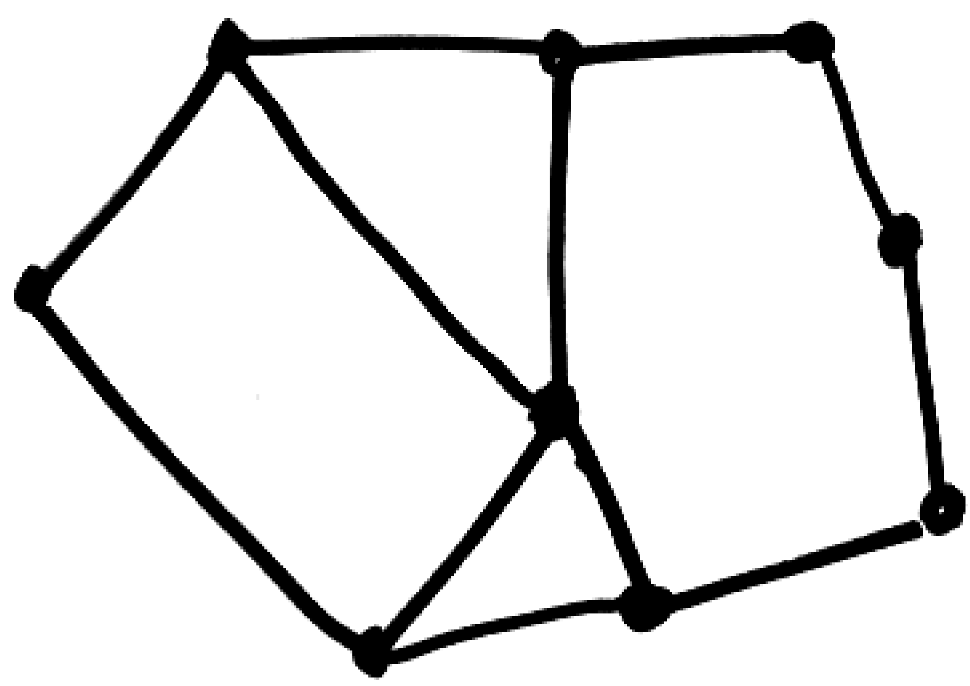


one more face

$$\boxed{V - E + F = 2}$$

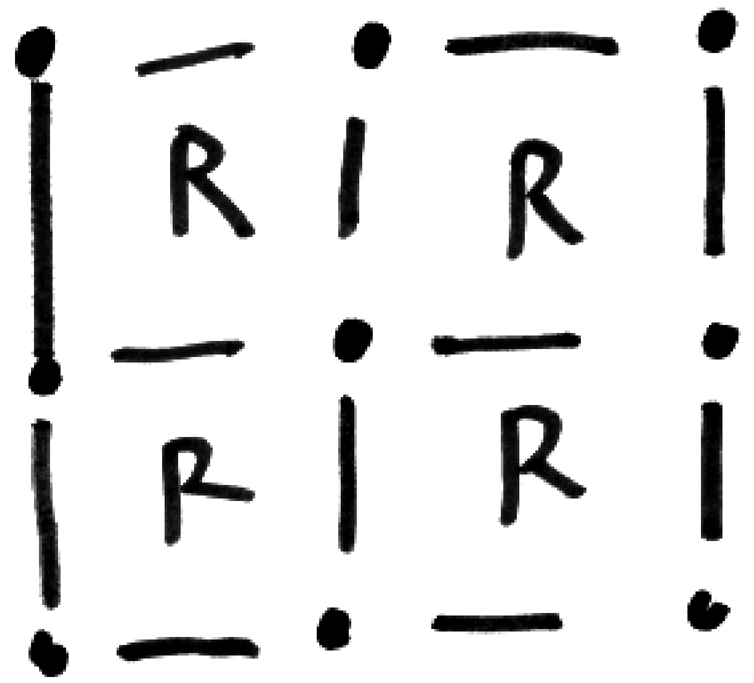
Euler's Formula

Planar graph



$V = 9$
$E = 12$
$F = 4$

$$V - E + F = 1$$



TURNS

$T = 5+4 = 9$

$B = 4$

$E = 12$

$E = 9+4-1$

<u>Turn</u> = complete set of consecutive moves by one player

D dots

T turns

E ~~lines~~ edges

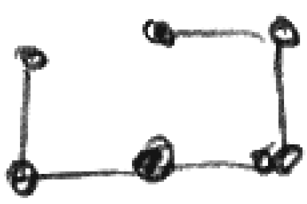
B Boxes total

<u>Chain Rule</u> Normal play ⇝ cycles + chains

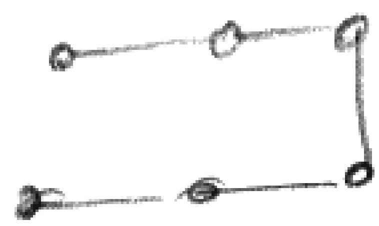
$D + CH \equiv L \mod 2$

↑ player forced into chains = <u>Loser</u>

<u>Doubletrap</u> offer opponent a 2-chain



or



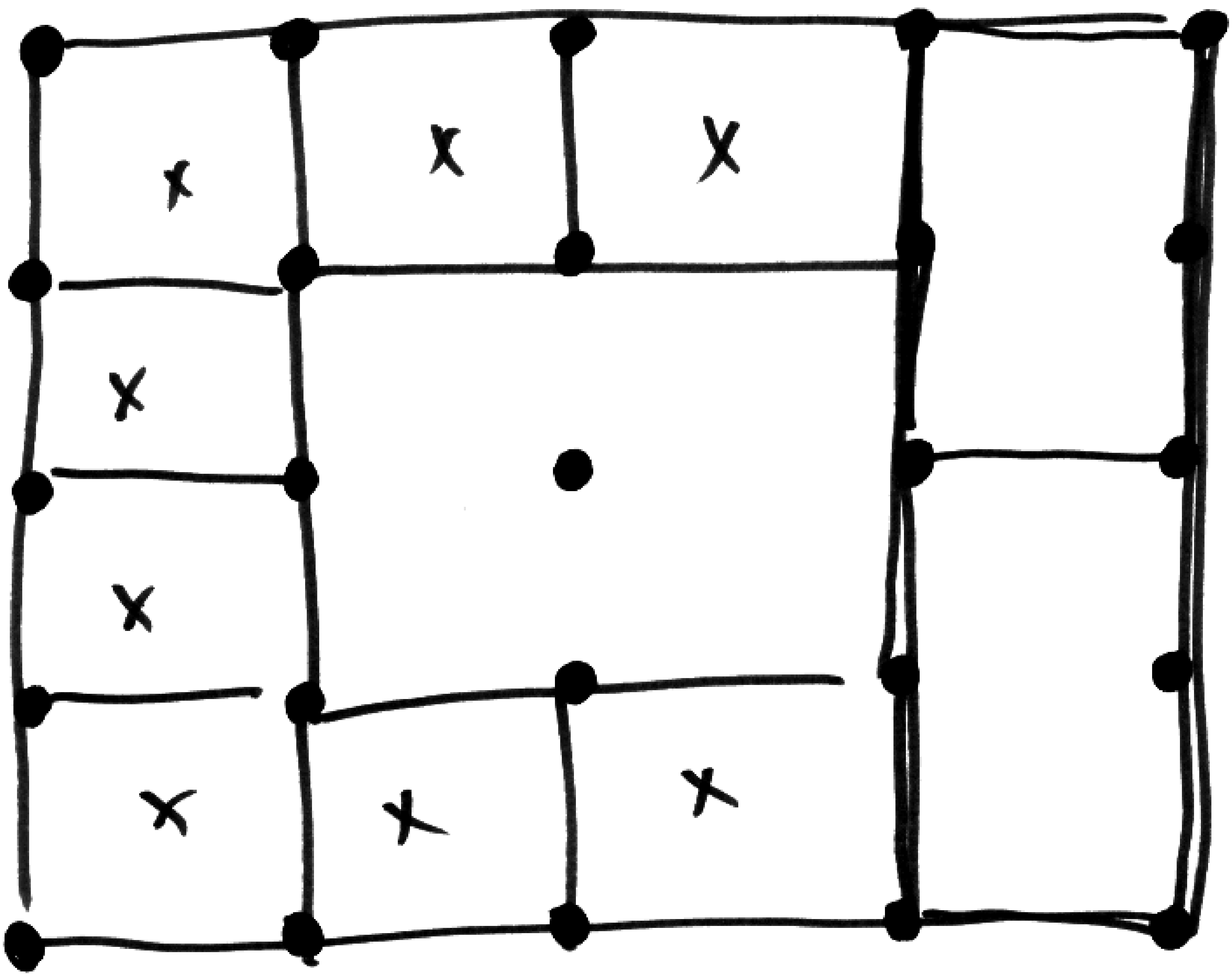
Allows opponent to change parity on chain count

In normal play the last player takes all boxes in last chain in last turn. <u>W</u>inner

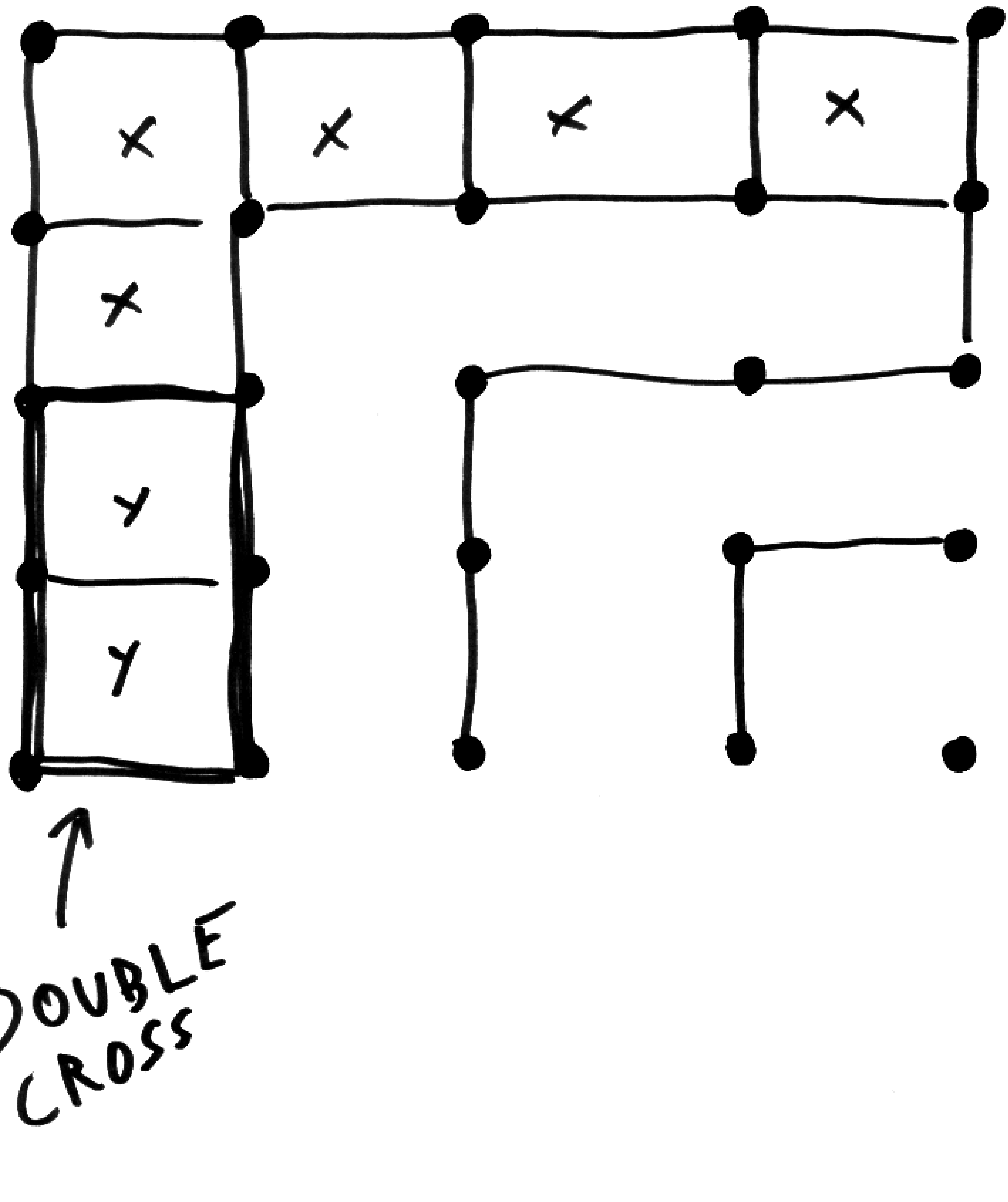
If $W = 1$ then <u>T</u>urns is odd

$W = 0$ " $T \equiv 0 \mod 2$

2 DC



you can't just leave one DC on a cycle; for endgame you leave two so you don't open a new chain/cycle


x
x
x
x
x
y
y
DOUBLE CROSS

Same counting with DC

There are DC edges filling a $\square$ DC

get 2 boxes w/o changing turn

$$E - DC = B - 2DC + T - 1$$

$$\rightarrow \quad E = B - DC + T - 1$$

$$\rightarrow \quad \boxed{D + DC = T}$$

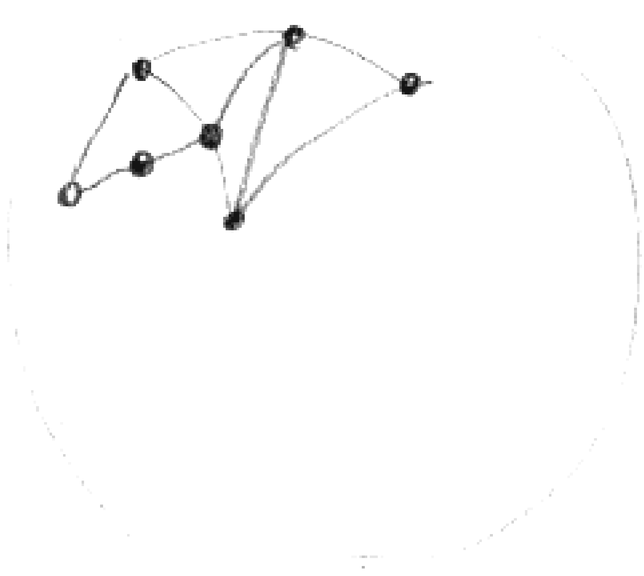
Euler (1707 – 1783)

<u>Euler's Fmla</u>

For any planar graph

$$V - E + F = 1$$

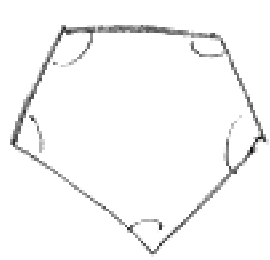
on sphere: one more face



geometry junkyard

17 proofs of this fmla

face



$$\sum_{i=1}^{k} \theta_i = (k-2)\pi$$

Each edge belongs to two faces so total angle $= (2E - 2F)\pi$

interior vertices contribute 2π

exterior vertices $2(\pi - \theta_v)$

Berlekamp

$D + DC = T$

$\Rightarrow$ Chain Rule

$DC = CH + 2CY - 1$

(if $CH > 0$?)

If cycle is last there is one DC. So fmla is OK in all cases

$DC \equiv CH + 1 \mod 2$

combined w/ theorem

$W \equiv T \equiv D + CH + 1 \mod 2$

(following I. Vardi)

Pf of Berlekamp's thm

If no DC E — 1 Box / gives turn to opponent

each move except last does

$E = B + T - 1$

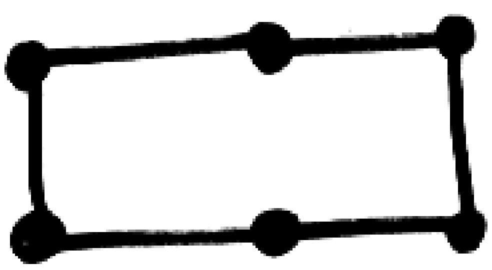
Thm of graph theory $E = B + D - 1$

$\Rightarrow T = D$

Key Fact (Berlekamp)

$$\boxed{D + DC = T}$$

DC = # double crosses



= Double Cross

Key Fact $\Rightarrow$ Chain Rule

$$DC = CH + 2CY - 1$$

$$W \equiv T \equiv D + CH + 1 \mod 2$$

$$L \equiv D + CH \mod 2$$

Turn

consecutive sequence moves
by some player.

Normal play
The last turn is by W
takes all boxes in the
last chain.

$T =$ # total number of
turns in game

$W \equiv T \bmod 2$

Chain Rule

$$\boxed{D + CH \equiv L \mod 2}$$

D = # Dots

CH = # (long) chains

L = parity of player forced to move into chains/cycles

Sketch of proof

Dots and Boxes

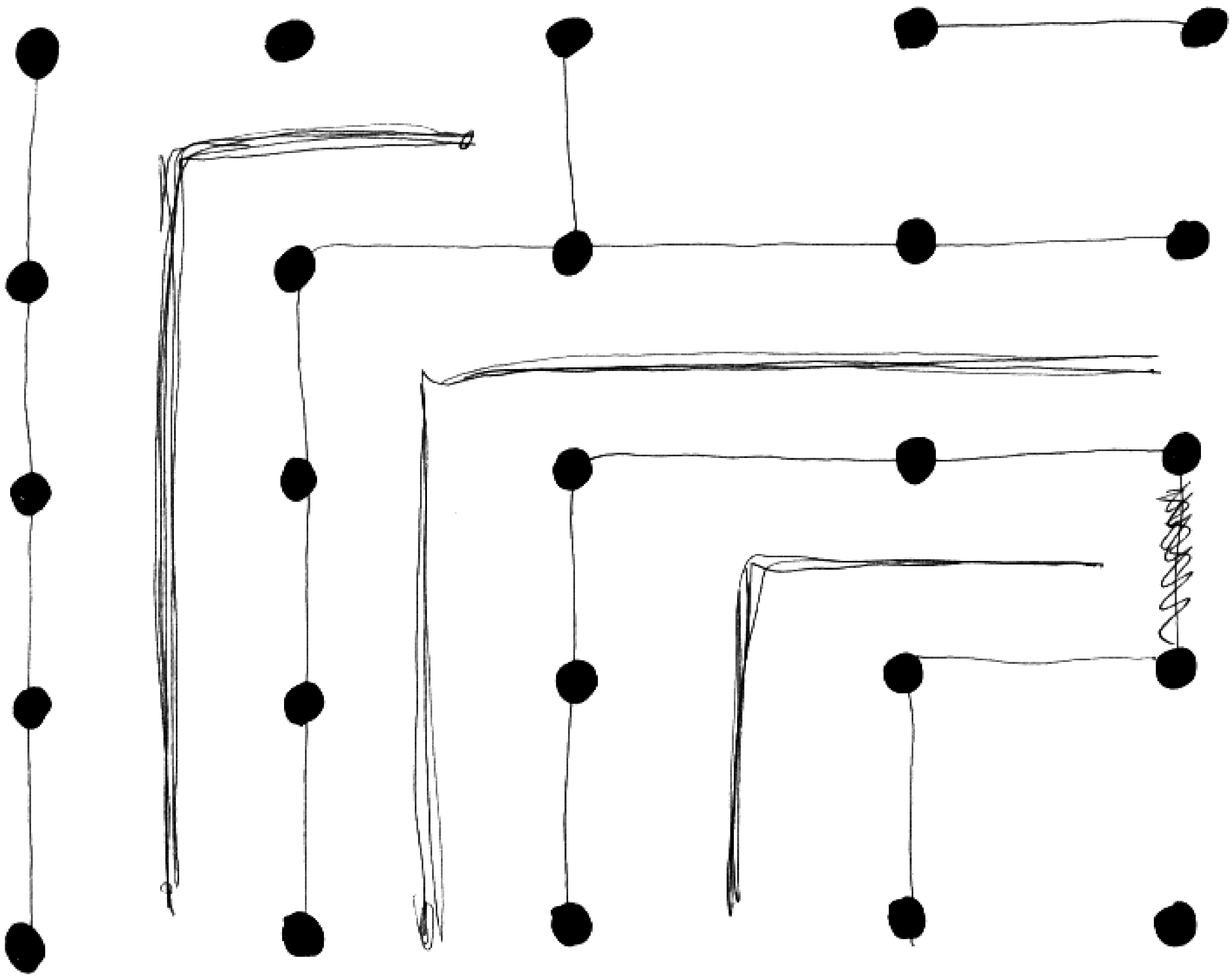
Endgame strategy:

- Force opponent to open chains/cycles

long chain: at least 3 boxes
long cycle: at " 4 "

- Fill boxes in chain/cycle

leave : 1 double cross / 2 double crosses



long chain : at least 3 boxes