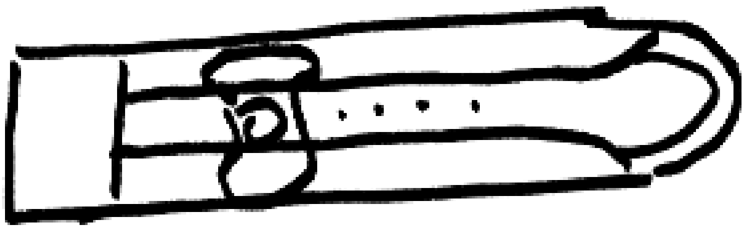
Jan 18, 2007

Reflected Gray Code

Chinese Rings



slide.

Brain

Engineer Bell Labs 30's TV.

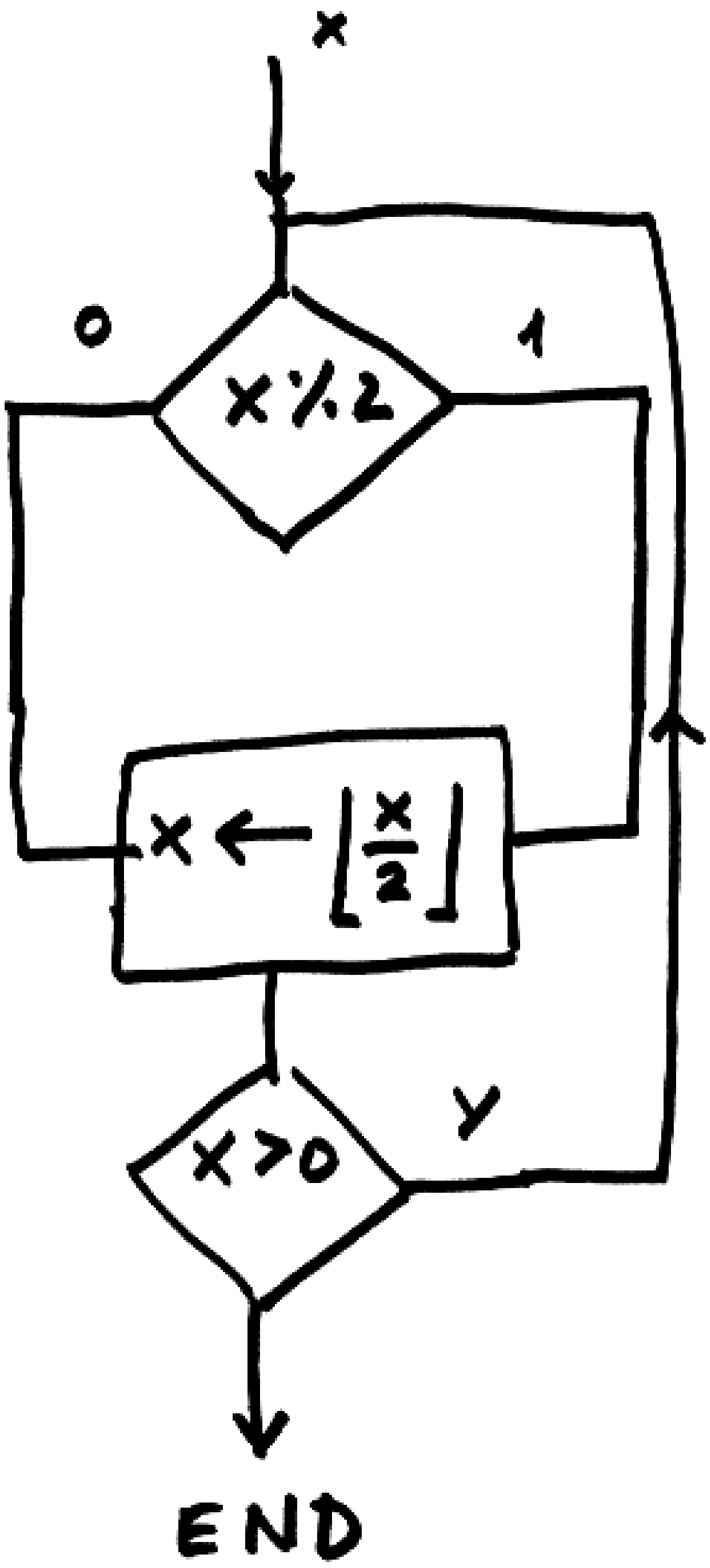
Binary Code

$0, 1, 2, \dots, 2^n - 1$

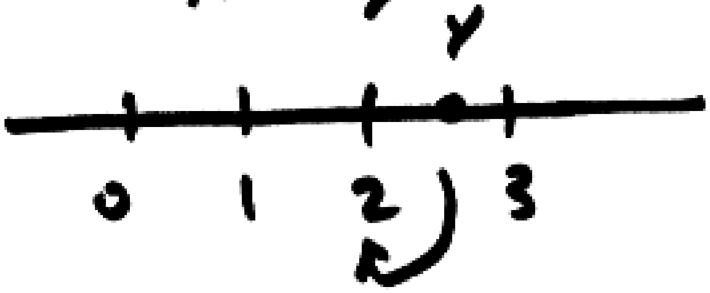
number x ⟿ code word = string of 0 and 1's.

$$13 = \begin{array}{cccc} 1 & 1 & 0 & 1_2 \\ \uparrow & \uparrow & \uparrow & \uparrow \\ 8 & 4 & 2 & 1 \end{array} \qquad 1 + 4 + 8 = 13$$

Number Theory



$\lfloor y \rfloor$ = largest integer smaller or equal than y



x = 13

$13 \% 2 = \boxed{1}$

$\left\lfloor \frac{13}{2} \right\rfloor = 6$

$6 \% 2 = \boxed{0}$

$\left\lfloor \frac{6}{2} \right\rfloor = 3$

$3 \% 2 = \boxed{1}$

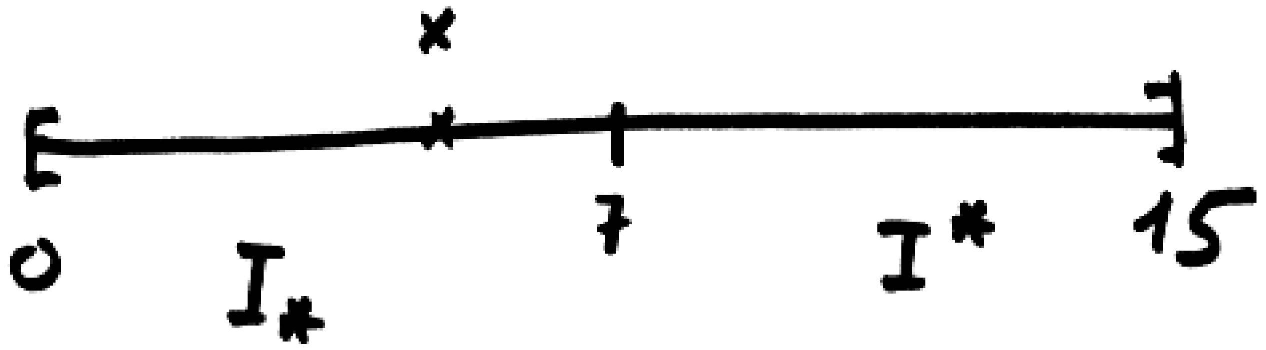
$\left\lfloor \frac{3}{2} \right\rfloor = 1$

$1 \% 2 = \boxed{1}$ $\left\lfloor \frac{1}{2} \right\rfloor = 0$

$13 = \overset{\longleftarrow}{1101_2}$

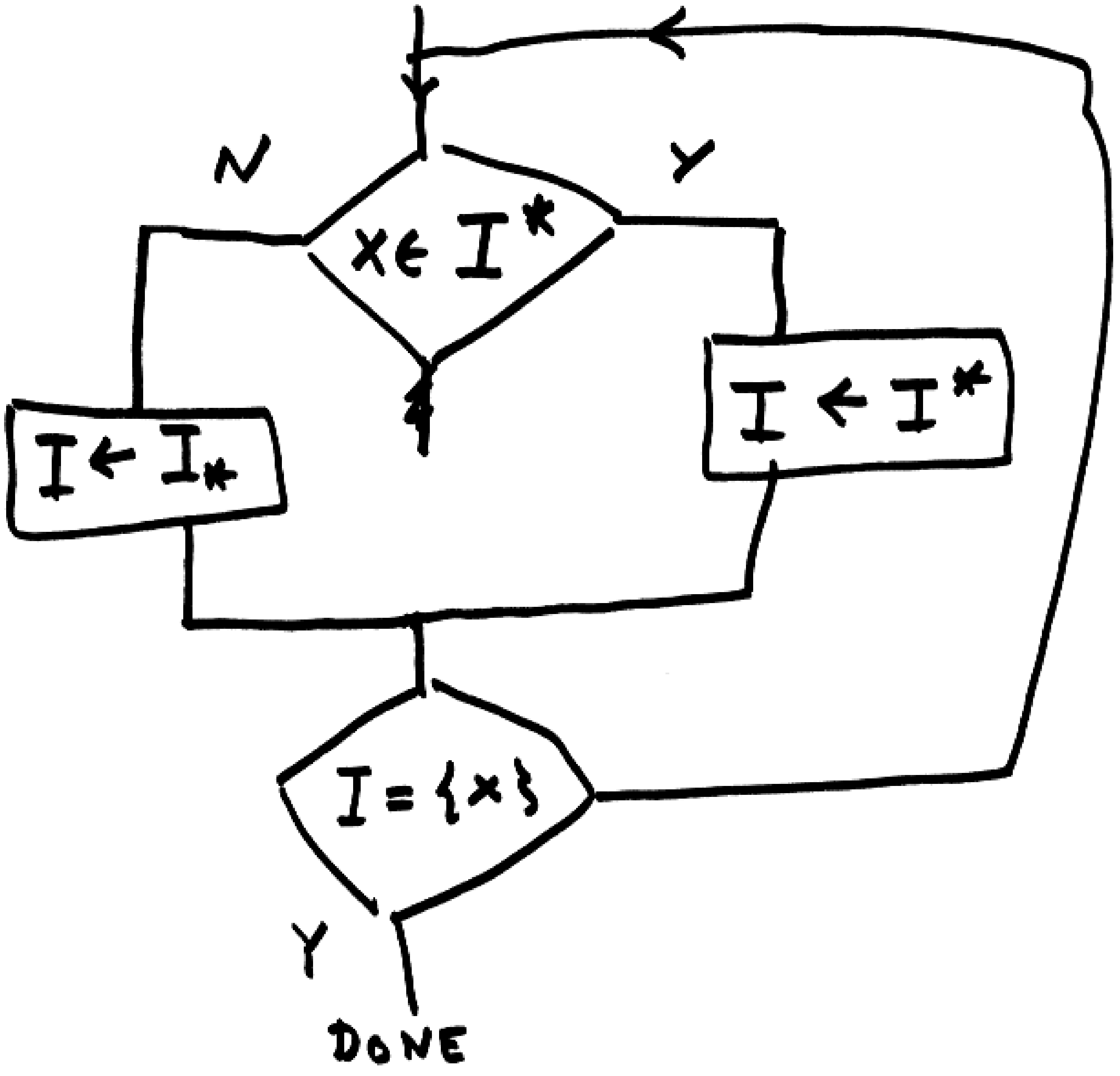
Dissection, Binary search

I =



$n=4$ bits

$I = 0, 1, 2, \ldots, 15$



$x = 13$

I

[0, …, 15] $13 > 7$ 1

[8, …, 15] $13 > 11$ 1

8, 9, 10, 11

[12, 13, 14, 15] $13 > 13$ 0

[12, 13] $13 > 12$ 1

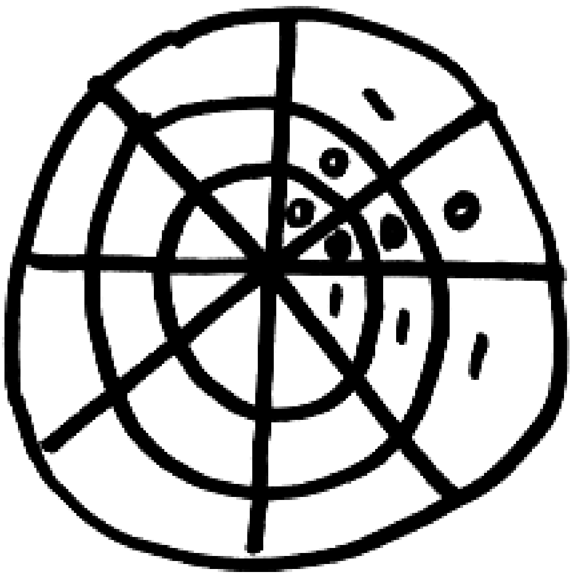
[13]

$13 = 1101_2$

Binary Code

	8	4	2	1	#changes
0	0	0	0	0	
1	0	0	0	1	1
2	0	0	1	0	2
3	0	0	1	1	1
4	0	1	0	0	3
5	0	1	0	1	1
6	0	1	1	0	2
7	0	1	1	1	1
8	1	0	0	0	4
9	1	0	0	1	1
10	1	0	1	0	2
11	1	0	1	1	1
12	1	1	0	0	3
13	1	1	0	1	1
14	1	1	1	0	2
15	1	1	1	1	1

Robot's arm



An error in reading will typically give a totally wrong answer.

It's better to have code words differ ~~by~~ at only one slot.

Binary Code

n

1 $0, 1$

2 $\underbrace{00, 01}_{\text{tag } 0}, \underbrace{10, 11}_{\text{tag } 1}$

3 $000, 001, 010, 011, 100, 101, 110, 111$

Jan 23, 2007

Binary code

0, 1, 2, ..., 15 $n = 4$ length

$$\begin{array}{c|c} 7 & 0111 \\ 8 & 1000 \end{array}$$

4 changes

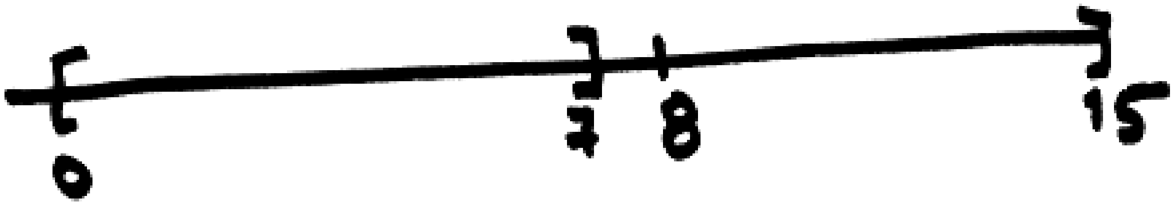
Gray code

A way to encode numbers such that any two consecutive words differ in exactly one slot.

Binary

Recursively

$0\,B_n \quad 1\,B_n$



$n=3$ 0, 1, 2 .., 7

$n=4$ 0, 1, 2, ... , 15

$n=1$ B_1: 0 . 1

$n=2$ B_2: 00, 01, 10, 11

$n=3$ B_3: $\underbrace{000,\ 001,\ 010,\ 011}$, $\underbrace{100,\ 101,\ 110,\ 111}$

Reflected gray Code

m : C_m

$C_m' = C_m$ backwards

$m+1$: $0C_m$, $1C_m'$

1 0 , 1

2 00, 01, 11, 10

3 000, 001, 011, 010, 110, 111, 101, 100

0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

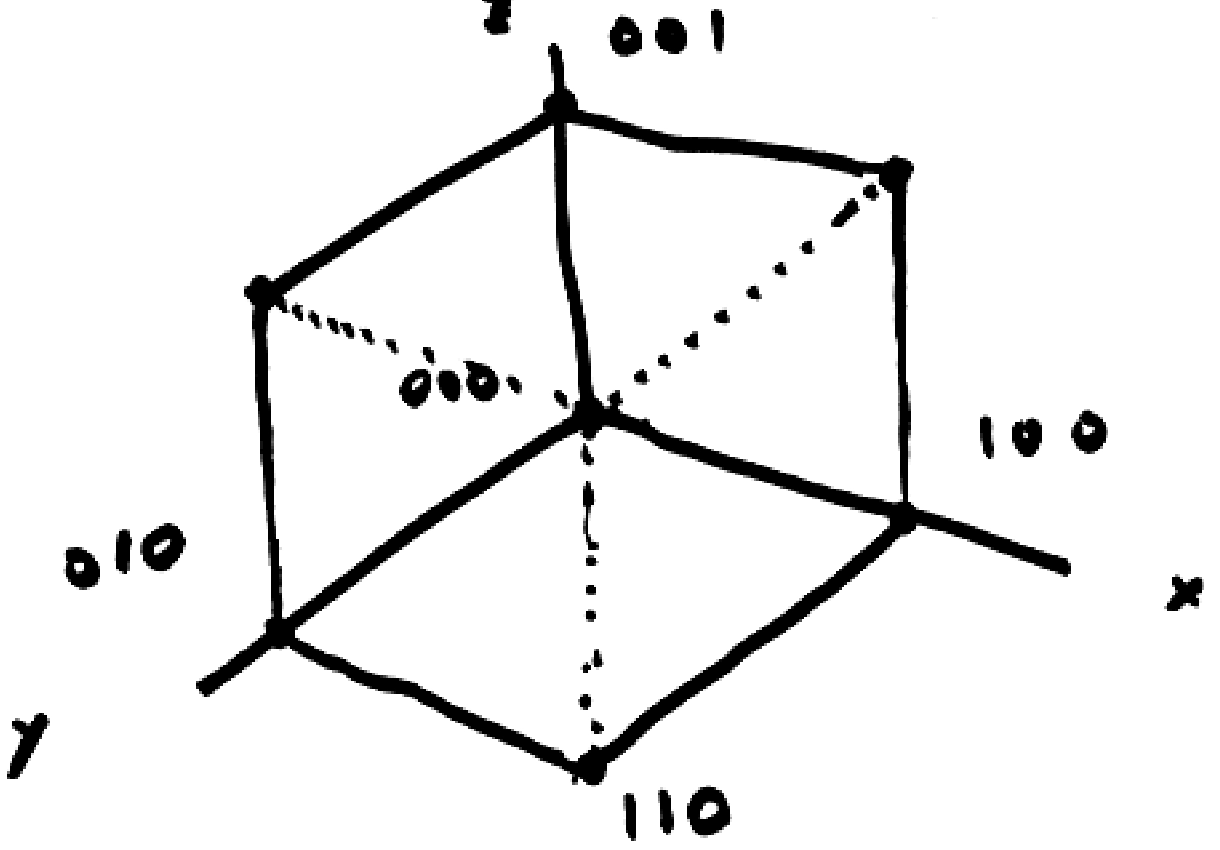
There is only two possible moves either

- change rightmost bit
- change the bit to the left of the rightmost 1

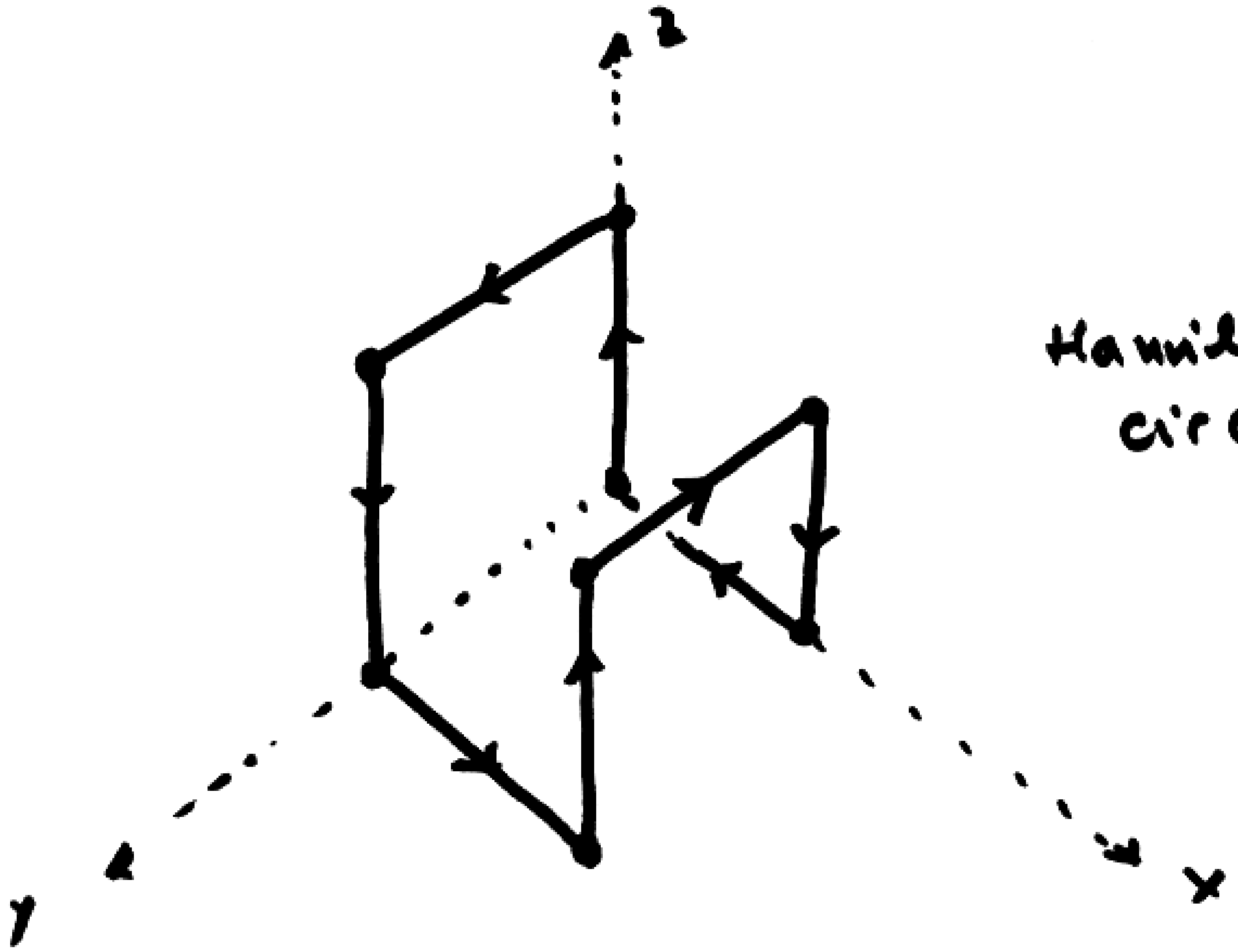
$$\cdots\cdots\ \underset{\underset{\text{change}}{\uparrow}}{\boxed{x}}\ \ \boxed{1}\ \ 0\cdots\cdots 0$$

Hamiltonian circuits

Represent length 3 binary words as points on a cube



(x, y, z)
words of length 3 ↔ vertices of cube.



Hamiltonian circuit

Path on the cube going through every vertex only once.

Gray code of length n

↕

Hamiltonian circuit in the n-cube.

Binary → Gray

$(b_{n-1}\ b_{n-2} \cdots\ b_1\ b_0)_2$ ← Binary

$\downarrow$

$(c_{n-1}\ c_{n-2} \ldots\ c_1\ c_0)$ ← Gray

E.g. 13 $(1101)_2$

$$c_j \equiv b_j + b_{j+1} \mod 2$$

+	0	1
0	0	1
1	1	0

mod 2 addition

$0 + 0 \equiv 0 \mod 2$
$0 + 1 \equiv 1 \mod 2$
$1 + 0 \equiv 1 \mod 2$
$1 + 1 \equiv 0 \mod 2$

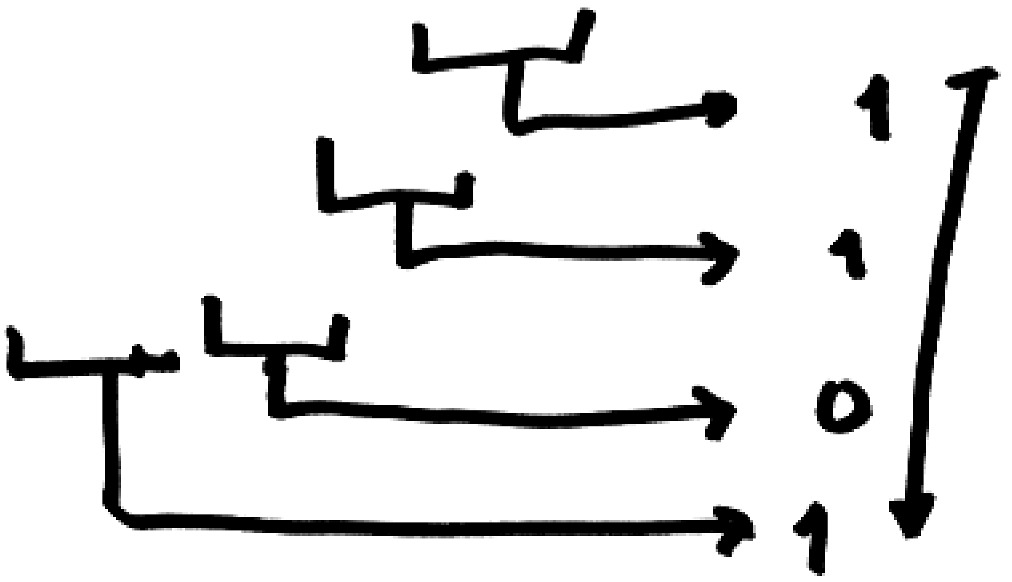
$1 \oplus 1 = 0$

$$c_j = b_j \oplus b_{j+1}$$

$1 \oplus 0 = 1$
$0 \oplus 1 = 1$
$0 \oplus 0 = 0$
$1 \oplus 1 = 0$

$$c_j = b_j \oplus b_{j+1}$$

13 …0 ¦ 1 1 0 1 binary



1011 GRAY

Jan 25, 2007

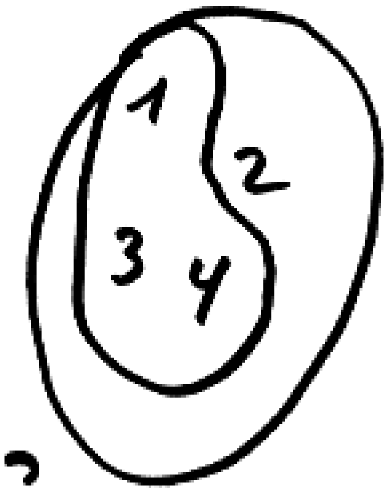
$n = 4$ $\quad 0, 1, 2, \ldots, 15$

0 ___
1 ___
2 ___
⋮ ⋮
15

↑ strings of 4 bits
words

$\{1, 2, 3, 4\}$

subset: $\{1, 3, 4\}$



How many subsets are there?

$\{1, 3, 4\} = \{1, 4, 3\}$

		1	2	3	4
$\{1,3,4\}$	↔	1	0	1	1
$\{2,3,4\}$	↔	0	1	1	1
ϕ	↦	0	0	0	0

subset ↔ word (string 4 bits)

Pblm #8

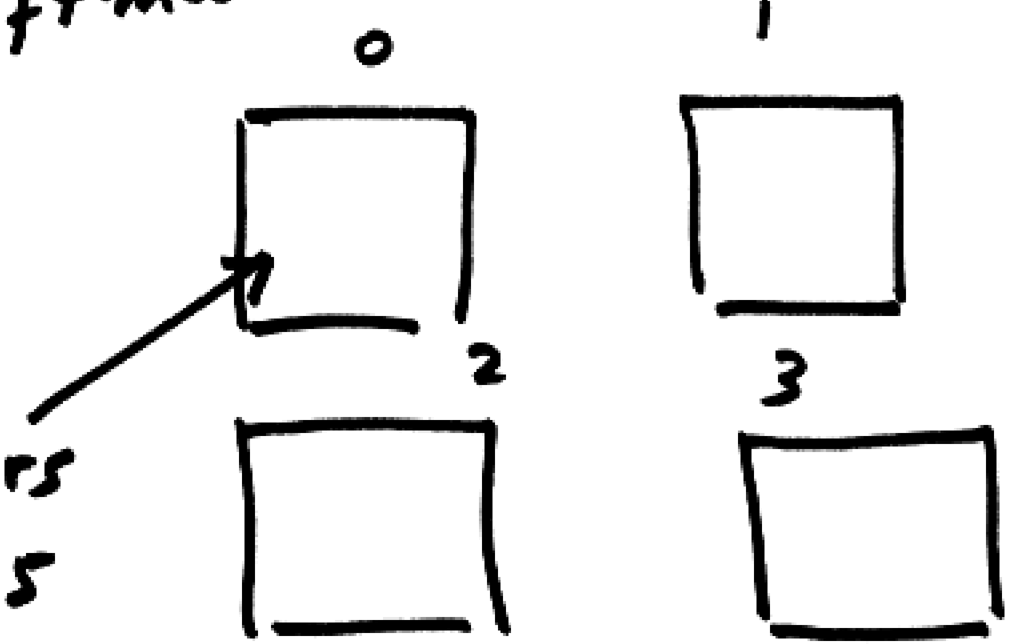
000
001
011
010
110
111
101
100

100
101
111
110
010
011
001
000

↪ change leftmost bit

Pblm #1

Binary



all numbers
0, 1, 2, ... 15
that have a 1
in the 0 slot

3210|

Pblm #4

$x = 2^k \cdot y$

$2 \nmid y$
(y is odd)

Highest power of 2
dividing x = K

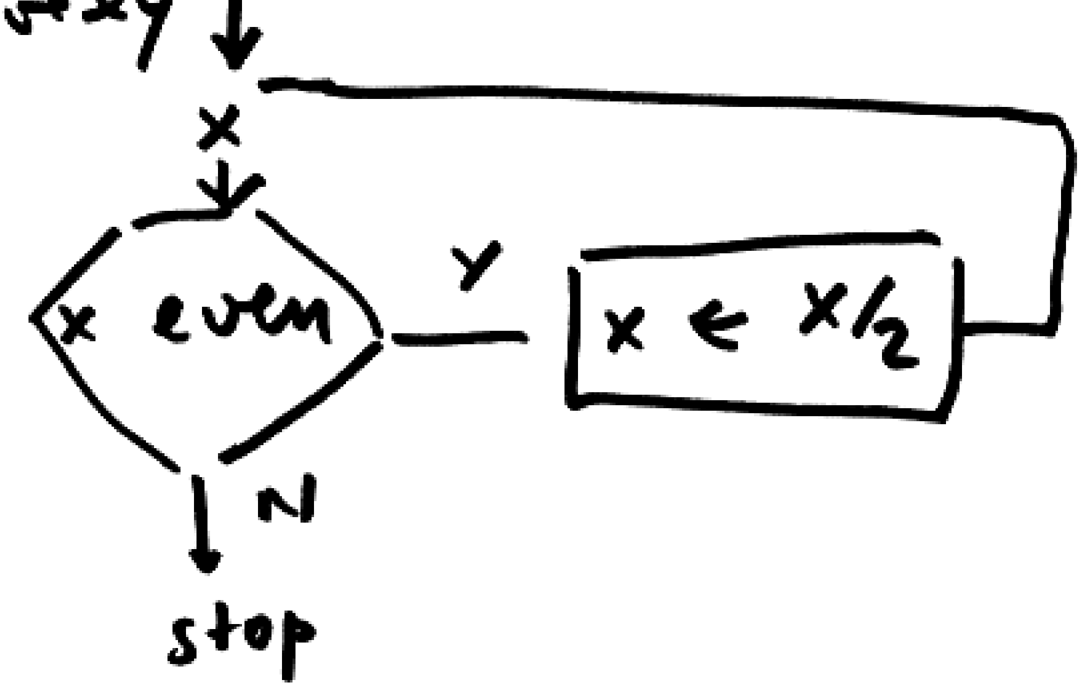
$12 = 2^2 \cdot 3$ ← odd

→ $k = 2$

$7 = 2^0 \cdot 7$

→ $k = 0$

Recursively



K = # times through this loop

x = 36 $\quad\quad\quad 36 = 2^2 \cdot 9$

↓

18

↓

$\underline{9}$

* * *

Two possible moves in Gray code

- $* * * * \cdots \boxed{\#}\ 1\ 0\ 0 \cdots\cdots 0$

  ↑ change

  $*$

- • change first (rightmost)

- 

- 

BINARY ⟷ GRAY

good way to encode/decode

BINARY ↦ GRAY

$b_{n-1} \cdots\cdots b_1 \, b_0 \longmapsto c_{n-1} \cdots\cdots c_1 \, c_0$

MOD 2 ADDITION OF BITS

⊕	0	1
0	0	1
1	1	0

0 ⟷ FALSE

1 ⟷ TRUE

$a \oplus b$

exclusive OR

a	b	c
0	0	0
0	1	1
1	0	1
1	1	0

$$a \oplus b = c$$

$$a = b \oplus c$$

$$b = a \oplus c$$

$$c_j = b_j \oplus b_{j+1}$$

(Think of all binary bits being 0 to the left)

E.g. BINARY GRAY

$$14 = (1110)_2 \longrightarrow 1001$$

GRAY $\longmapsto$ BINARY

$b_3 \, b_2 \, b_1 \, b_0$

$$c_0 = b_0 \oplus b_1$$

$$c_1 = b_1 \oplus b_2$$

$$c_2 = b_2 \oplus b_3$$

$$c_3 = b_3 \oplus b_4 = b_3 \qquad (b_4 = 0)$$

$$b_3 = c_3$$

$$b_2 = c_2 \oplus c_3$$

$$b_1 = c_1 \oplus b_2 = c_1 \oplus c_2 \oplus c_3$$

$$b_0 = c_0 \oplus c_1 \oplus c_2 \oplus c_3$$

In general

$$b_j = c_j \oplus c_{j+1} \oplus \cdots$$

E.g. GRAY $\longmapsto$ BINARY

$$1111 \longmapsto (1010)_2 = 10$$

$$11111 \longmapsto (10101)_2 = 16+4+1=21$$

$$111111 \longmapsto (101010)_2$$

$$\underset{32}{1}\,0\,\underset{8}{1}\,0\,\underset{2}{1}\,0$$

$$32+8+2=42$$

$$1, 2, 5, 10, 21, 42$$

↑ # steps to solve spin-out or chinese rings puzzle with 4, 5, 6

$111 \mapsto 101$

5

$11 \mapsto 10$

n even $\frac{2}{3}(2^{n}-1)$

n odd $\frac{1}{3}(2^{n+1}-1)$

Rule to get out : (right to left)

n even move 2nd bit

n odd " 1st bit

Jan 30, 2007

Chinese rings

initial position $11\cdots1$

all rings are on

GRAY	BINARY	
$11\cdots1$	$10\ \cdots 010$	n even
	$1 \cdots 0101$	n odd

$$\underbrace{1 \oplus 1 \oplus \cdots \oplus 1}_{n} = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

What number is this?

n odd

$$(10\cdots\cdots10101)_2 = ?$$

$$\cdots\cdots + 16 \quad 4 + 1$$

$$1 + 4 + 16 + \cdots +$$

$$= 1 + 4 + 4^2 + 4^3 + \cdots + 4^k$$

geometric series

k	
0	1
1	$1 + 4 = 5$
2	$1+4+4^2 = 21$
3	$1+4+4^2+4^3 = 85$

$$\vdots$$

$$\frac{4^{k+1} - 1}{4 - 1}$$

geometric series, $a \neq 1$ number

$$S_k := 1 + a + a^2 + \dots\dots + a^k = \frac{a^{k+1} - 1}{a - 1}$$

$$S_{k+1} = 1 + a + \dots\dots + a^{k+1}$$

$$= 1 + a\,(1 \underbrace{+ a + \dots\dots}_{S_k} + a^k)$$

$$S_{k+1} = 1 + a S_k$$

verify $\quad S_k \overset{?}{=} \dfrac{a^{k+1} - 1}{a - 1}$

$$(a-1)\,S_K \overset{?}{=} a^{k+1}-1$$

$$\begin{aligned} a\,S_K - S_K &= a(1+a+\cdots+a^k) \\ &\quad -(1+a+\cdots+a^k) \\ &= a+a^2+\cdots+a^{k+1} \\ &\quad -(1+a+\cdots+a^k) \\ &= a^{k+1}-1 \ \checkmark \end{aligned}$$

Find # steps to solve the puzzle

n odd

$$\frac{1}{3}(2^{n+1}-1) = \frac{2}{3}\cdot 2^n - \frac{1}{3}$$

$$\uparrow$$

n even

\# position

$$\downarrow$$

$$\frac{2}{3}(2^n-1) = \frac{2}{3}\times 2^n - \frac{2}{3}$$

2^n = # of positions in puzzle

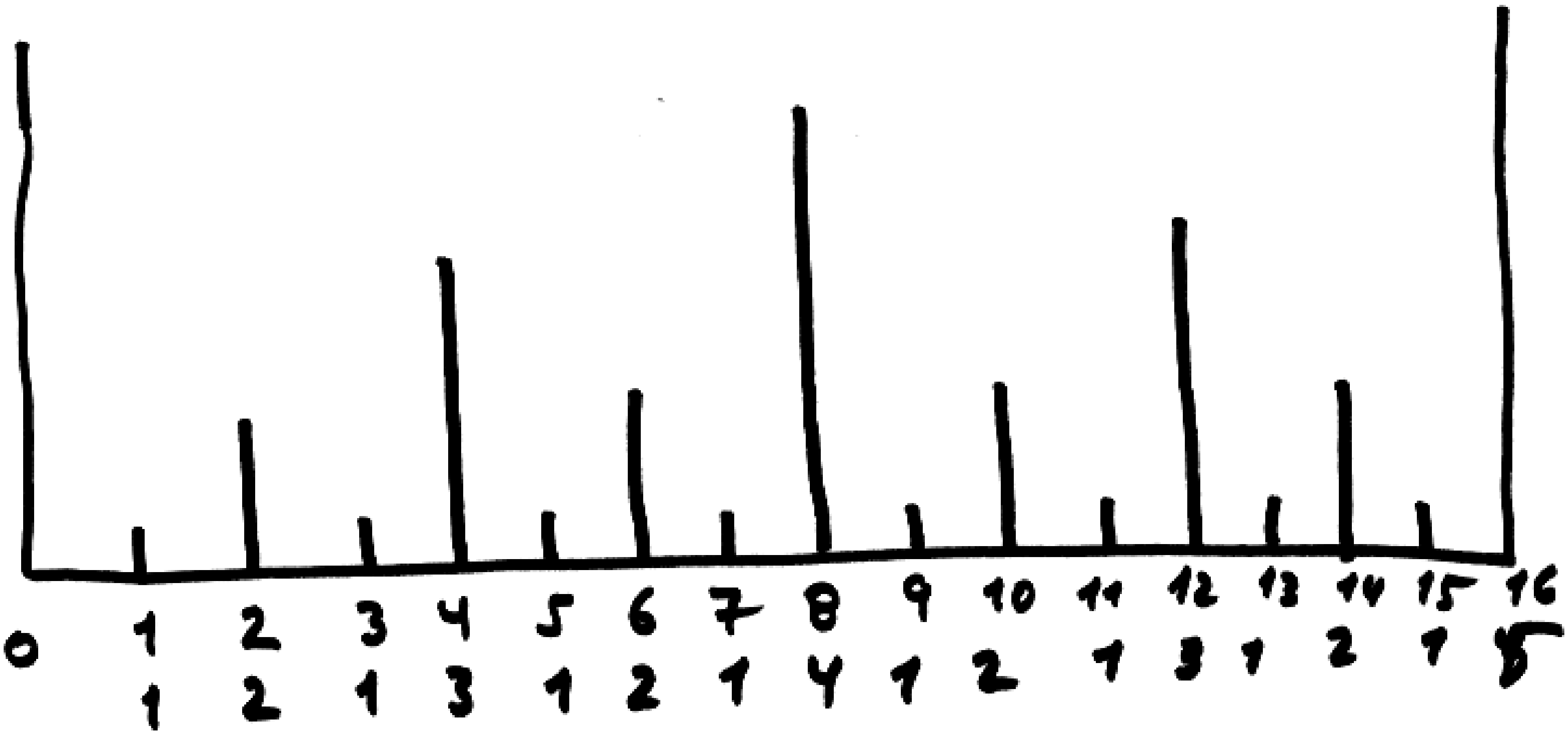
\# steps $\approx \frac{2}{3}\times$ positions

Ruler function

$\rho(m) =$ bit that changes in gray code $m-1 \mapsto m$

$=$ # bits that change in binary code $m-1 \mapsto m$

m	GRAY 4 3 2 1	BINARY	$\rho(m)$
0	0 0 0 0	0 0 0 0	
1	0 0 0 1	0 0 0 1	1
2	0 0 1 1	0 0 1 0	2
3	0 0 1 0	0 0 1 1	1



$\rho(m)$ = highest power of 2 dividing m + 1

$$12 = 2^2 \times 3$$

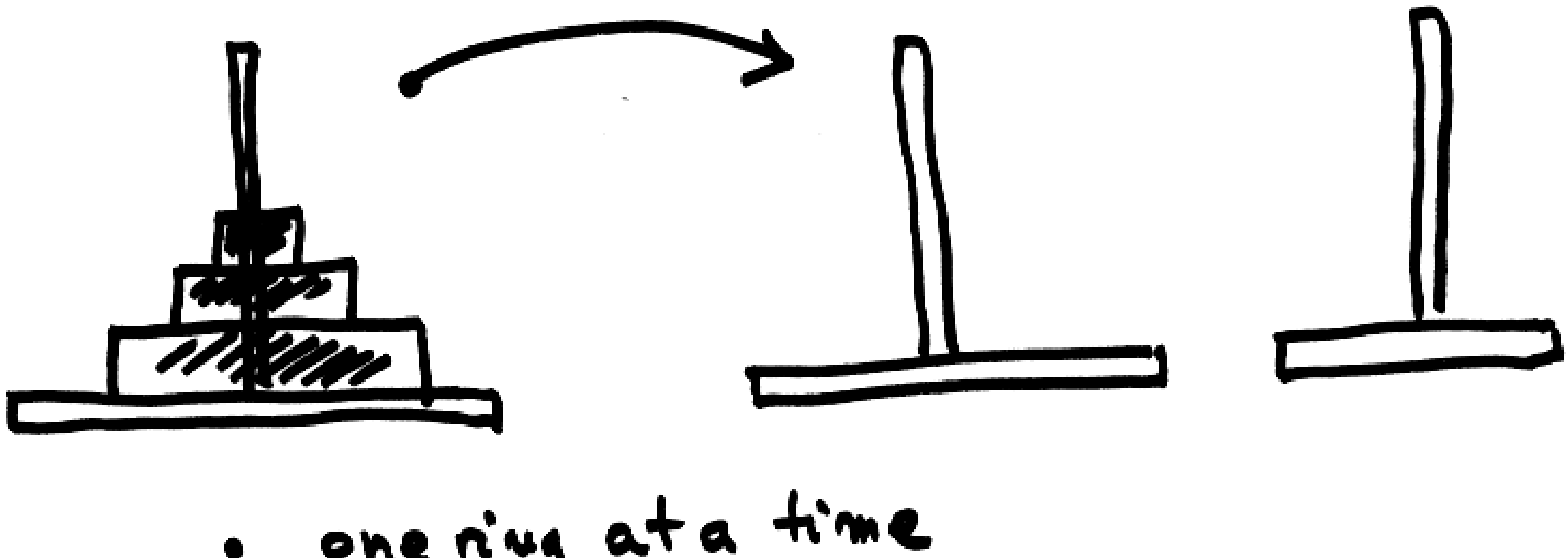
↖ 2 highest power of 2

$$\rho(12) = 3$$

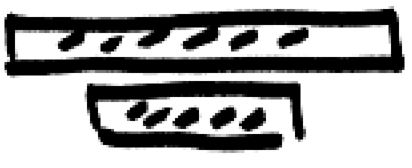
This gives a recipe to construct the Gray code (solve the lights puzzle or pick a binary lock)

* * *

Hanoi Towers



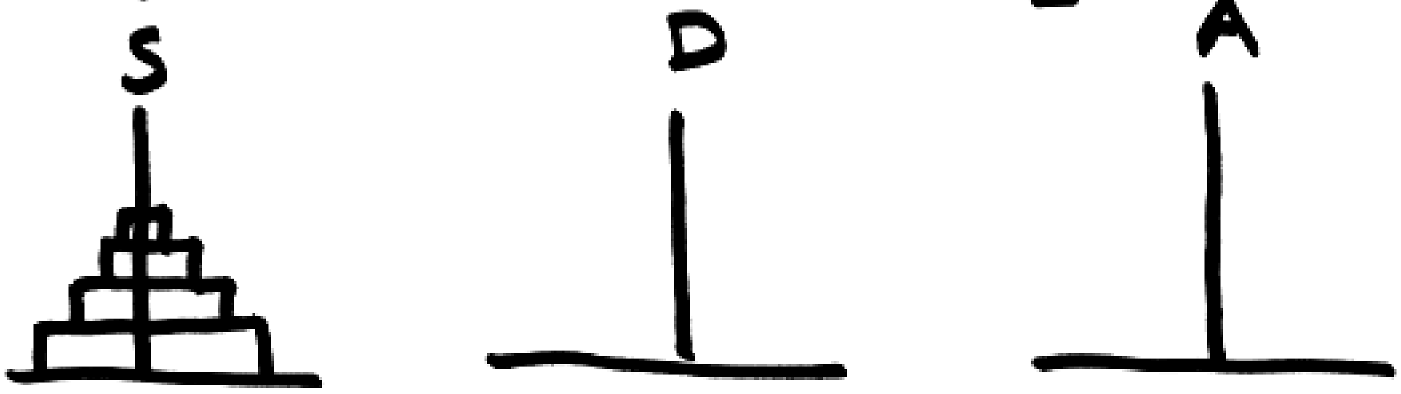
- one ring at a time
- with no:



Unique optimal solution

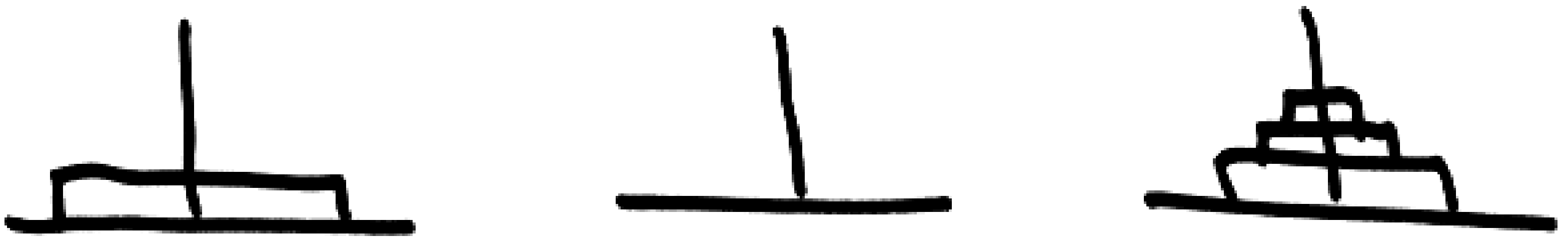
(optimal = least number of steps)

Suppose we know how to solve it if we had 3 rings.



- Move top 3 disks

$S \longmapsto A$



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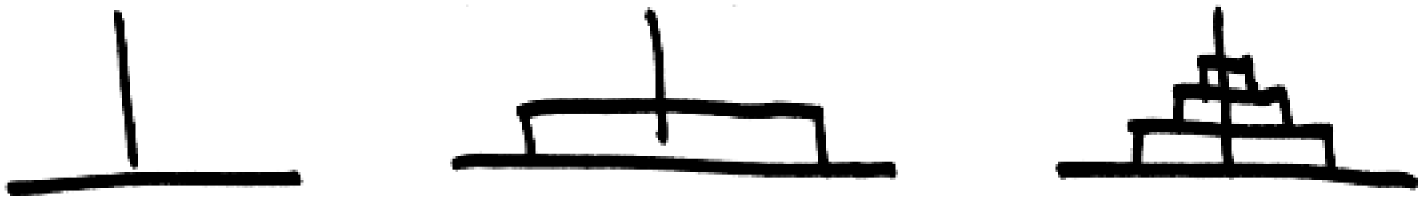
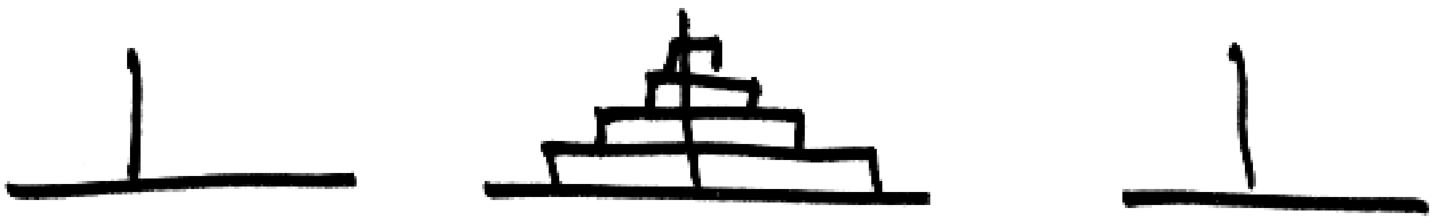


Move last disk to destination

- Move top 3 disks to destination



hanoi (n, S, D, A) =

- hanoi(n-1, S, A, D)
- move disk n from S to D
- hanoi(n-1, A, D, S)

Recursive procedure

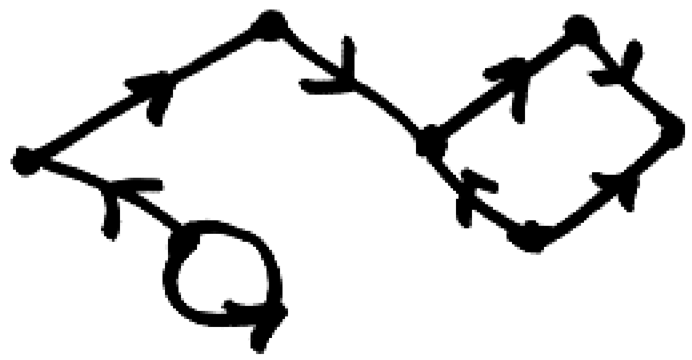
n	h_n	
1	1	2 - 1
2	3	4 - 1
3	7	8 - 1
4	15	16 - 1
5	31	32 - 1

Recursion: $h_n = 2h_{n-1} + 1$

In fact: $h_n = 2^n - 1$

What is the graph of the puzzle?

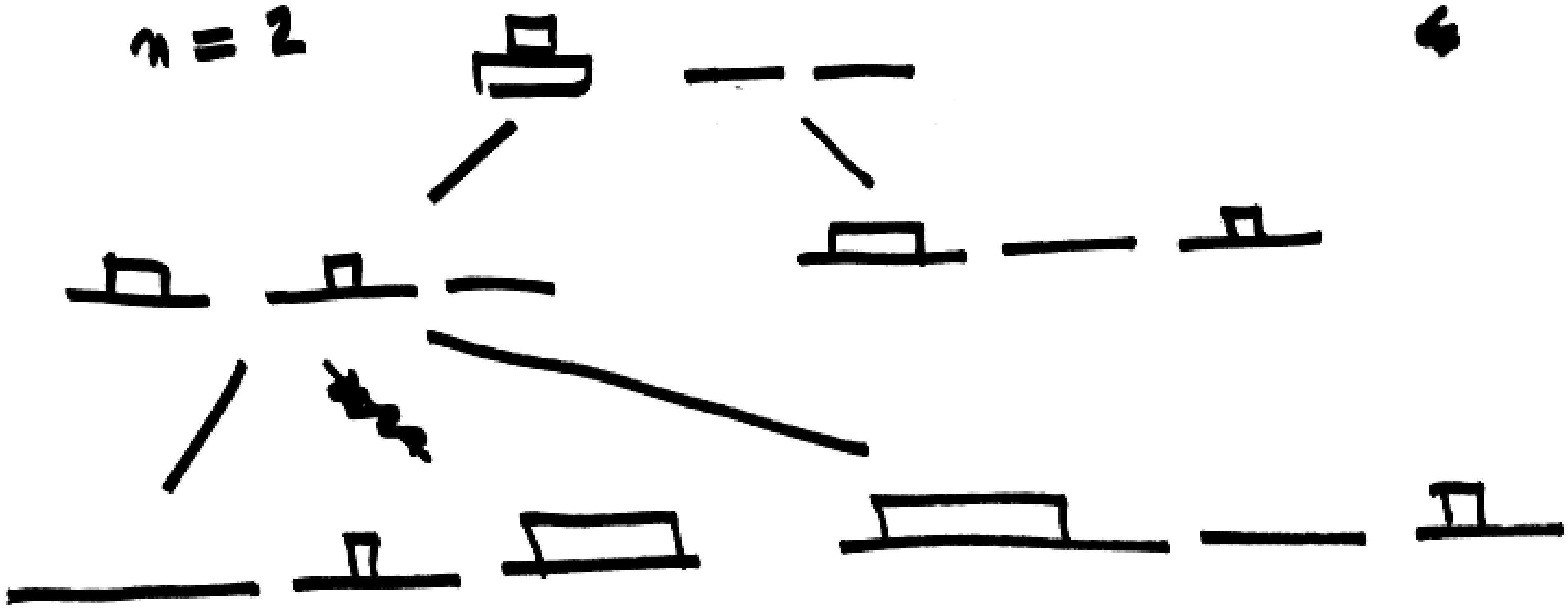
A <u>graph</u> : collection of vertices & edges



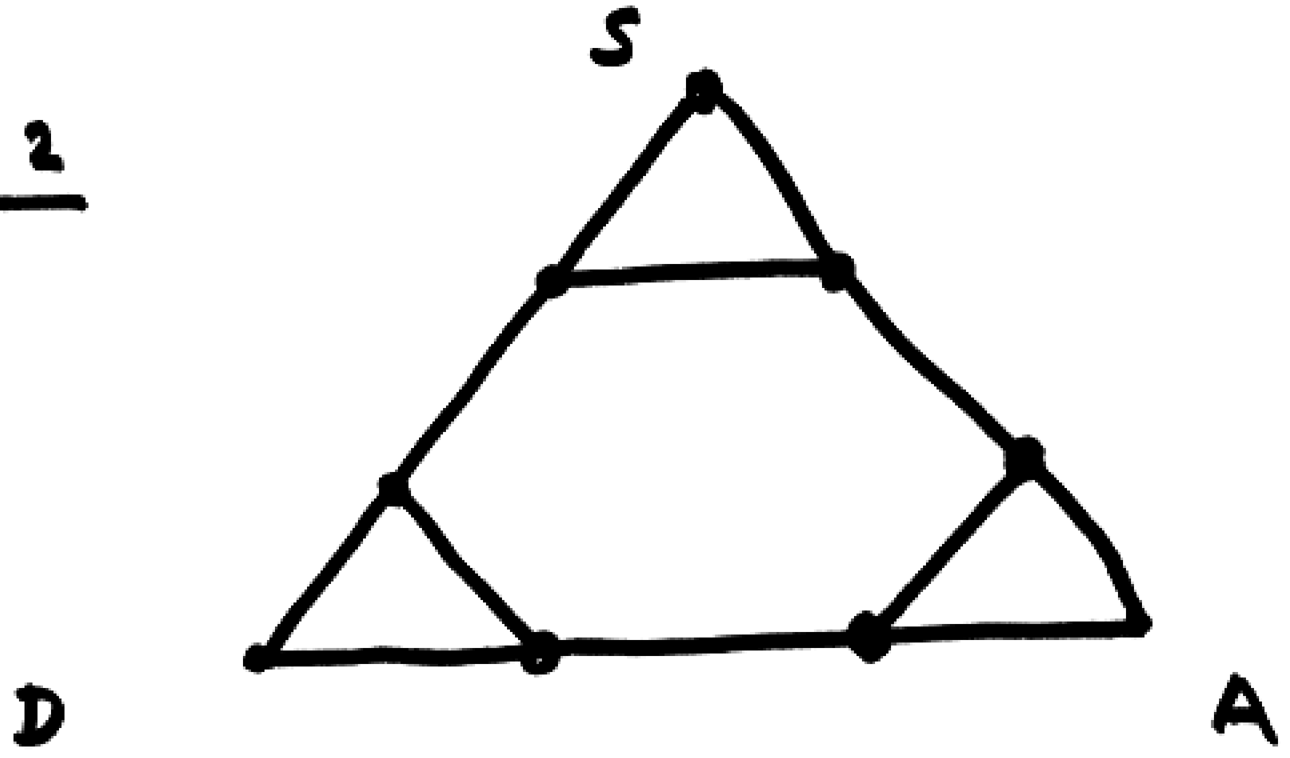
positions ↔ vertices
moves ↔ edges

Chinese rings graph

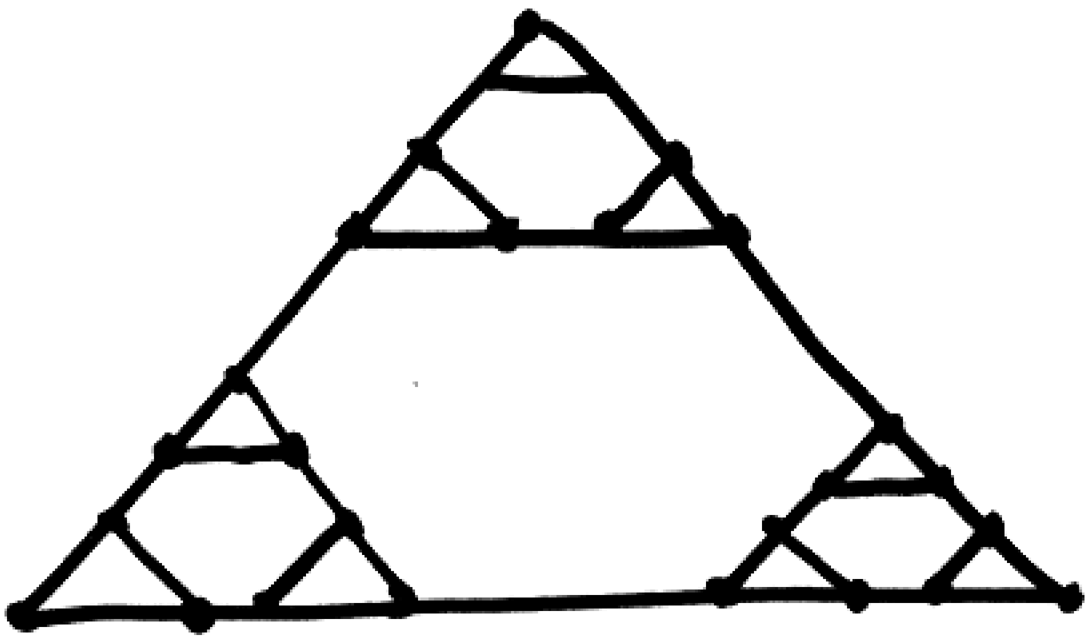
n = 2



n = 2



n = 3



Feb 1, 2007

Recursion

$n! = n \cdot (n-1) \cdots \quad 3 \cdot 2 \cdot 1$

$= \#$ of permutations of n things

$n = 3 \qquad 3! = 3 \cdot 2 \cdot 1 = 6$

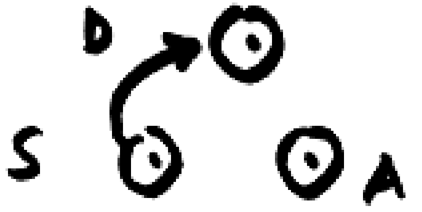
1 2 3
2 1 3
3 2 1
1 3 2
2 3 1
3 1 2

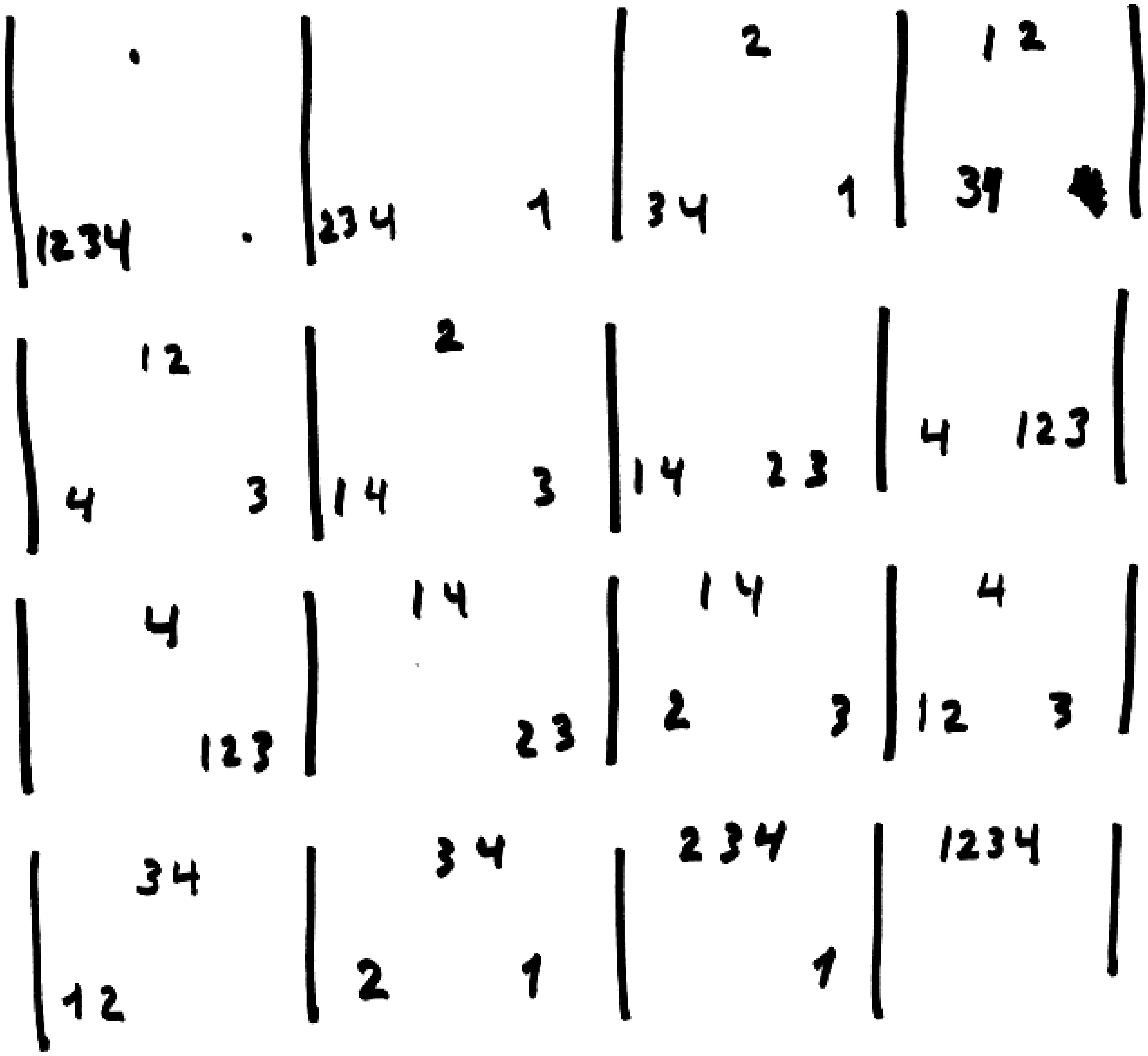
Recursively.

~~[illegible]~~

$$\text{factorial}(n) = \begin{cases} n * \text{factorial}(n-1) & n > 1 \\ 1 & n = 1 \end{cases}$$

n = 4



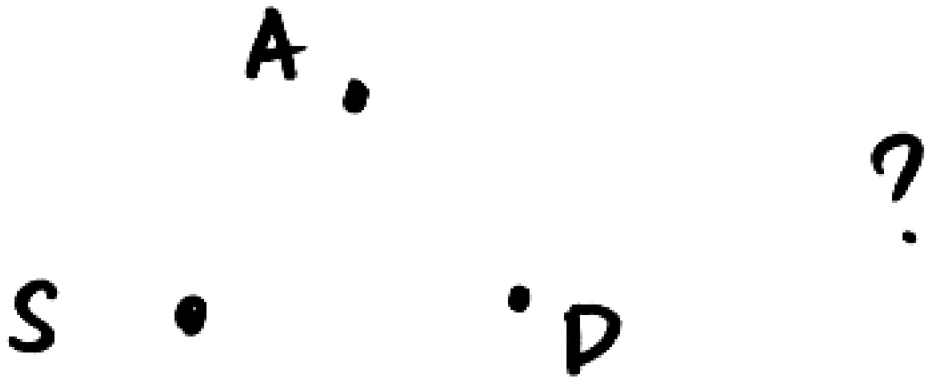


sequential

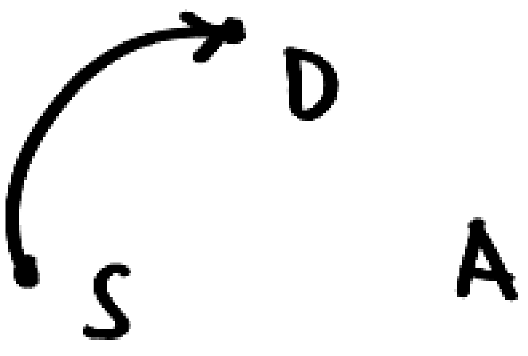
- disk 1 ↻ every other move
- in other moves: move disk <u>other</u> than 1.

Remarks

- disk that moves is given by ruler function.
- odd ↺ ↻ even



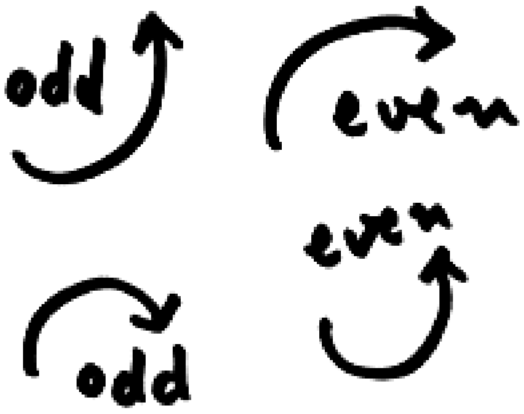
move in opposite direction.



n disks

n even

n odd



PUZZLE ⇝ graph

position ↔ vertex

move ↔ edge

Hanoi towers

position = arrangement of disks in a legal way.

To give a position is enough to say what disks are where.

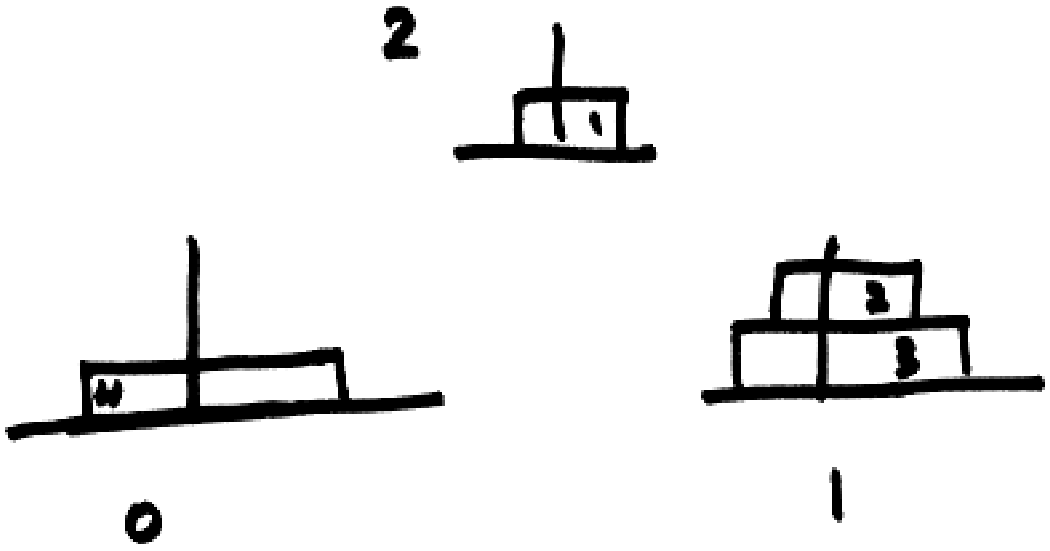
encode a position

$$(p_1, p_2, \dots, p_n)$$

ternary bits $\quad p_i = 0, 1, 2$

$p_i :=$ peg # where disk i is.

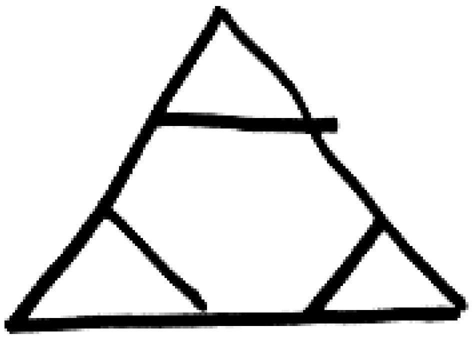
$n=4$



$$\updownarrow$$

$$(2, 1, 1, 0)$$

graph of puzzle



$\binom{n}{k} =$

$(1+x)^2 = 1 + 2x + x^2$ 1 2 1

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$ 1 3 3 1

Pascal's triangle

k →

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

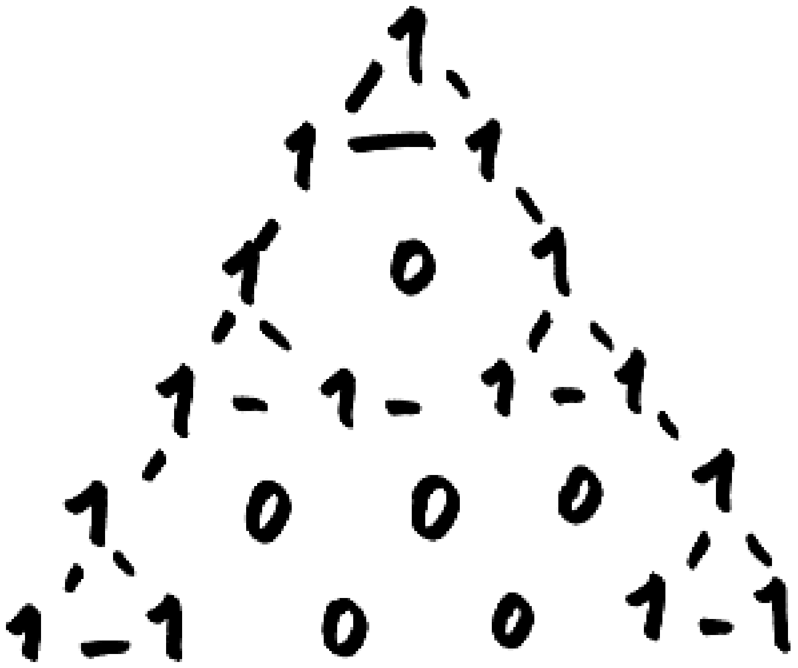
↓ n

$\binom{5}{2} = 10$

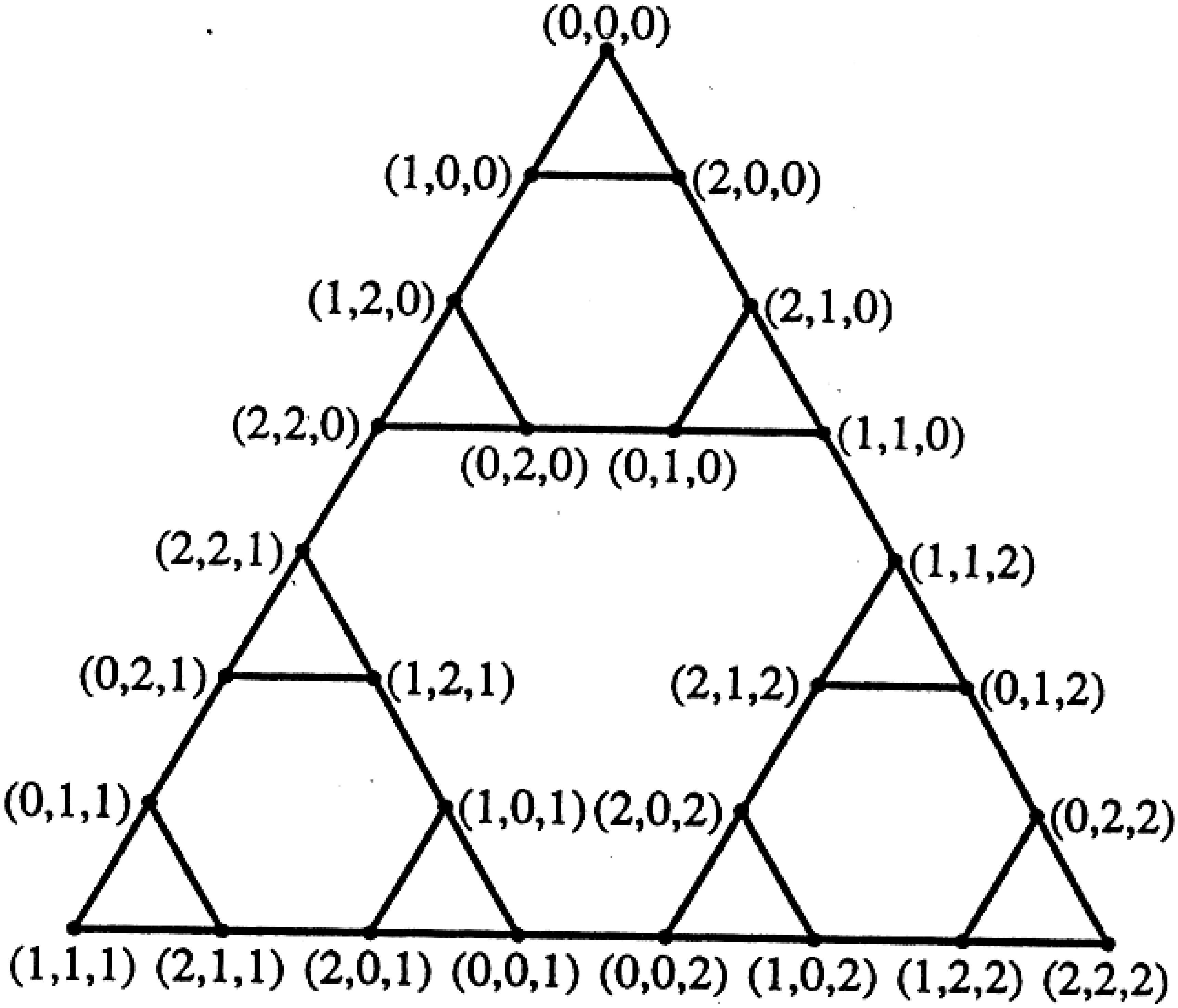
$\binom{n}{k}$ = n chose k

= # of ways to choose k things out of n

Pasal triangle w/parity

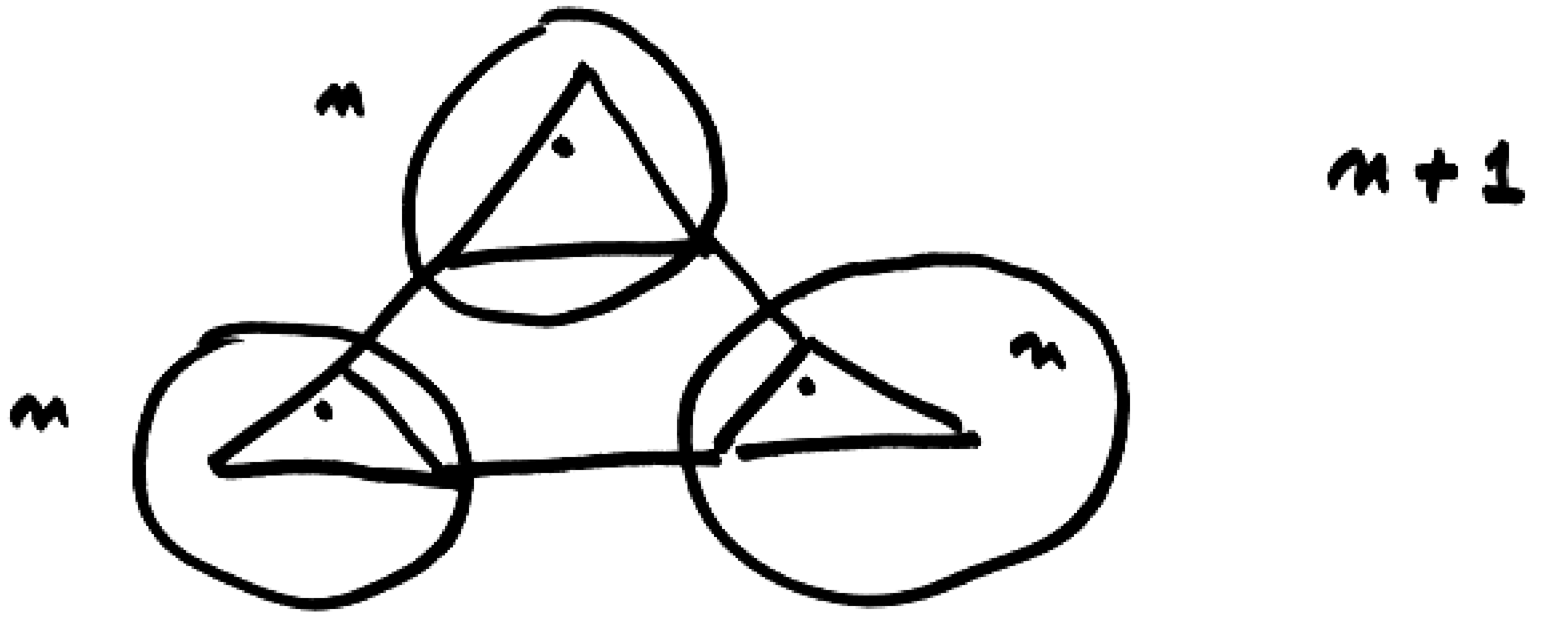



LA TOUR D'HANOÏ
VÉRITABLE CASSE-TÊTE ANNAMITE
BREVETÉ S.G.D.G.
JEU
RAPPORTÉ DU TONKIN
PAR LE PROFESSEUR N. CLAUS (DE SIAM)
MANDARIN
DU COLLÈGE LI-SOU-STIAN
A.U



Graph of Hanoi Towers puzzle with three disks.

Feb 06, 2007



positions in case n+1

= 3 × positions in case n

$C_n :=$ # positions in case n

$$\begin{cases} C_{n+1} = 3\, C_n \\ C_1 = 3 \end{cases}$$

$3, 3^2, 3^3, \ldots$

Claim $C_n = 3^n$

$C_{n+1} = 3^{n+1} = 3 \times C_n$

positions $\longleftrightarrow$ $(p_1, p_2, \dots, p_n)$

$p_i = 0, 1, 2$
= peg # where
disk i is

$\underline{n=1}$

(0) , (1) , (2)

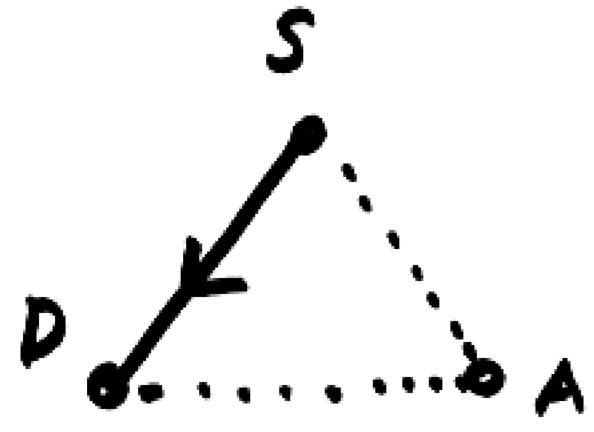
$\underline{n=2}$

$(0,0)$, $(1,0)$, $(2,0)$
$(0,1)$, $(1,1)$, $(2,1)$
$(0,2)$, $(1,2)$, $(2,2)$

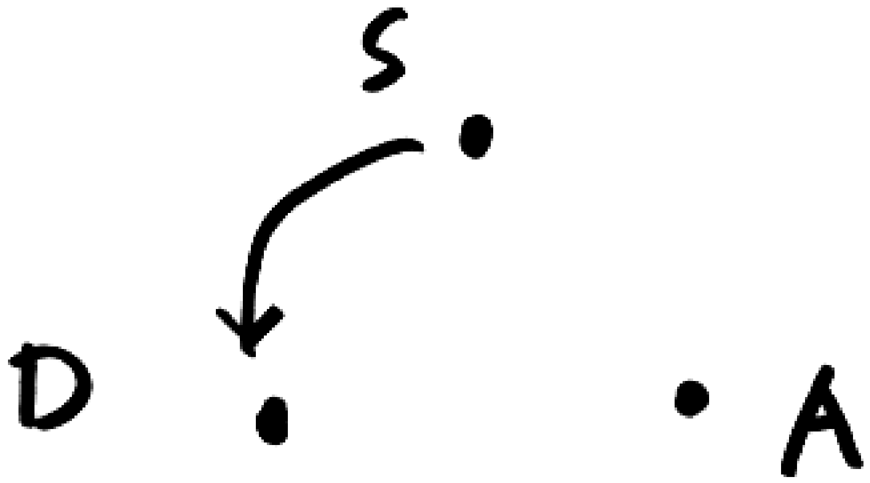
$\underline{n=3}$ $(0,0,*)$ $(1,0,*)$, $(2,0,*)$

$\vdots$

Solving puzzle optimally
what happens with last
disk , n ?



Disk n just moves from S → D ?



n=7 $\quad \circlearrowleft 7 \Rightarrow$ disk # 1 $\circlearrowleft$

PASCAL TRIANGLE

$\binom{n}{1}$ $\binom{n}{2}$ $\binom{n}{3}$ ⋮

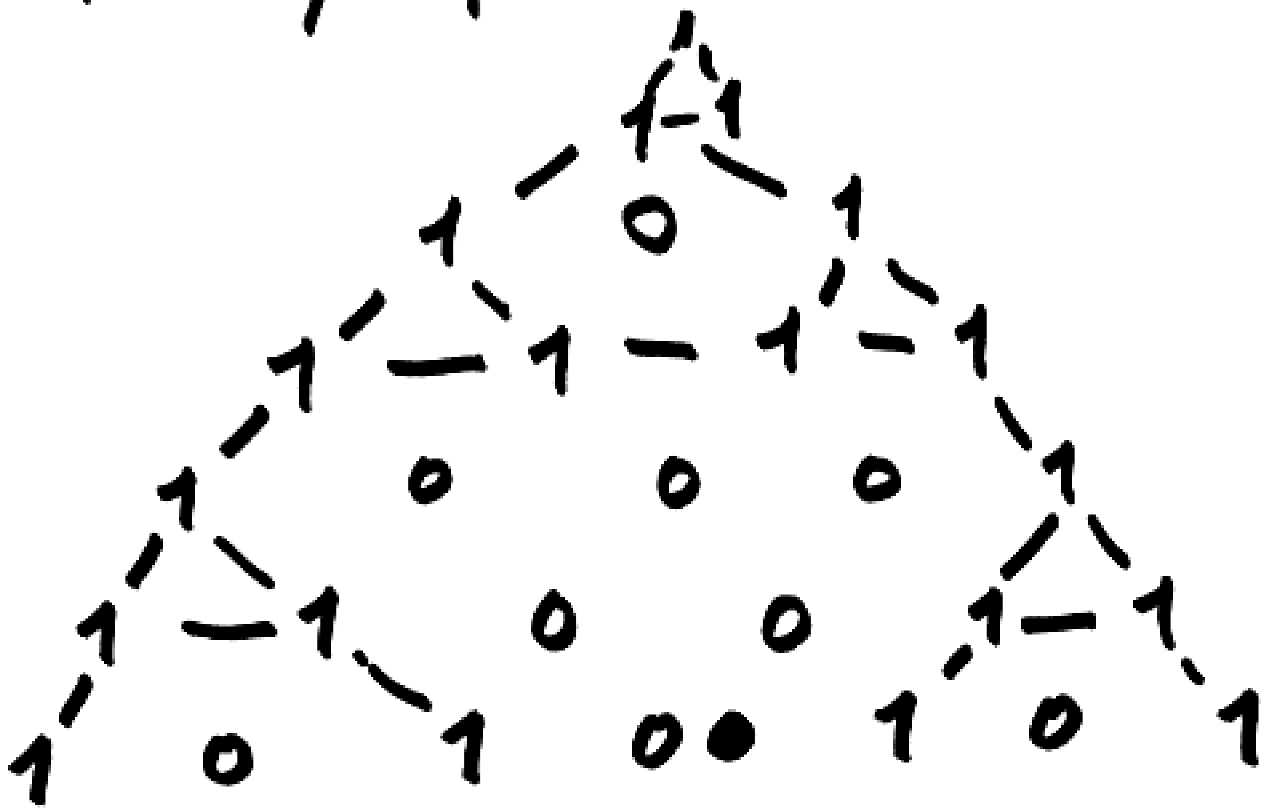
n	
	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 ④ ⑥ 4 1
5	1 5 ⑩ 10 5 1
6	1 6 15 20 15 6 1

$\binom{n}{k}$

"n chose k"

$0 \le k \le n$

Parity of the Pascal triangle



$$(1+x)^{3} = 1 + 3x + 3x^{2} + 1$$

$$(1+x)^{4} = 1 + 4x + 6x^{2} + 4x^{3} + x^{4}$$

$$\vdots$$

$$\binom{n}{k} = \#\left\{\begin{array}{l}\text{ways to choose k things}\\ \text{out of } n\end{array}\right\}$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} =$$

1, 2, 3, 4

$n = 4$

1, 2	2, 3	3, 4
1, 3	2, 4	
1, 4		

$$\binom{4}{2} = 6$$

$$\binom{n}{2} = \frac{1}{2}\ \underset{\uparrow}{n} \times (n - 1)$$

1st choice

2nd choice

account for the order

otherwise we are double counting.

$\underline{n=4}$

$\boxed{1, 2}$	$\boxed{2, 1}$	3, 1	4, 1
1, 3	2, 3	3, 2	4, 2
1, 4	2, 4	3, 4	4, 3

$$\binom{n}{2} = \frac{n \times (n-1)}{2}$$

$$\binom{4}{2} = \frac{4 \times 3}{2} = 6$$

$n = \underline{5}$ $\qquad \binom{5}{2} = 10$

1, 2
1, 3
1, 4
1, 5

2, 3
2, 4
2, 5

3, 4
3, 5

4, 5

$$\frac{n \times (n-1)}{2} = \frac{5 \times 4}{2} = 10$$

$$\binom{n}{3} = \frac{1}{6}\, \underset{\overset{\uparrow}{1^{st}}}{n}\, \underset{\cancel{3}\;\overset{\uparrow}{2^{nd}}}{(n-1)}\, \underset{\overset{\nwarrow}{3^{rd}}}{(n-2)}$$

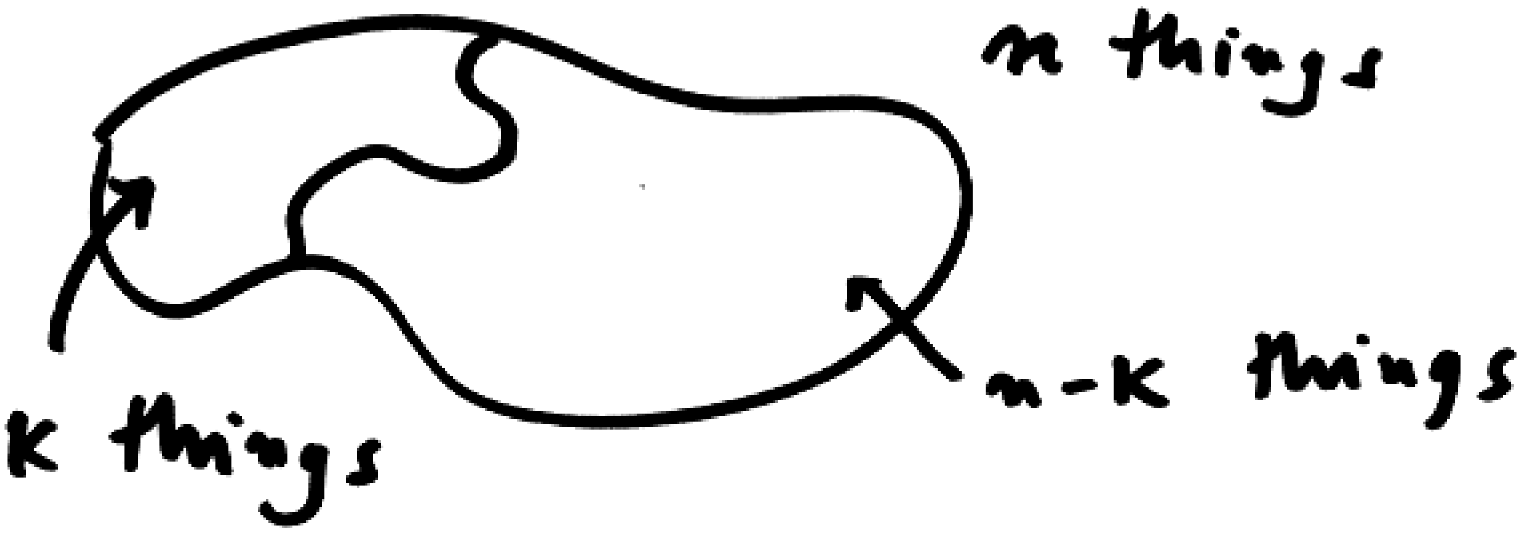
overcounting e.g. $\{1, 2, 3\}$

$$6 \text{ times} \left\{ \begin{array}{l} 1, 2, 3 \\ 1, 3, 2 \\ 2, 1, 3 \\ 2, 3, 1 \\ 3, 1, 2 \\ 3, 2, 1 \end{array} \right.$$

$$\binom{n}{4} = \frac{1}{4!} n(n-1)(n-2)(n-3)$$

$$\binom{5}{3} = \frac{5\times 4\times 3}{3\times 2\times 1} = 10$$

Symmetry in Pascal's triangle



Picking k ↔ Picking n-k

E.g. $n = 5$

Pick two things $\quad \binom{5}{2} = 10$

↓

Pick 3 things $\quad \binom{5}{3} = \frac{5\times 4\times 3}{3\times 2\times 1} = 10$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

$$= \frac{n(n-1)\cdots\cdots 2\ 1}{k!\ (n-k)!}$$

$$\binom{7}{4} = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5\cdot 4\cdot \overbrace{3\cdot 2\cdot 1}}{4\cdot 3\cdot 2\cdot 1\ \cdot \underbrace{3\cdot 2\cdot 1}}$$

$$= \frac{7!}{4!\,3!}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

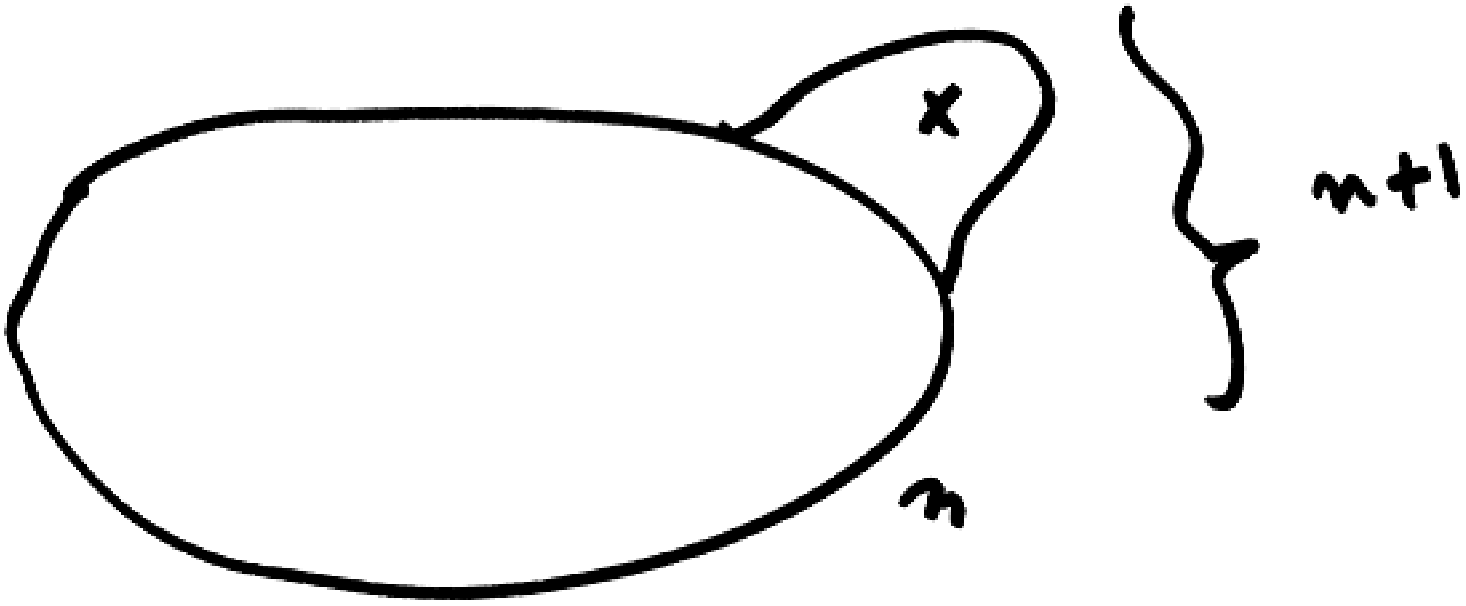
$$k \longleftrightarrow n-k$$

$$k \longmapsto n-k$$

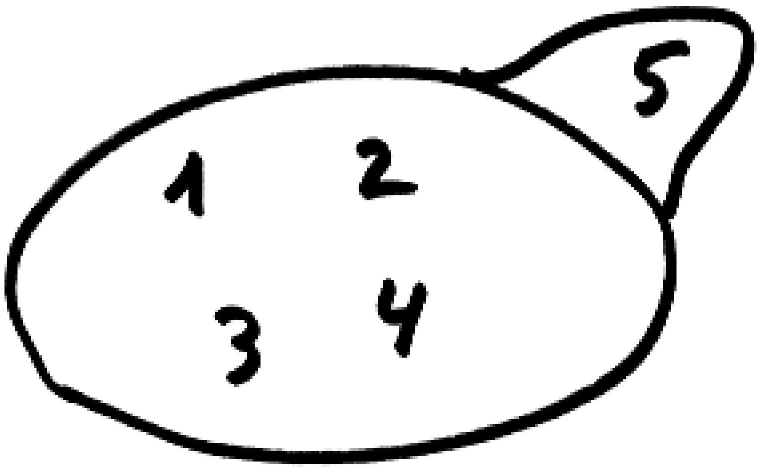
$$n-k \longmapsto n-(n-k)=k$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\boxed{\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}}$$



n=4



Excluding 5	Including 5
1, 2 2, 3 3, 4	5, 1
1, 3 2, 4	5, 2
1, 4	5, 3
$\binom{4}{2}$	5, 4
	$\uparrow$ $\binom{4}{1}$

$$\binom{5}{2}=\binom{4}{2}+\binom{4}{1}$$

15 - puzzle

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

S. LLoyd

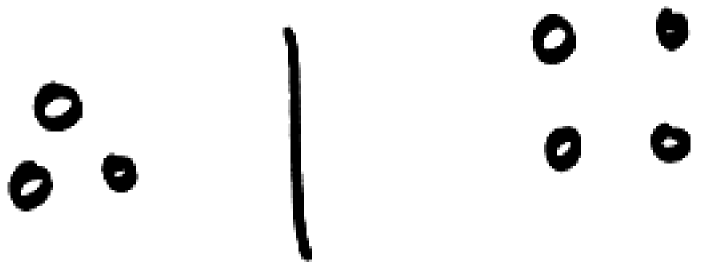
1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

cannot bedone!

Feb 8, 2007

NIM

Two piles



good move →

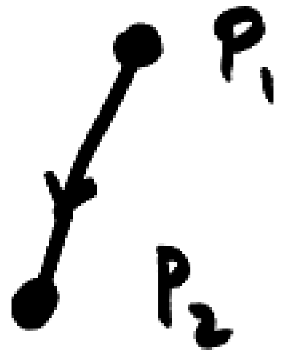
Strategy: achieve same number in both piles.

How does this extend to more piles?

Ch. Bouton 1901

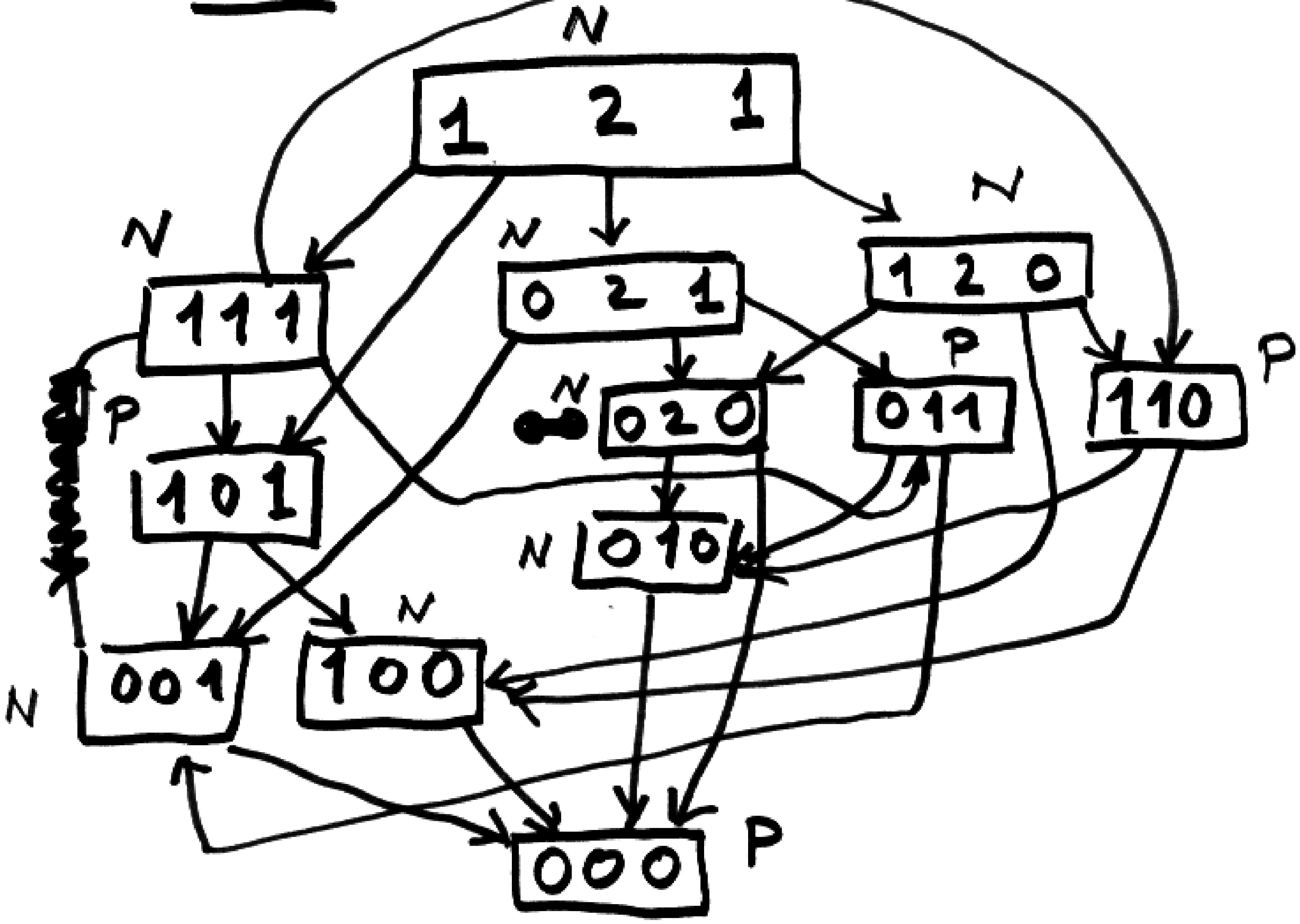
Theory applies to impartial games.

graph : vertices ↔ position
edge ↔ move



$P_1 \rightsquigarrow P_2$

NIM



Strategy

- Reach P
- opponent moves to N

Impartial game

- Two players, alternate
- same moves
- No chance
- complete information
- no ties / endgame

Player unable to move loses.
(Normal play)
(opposite " wins)
misére play

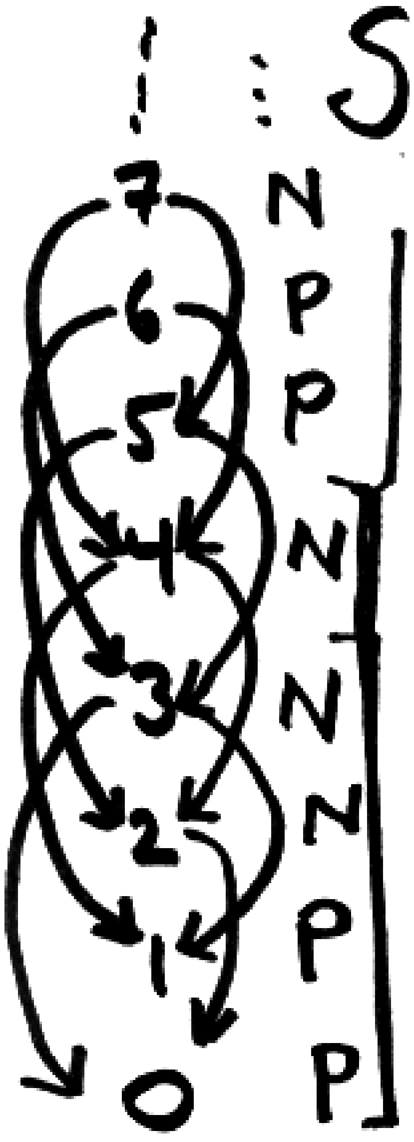
Substraction games

one pile

you take s objects from the pile where

$s \in S$

E.g. $S = \{2,3\}$



positions labels repeat in the

P P N N N

pattern

E.g. if pile has 22 things

take 2 to reach a P position

NiM addition

(Nimbers)

$n \oplus m$

write n, m in binary

E.g. $n = 3$, $m = 5$

$$\begin{array}{ll} m & 1\,0\,1 \\ n & \underline{0\,1\,1} \\ & 1\,1\,0 \leftarrow \quad n \oplus m = 6 \end{array}$$

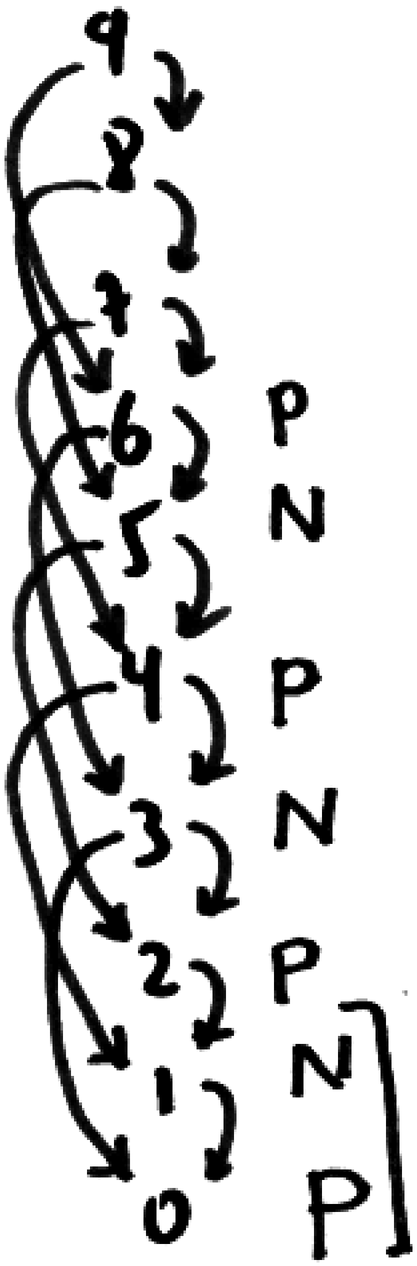
$3 \oplus 5 = 6$

$n = m \longleftrightarrow n \oplus m = 0$

$$\begin{array}{l} 1\,1\,0\,1 \\ \underline{a\,b\,c\,d} \\ 0\,0\,0\,0 \end{array} \longleftrightarrow \begin{array}{l} d = 1 \\ c = 0 \\ b = 1 \\ a = 1 \end{array}$$

$(n_1, \dots, n_k)$ P-position $\Leftrightarrow n_1 \oplus \dots \oplus n_k = 0$

E.g. $S = \{1, 3\}$



$\underbrace{PN}_{\text{pattern repeats}}$

What are the P positions in NIM?

two piles: (n, m)

$n = m \longleftrightarrow$ P position

<u>More piles</u>

Feb 13, 2007

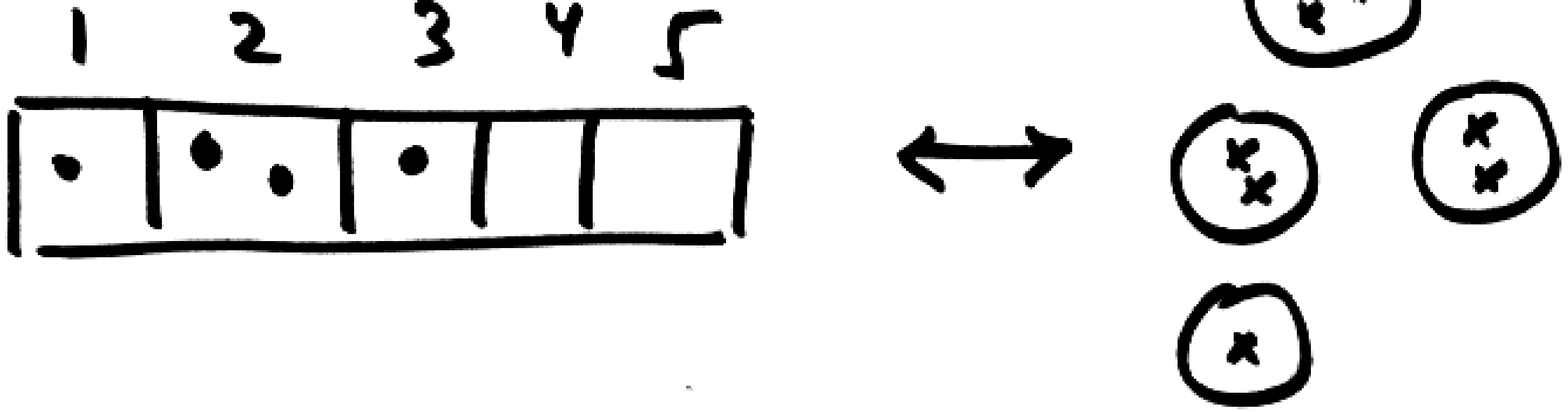
Nimble

Rule Move one penny to the left (any # of steps)

Nimble ↔ Nim

penny ↔ pile

position ↔ size



$$\begin{array}{c} 001 \\ 011 \\ 100 \end{array} \qquad \begin{array}{c} 001 \\ 010 \\ 011 \end{array} \qquad \begin{array}{c} 001 \\ 011 \end{array} \qquad \begin{array}{c} 000 \end{array}$$

Write position of H's

$n_1, n_2, \ldots, n_K$

P position $\longleftrightarrow$ $n_1 \oplus \cdots \oplus n_K = 0$

$$\begin{array}{rr} 1 & 001 \\ 3 & 011 \\ 4 & 100 \\ \hline 6 & 110 \end{array} \rightarrow \text{N position}$$

Strategy : make it a P-position

Move H T H H T T

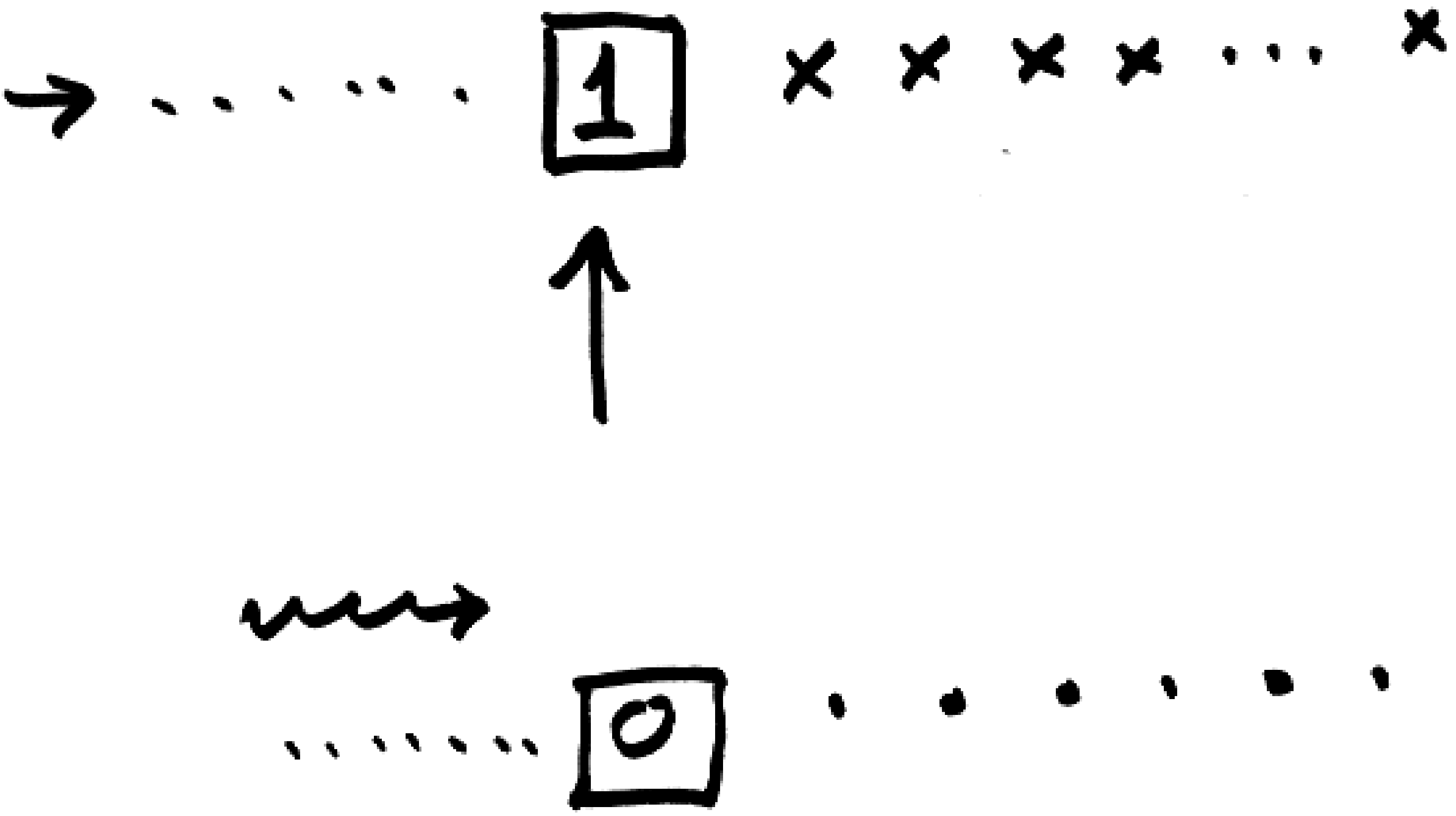
$\downarrow$

H H H T T T

$$\begin{array}{c} 001 \\ 010 \\ 011 \\ \hline 000 \end{array} \rightarrow \text{P-position}$$

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 1 & 0 \end{array} \qquad N \rightarrow P$$

- identify leftmost column with sum of 1.
- In that column find a row with a 1.
- Change that row to get 0 sum



$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & \boxed{1} & 0 & 0 \\ \hline & 1 & 1 & 0 \end{array} \longrightarrow \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ \hline & 0 & 0 & 0 \end{array}$$

NIM

$n_1, n_2, \ldots, n_K$ # of objects in each pile

P-position $\Leftrightarrow$ $n_1 \oplus \cdots\cdots \oplus n_K = 0$

- A move from $n_1 \oplus \cdots\cdots \oplus n_K = 0$ will mess this up
- Any N-position canbe made into P.

E.g. $15, 13, \underset{\substack{\downarrow \\ 2}}{5}$

15, 13, 5

$\downarrow$

10

①

$$\begin{array}{r|cccc}
 & 8 & 4 & 2 & 1 \\
\hline
15 & 1 & 1 & 1 & 1 \\
13 & 1 & 1 & 0 & 1 \\
5 & 0 & \textcircled{1} & 0 & 1 \\
\hline
 & 0 & 1 & 1 & 1
\end{array}
\longrightarrow
\begin{array}{cccc|l}
1 & 1 & 1 & 1 & 15 \\
1 & 1 & 0 & 1 & 13 \\
0 & 0 & 1 & 0 & 2 \\
\hline
0 & 0 & 0 & 0 &
\end{array}$$

N-position

P-position

②

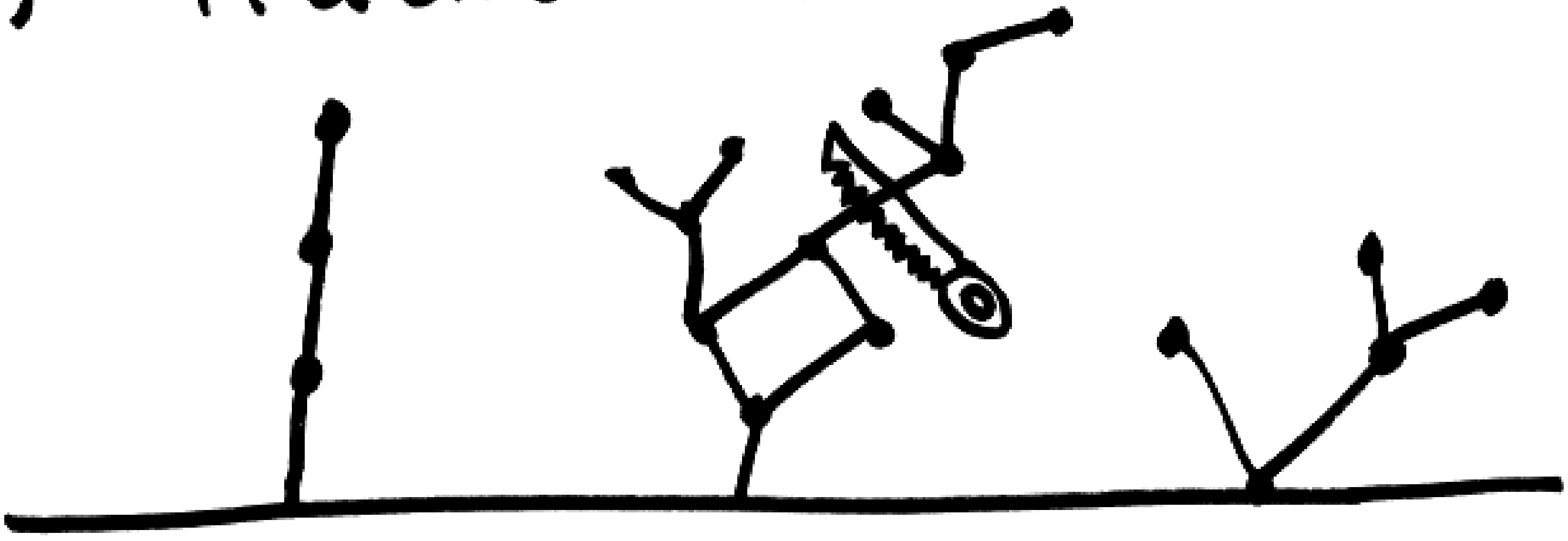
$$\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & \textcircled{1} & 0 & 1 \\
0 & 1 & 0 & 1 \\
\hline
0 & 1 & 1 & 1
\end{array}
\longrightarrow
\begin{array}{cccc|l}
1 & 1 & 1 & 1 & 15 \\
1 & 0 & 1 & 0 & 10 \\
0 & 1 & 0 & 1 & 5 \\
\hline
0 & 0 & 0 & 0 &
\end{array}$$

③

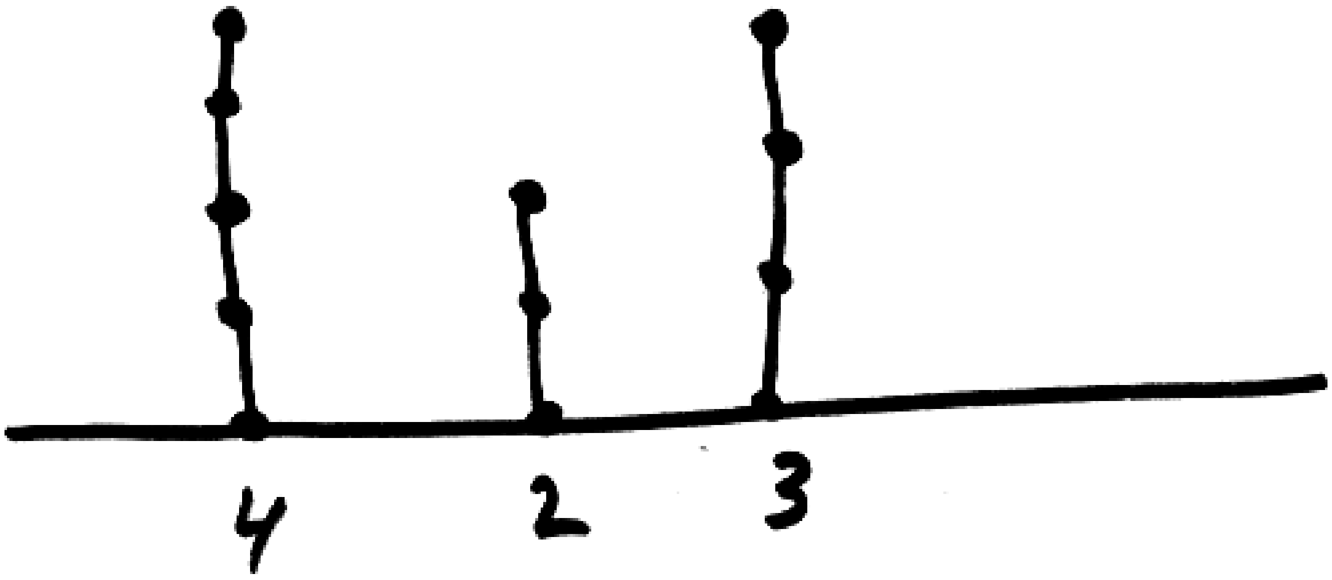
$$\begin{array}{cccc}
1 & \textcircled{1} & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
\hline
0 & 1 & 1 & 1
\end{array}
\longrightarrow
\begin{array}{cccc|l}
1 & 0 & 0 & 0 & 8 \\
1 & 1 & 0 & 1 & 13 \\
0 & 1 & 0 & 1 & 5 \\
\hline
0 & 0 & 0 & 0 &
\end{array}$$

Other examples of impartial games

3) Hackenbush



Move: Hack a piece off!



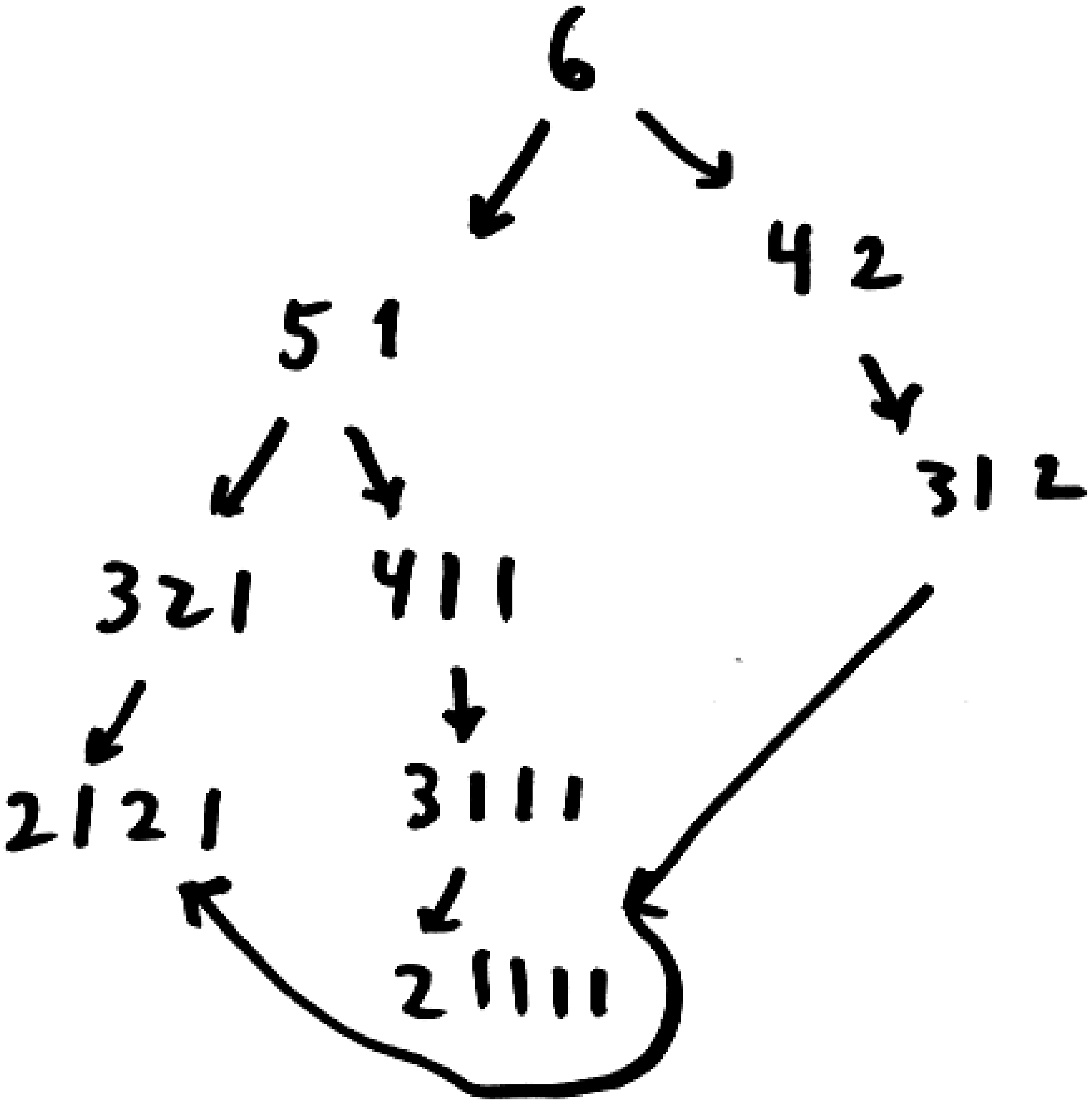
Each bamboo shoot ⟷ pile

segment ⟷ penny

4) Grundy's game

Start pile w/ n things

Rule Pick a pile and divide it into two unequal piles



* * *

sum of games

Γ_1, Γ_2 impartial games

$\Gamma = \Gamma_1 \oplus \Gamma_2$

A <u>move</u> in Γ is either a move Γ_1 or a move in Γ_2

Similarly $\Gamma_1, \ldots, \Gamma_k$

$\Gamma = \Gamma_1 \oplus \cdots \oplus \Gamma_k$

E.g. Nim with k-piles is the sum $\oplus$ of k 1-pile games.

<u>Example</u> $\Gamma_1 =$ substraction game

$S = \{1, 2\}$

$\Gamma_2 =$ substraction

$S = \{1, 3\}$

Position $\Gamma = \Gamma_1 \oplus \Gamma_2$

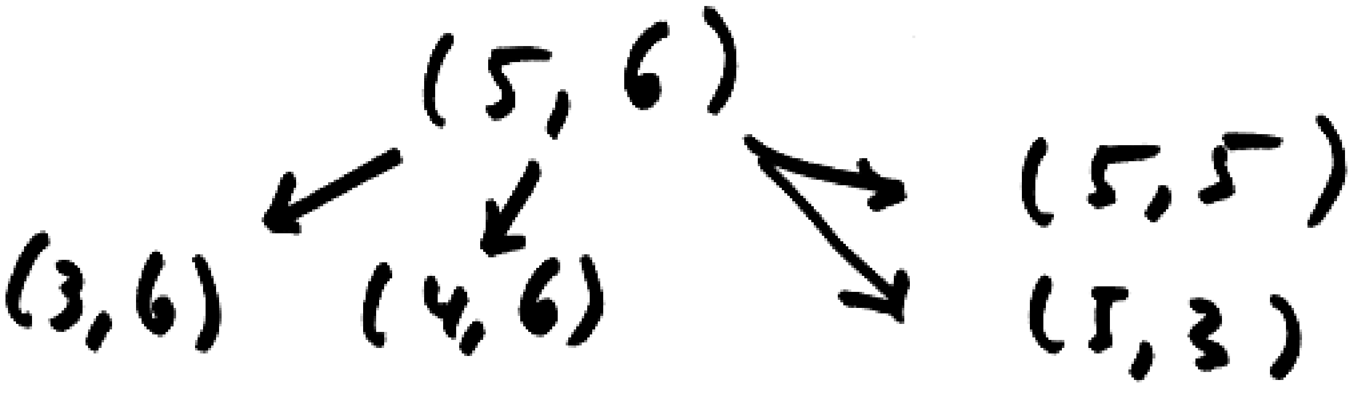
two piles $n_1 \to \Gamma_1$

$n_2 \to \Gamma_2$

Γ_1	Γ_2
5	6

Pick Γ_1:
$$\begin{tikzcd} & 4 \\ 5 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd}$$

Pick Γ_2:
$$\begin{tikzcd} & 5 \\ 6 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd}$$



Feb 15, 2007

Impartial games

Γ_1, Γ_2

$\Gamma = \Gamma_1 \oplus \Gamma_2$

A <u>move</u> in Γ is a move in either Γ_1 or Γ_2

E.g. Nim with k-piles

$$\underbrace{\Gamma \oplus \cdots \oplus \Gamma}_{k\text{-times}}$$

of Γ = Nim w/ one pile

Labeling P/N positions in Γ_1 and Γ_2 is not enough to find the label in $\Gamma_1 \oplus \Gamma_2$

Γ_1 = substraction game $\quad S = \{1,2\}$

Γ_2 = " $\quad S = \{1,3\}$

$$\Gamma = \Gamma_1 \oplus \Gamma_2$$

$$n_1 , \; n_2$$

$\boxed{\Gamma_1}$ $\quad S = \{1,2\}$

⋮	
5	N
4	N
3	P
2	N
1	N
0	P

$$n \text{ P-position} \Updownarrow 3 \mid n$$

$\boxed{\Gamma_2}$ $\quad S = \{1,3\}$

⋮	⋮
5	N
4	P
3	N
2	P
1	N
0	P

$$n \text{ P-position} \Updownarrow 2 \mid n$$

$\Gamma = \Gamma_1 \oplus \Gamma_2$

Γ_2								
6	⋮							
5	N							
4	P	N	N	P				
3	N	P	N	N				
2	P	N	N	P	N			
1	N	P	N	N	P			
0	P	N	N	P	N	N	....	
	0	1	2	3	4	5	6	7 Γ_1

1st column

$$\begin{tikzcd} P \arrow[r] & \textcircled{P} \\ & P \arrow[u] \end{tikzcd}$$

1st row

$$\begin{tikzcd} N \arrow[r] & \textcircled{?} \\ & N \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} N \arrow[r] & \textcircled{N} \\ & P \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] & \textcircled{N} \\ & N \arrow[u] \end{tikzcd}$$

We need something more elaborate than just N/p labels for each individual game.

We'll define a numerical value to a position in an impartial game. G (position) = 0, 1, 2, 3, ...

P - position $\longleftrightarrow$ G value = 0

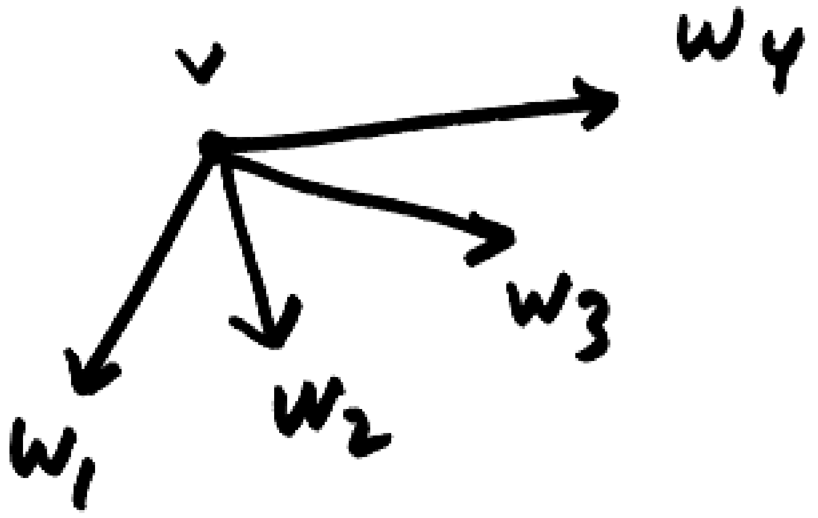
N - position $\longleftrightarrow$ G - value > 0

In the case of Nim k-piles

$n_1, \dots, n_k$

$$G = n_1 \oplus \cdots \oplus n_k$$

<u>Grundy function</u>



$$G(v) = \overset{\text{mex}}{\{}G(w_1), G(w_2), G(w_3), G(w_4)\}$$

$$G(v) := \text{mex}\{G(w) \mid v \mapsto w\}$$

mex = minimum excludant

$S \subseteq \{0, 1, 2, 3, \dots\}$

mex(S) := smallest number which is NOT in S

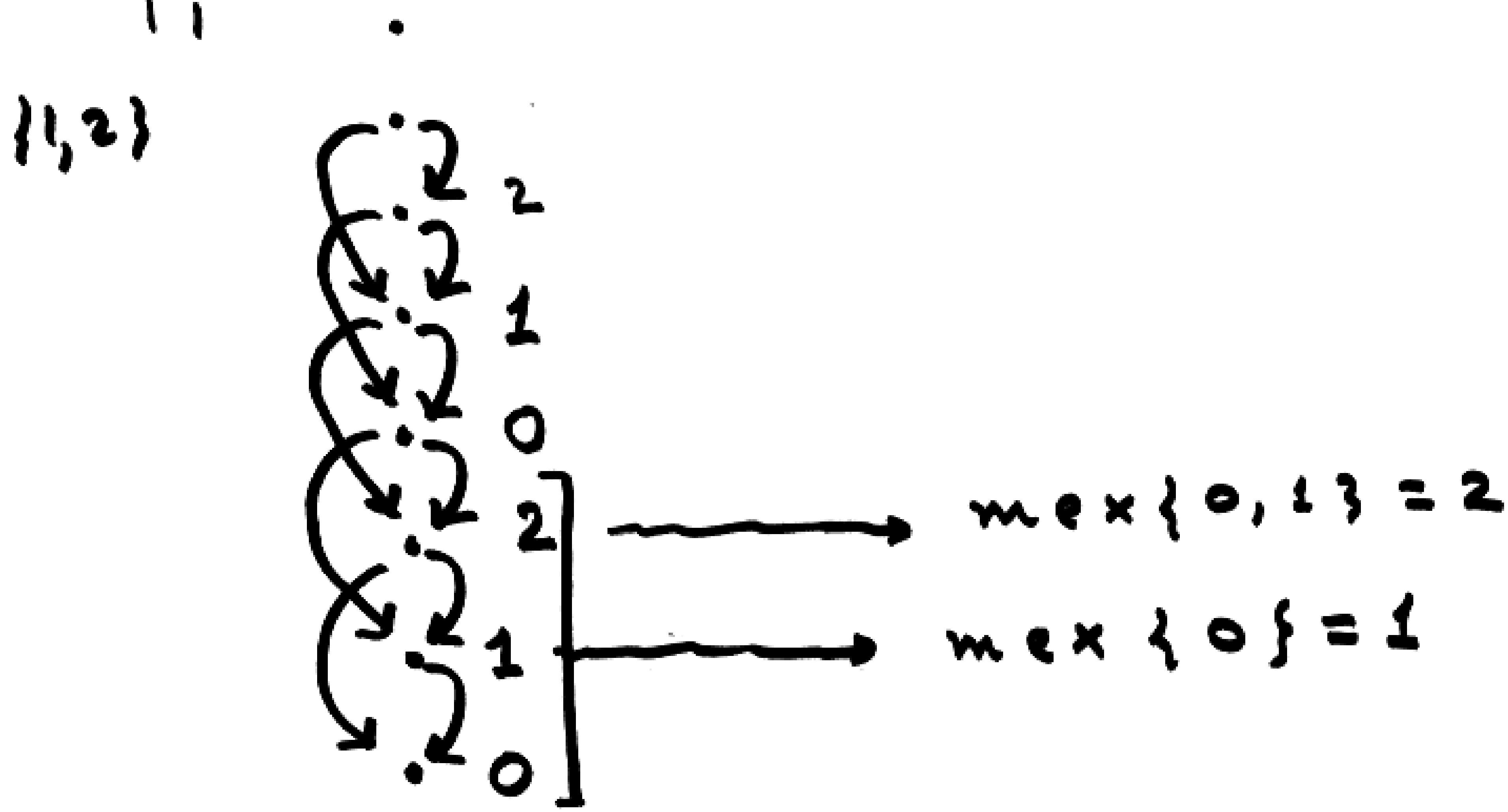
$\text{mex}\{0, 1, 4, 6, 9\} = 2$

$\text{mex}\{\phi\} = 0$

$$\boxed{\text{mex}\{S\} = 0 \quad \leftrightarrow \quad 0 \text{ is not in } S}$$

$\text{mex}\{0, *, *, * \dots\} > 0$

Γ_1



$$G(V) = 0 \quad \longleftrightarrow \quad P\text{-position}$$

$$\Updownarrow$$

$$\text{mex}\{G(w) \mid V \mapsto w\} = 0$$

$$\Updownarrow$$

$$0 \neq G(w) \quad \mid v \mapsto w$$

$$G(v) > 0 \quad \longleftrightarrow \quad \text{N-position}$$

$$\updownarrow$$

$$\operatorname{mex}\{G(w) \mid v \mapsto w\} > 0$$

$$\updownarrow$$

at least on child $v \mapsto w$
has $G(w) = 0$

(Sprague - Grundy)

<u>THEOREM</u>

$\Gamma_1, \Gamma_2, \dots, \Gamma_k$, $\Gamma := \Gamma_1 \oplus \cdots \oplus \Gamma_k$

$$G_\Gamma = G_{\Gamma_1} \oplus \cdots \oplus G_{\Gamma_k}$$

E.g. Γ_1 substraction $S = \{1, 2\}$
Γ_2 " $S = \{1, 3\}$

$G_{\Gamma_1}(n) = \{ 0, 1, 2, 0, 1, 2, \dots$

$G_{\Gamma_2}(n) = 0, 1, 0, 1, 0, 1, \dots$

5	1	0						
4	0	1						
3	1	0	3	1				
2	0	1	2	0	1			
1	1	0	3	1	0	3	1	
0	0	1	2	0	1	2	0	1
	0	1	2	3	4	5	6	7

⑧
$$\begin{array}{r} 10 \\ 01 \\ \hline 11 \end{array}$$

$\Gamma_1 \quad \Gamma_2$

$(5, 6)$

$G_{\Gamma_2}(6) = 0$

$G_{\Gamma_1}(5) = 2$

$G_{\Gamma}(5, 6) = 2 \oplus 0 = 2$

→ N-position

winning move $2 \longrightarrow 0$

$5 \longmapsto 3$

Feb 20, 2007

15-puzzle

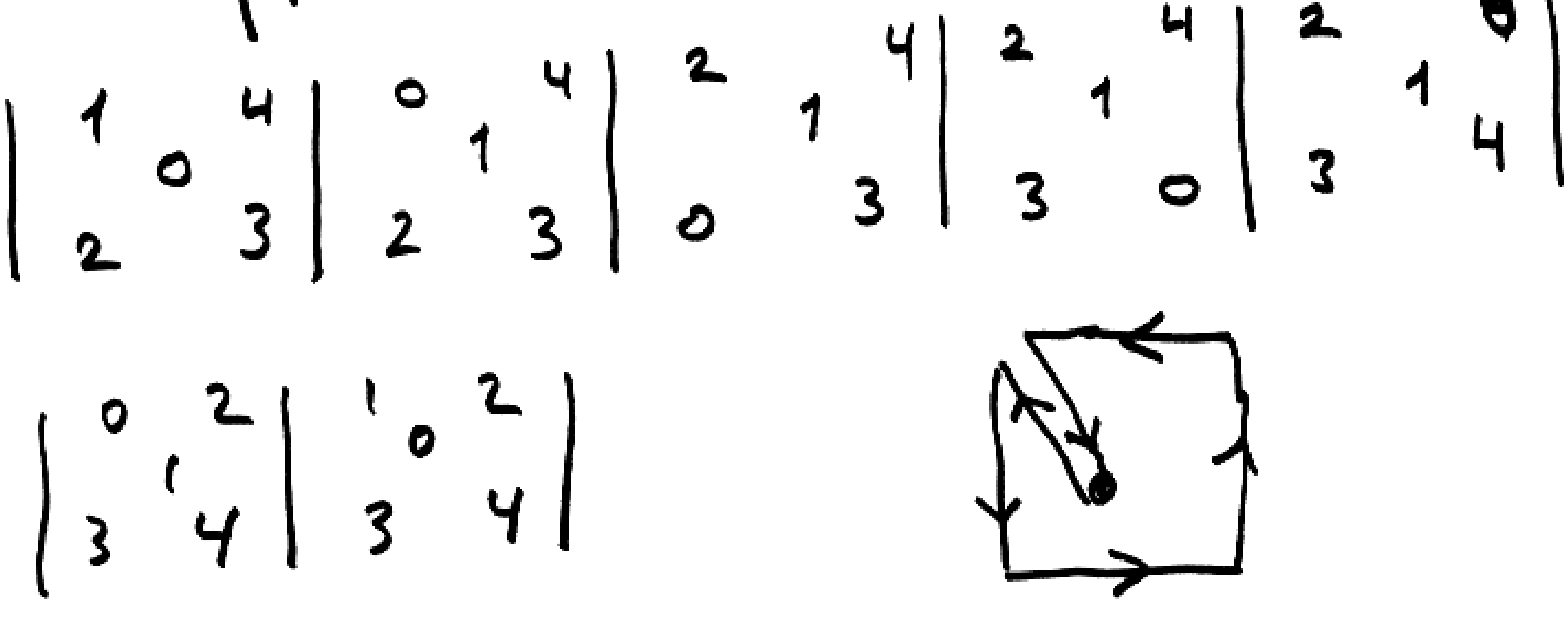
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

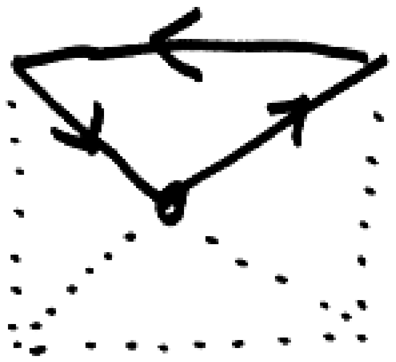
Goal: scrambled position to this

Move: Exchange a number w/blank (if neighbors)

Moves permute the numbers.

$$\sigma := \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$



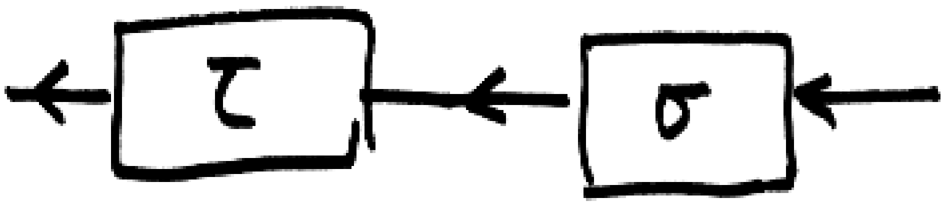


$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

transposition (swap two numbers)

Permutations can be "multiplied"

$$\underset{\text{2nd}}{\tau} \cdot \underset{\text{1st}}{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$



$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

<u>Note</u> $\sigma \cdot \tau \neq \tau \cdot \sigma$

Not commutative

Every permutation has inverse

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$\sigma^{-1} \cdot \sigma = 1$ ← permutation don't do anything

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\sigma \cdot \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \underline{1}$$

$$\sigma \cdot 1 = \sigma$$

$$1 \cdot \sigma = \sigma$$

<u>All</u> permutations of n things $= S_n$

$$|S_n| = n!$$

15-puzzle : permutations of 15 #'s w/ blank in its final position. ?

$S = \{4, 10, 12\}$

0	1	2	3	4	5	6	7	8	9
0	0	0	0	1	1	1	1	0	0

10	11	12	13	14	15	16	17	18	19
2									

Feb 22, 2007

Permutations

- 1, identity, do-nothing
- $\sigma \rightsquigarrow$ inverse σ^{-1}

  $\sigma \cdot \sigma^{-1} = \sigma^{-1} \cdot \sigma = 1$
- $(\sigma \cdot \tau) \cdot \rho = \sigma \cdot (\tau \cdot \rho)$

  associative

  $= \sigma \cdot \tau \cdot \rho$

A group G (of permutations of n things) is a set of permutations

- $1 \in G$
- $\sigma \in G$, $\sigma^{-1} \in G$
- $\sigma, \tau \in G$, $\sigma \cdot \tau \in G$

For example

1) $G = S_n$ (all permutations)

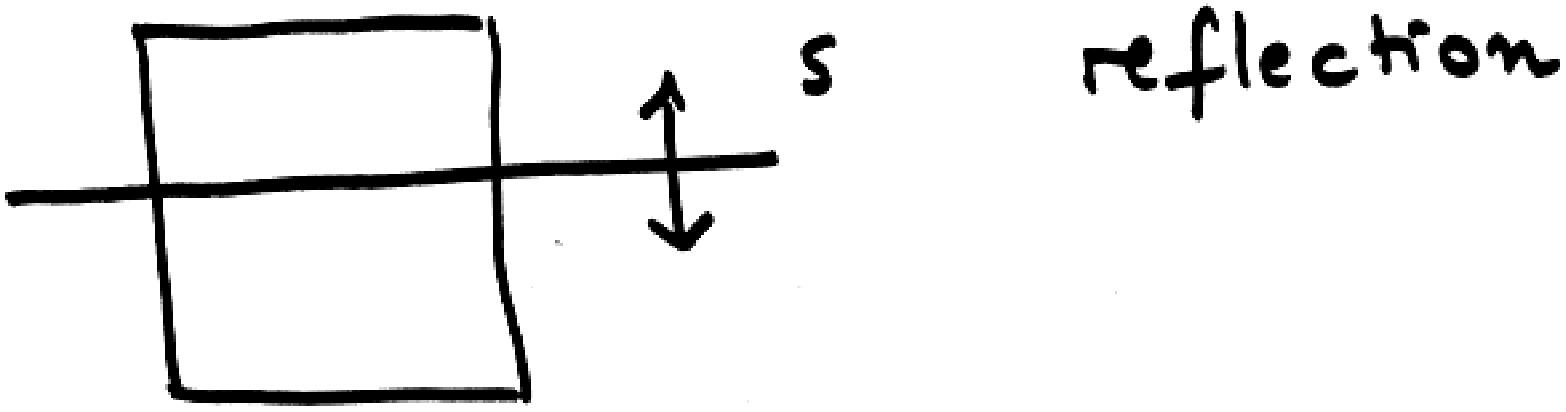
2) $G = \{1\}$

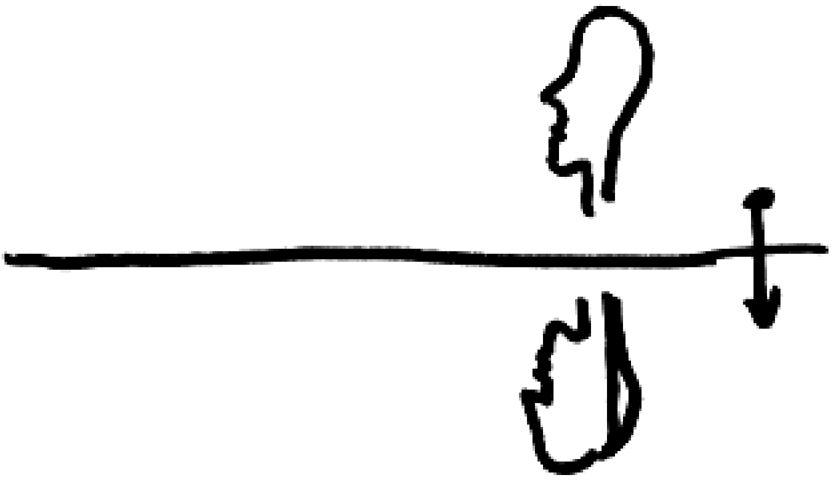
trivial group

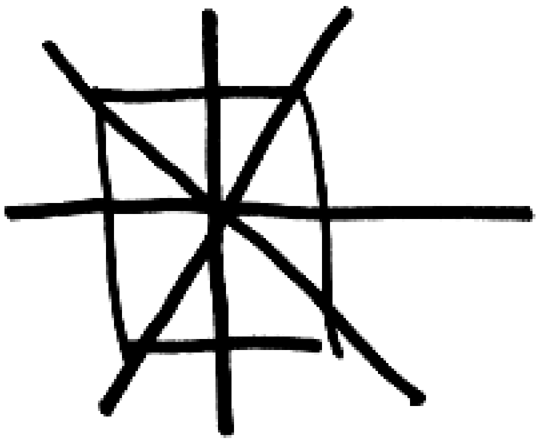
$$1^{-1} = 1$$

$$1 \cdot 1 = 1$$

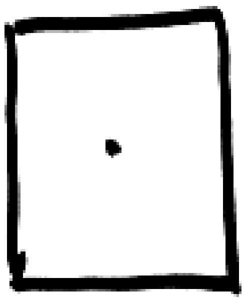
Symmetries of the square



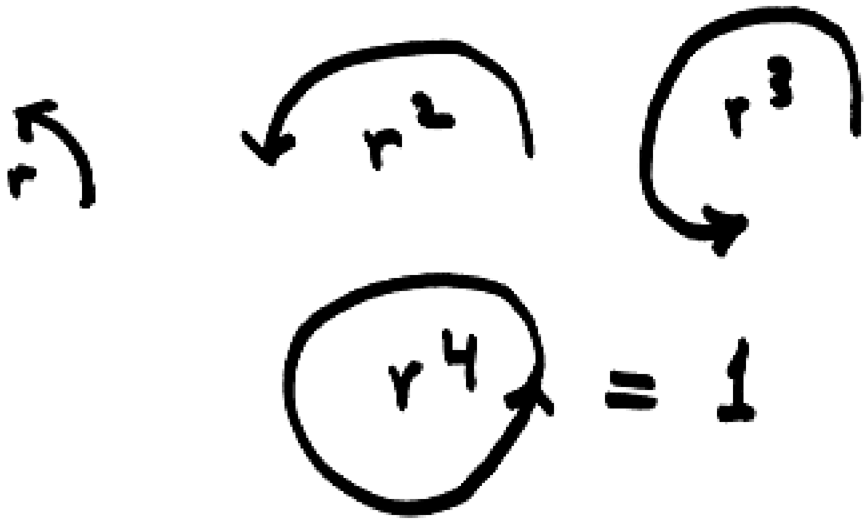




4 reflections



4 rotations



$r^2 = r \cdot r$

$r^3 = r \cdot r \cdot r$

$r^4 = r \cdot r \cdot r \cdot r$

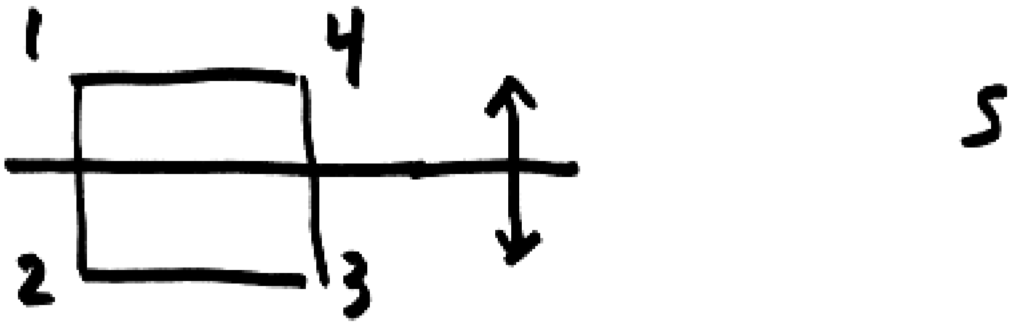
These are all the symmetries of the square.

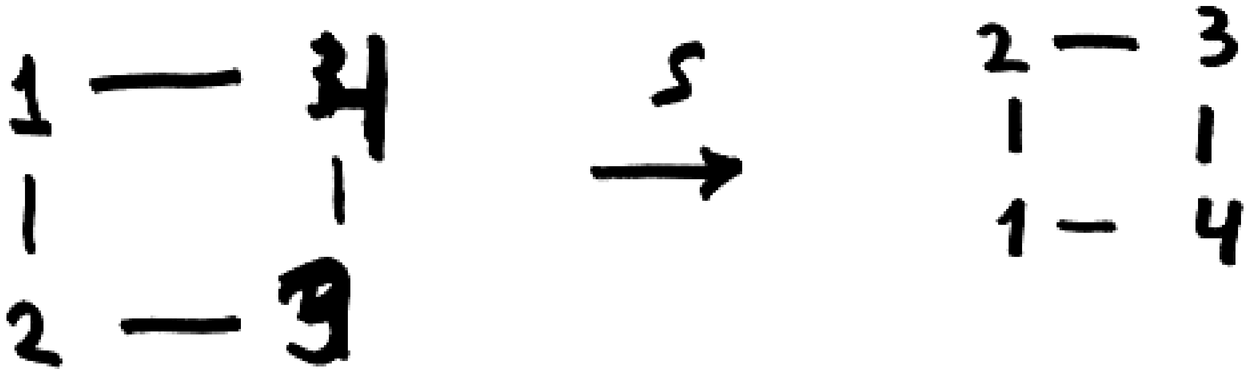
Total of 8 symmetries.

These symmetries form a group.

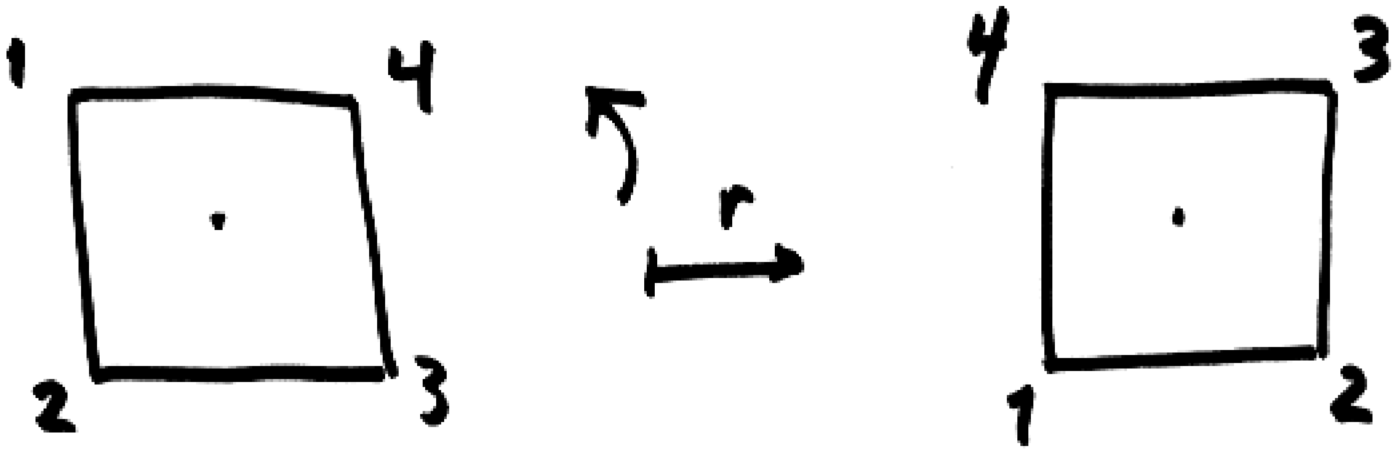
D_4 (dihedral group)

Each symmetry gives a permutations of the vertices





$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$



$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

The eight symmetries of the square result in eight permutations of the vertices.

Total number of possible permutations is 24.

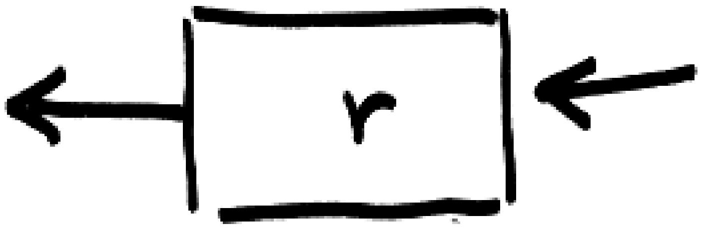
I.e. NOT every permutation is obtained as a symmety of the square.

The 8 permutation form a group.

* * *

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

cycle notation

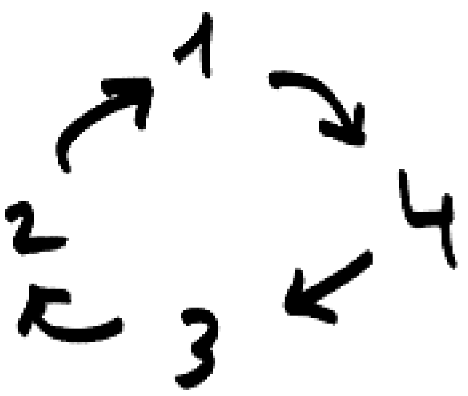


chase the numbers:

$$1 \longmapsto 4 \longmapsto 3 \longmapsto 2 \longmapsto 1 \longmapsto 4 \longmapsto 3 \ldots$$

Write r as a bunch of cycles

$$r = (1\ 4\ 3\ 2)$$



$$r = (2\ 1\ 4\ 3)$$

r is a 4-cycle.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$

In cycle notation:

$$\sigma = \underset{\substack{\uparrow \\ \text{3-cycle}}}{(1\ 5\ 2)}\ \underset{\substack{\uparrow \\ \text{2-cycle}}}{(3\ 4)}$$

Note that the cycles are disjoint

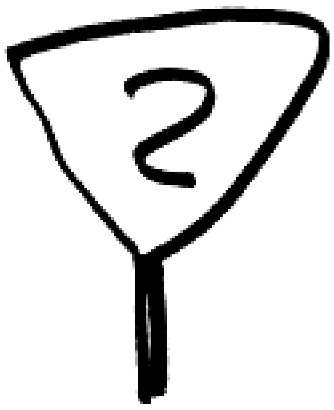
$$\sigma = (3\ 4)\ (1\ 5\ 2)$$
$$= (4\ 3)\ (5\ 2\ 1)$$

$$\sigma = (3\ 4) \cdot (1\ 5\ 2) = (1\ 5\ 2) \cdot (3\ 4)$$

cycles commute
disjoint

disjoint
a , b are cycles

$$\boxed{\underline{a \cdot b = b \cdot a}}$$



NON disjoint cycles
may not commute.

$a = (1\ 2)$
$b = (2\ 3)$

$$a \cdot b = (1\ 2) \cdot (2\ 3) = (1\ 2\ 3)$$
$$b \cdot a = (2\ 3)(1\ 2) = (1\ 3\ 2)$$

$$(1\ 2\ 3) \neq (1\ 3\ 2)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

cycle notation

$(4\ 2)\ (3\ 1)\ \ (5)$

Typically not write 1-cycles

$\sigma = (4\,2)\,(3\,1)$

& identity in S_5: $(1)\,(2)\,(3)\,(4)\,(5)$

$= (\ )$

Feb 27, 2007

Permutations σ, τ

$$\underset{\overset{\uparrow}{2^{\text{nd}}}}{\sigma} \cdot \underset{\overset{\uparrow}{1^{\text{st}}}}{\tau}$$

Eventually drop $\cdot$ altogethor

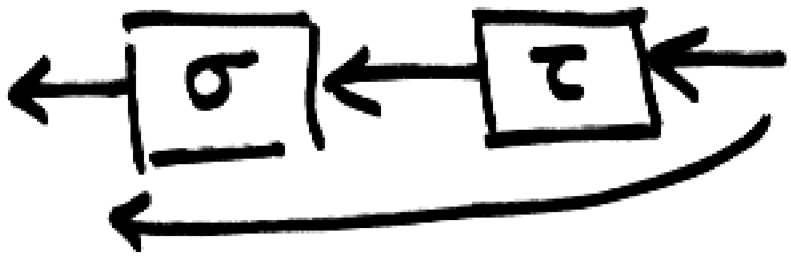
$$\sigma\tau$$

$$\sigma = (1\,2\,3)(4\,5)$$

$$\tau = (1\,4\,3)(2\,5\,6)$$

$\sigma, \tau \in S_6$

$$\sigma \cdot \tau$$



$$\sigma\tau = (3\,2\,4\,1\,5\,6)$$

$$\begin{array}{ccccc}
2 & \xleftarrow{\sigma} & 1 & \xleftarrow{\tau} & 3 \\
4 & \xleftarrow{\sigma} & 5 & \xleftarrow{\tau} & 2 \\
1 & \xleftarrow{\sigma} & 3 & \xleftarrow{\tau} & 4 \\
5 & \xleftarrow{\sigma} & 4 & \xleftarrow{\tau} & 1 \\
6 & \xleftarrow{\sigma} & 6 & \xleftarrow{\tau} & 5 \\
3 & \xleftarrow{\sigma} & 2 & \xleftarrow{\tau} & 6
\end{array}$$

__Defn__ σ, τ __commute__ if

$$\sigma\tau = \tau\sigma$$

Typically does __not__ hold.

$\tau^2\sigma$

$$\leftarrow \boxed{\tau} \leftarrow \boxed{\tau} \leftarrow \boxed{\sigma} \leftarrow$$

$$(1\,2\,3) \neq (1\,3\,2)$$

$\sigma \in S_n$ the **order** of σ

is the smallest positive power

$$\sigma^k = \underbrace{\sigma \cdot \cdots \cdot \sigma}_{k \text{ times}}$$

which is the identity.

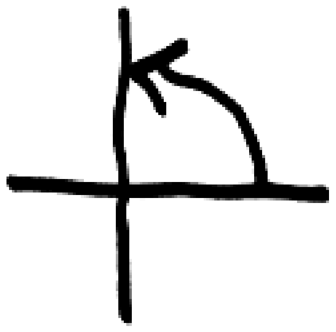
$\sigma = (4\ \ 6)$ $\qquad$ $\sigma^2 = \sigma \cdot \sigma = 1$

$\sigma^{-1} = \sigma$

order 2

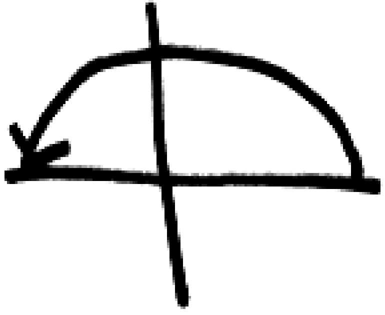
1	2	3	4	5	6
1	2	3	6	5	4
1	2	3	4	5	6

rotation

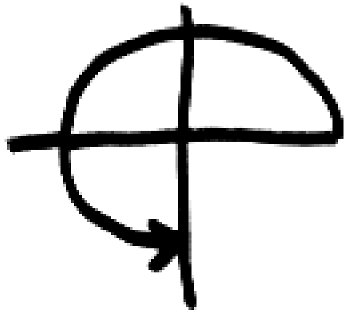


r

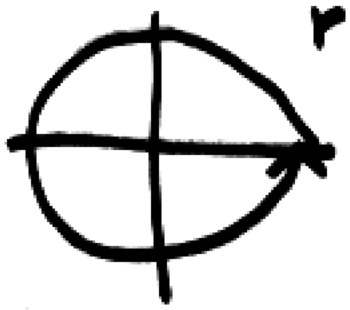
r^2



r^3



$r^4 = 1$



order = 4

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

$$= (1\,4)(2\,5\,3)$$

$$\rho = (1\,4)\,(2\,5\,3)$$

$$\rho^2 = (1)(4)\,(2\,3\,5)$$

$$\rho^3 = (1\,4)\,(2)(3)(5) \qquad = \rho \cdot \rho^2$$

$$\rho^4 = (1)(4)\,(2\,5\,3) \qquad = \rho \cdot \rho^3$$

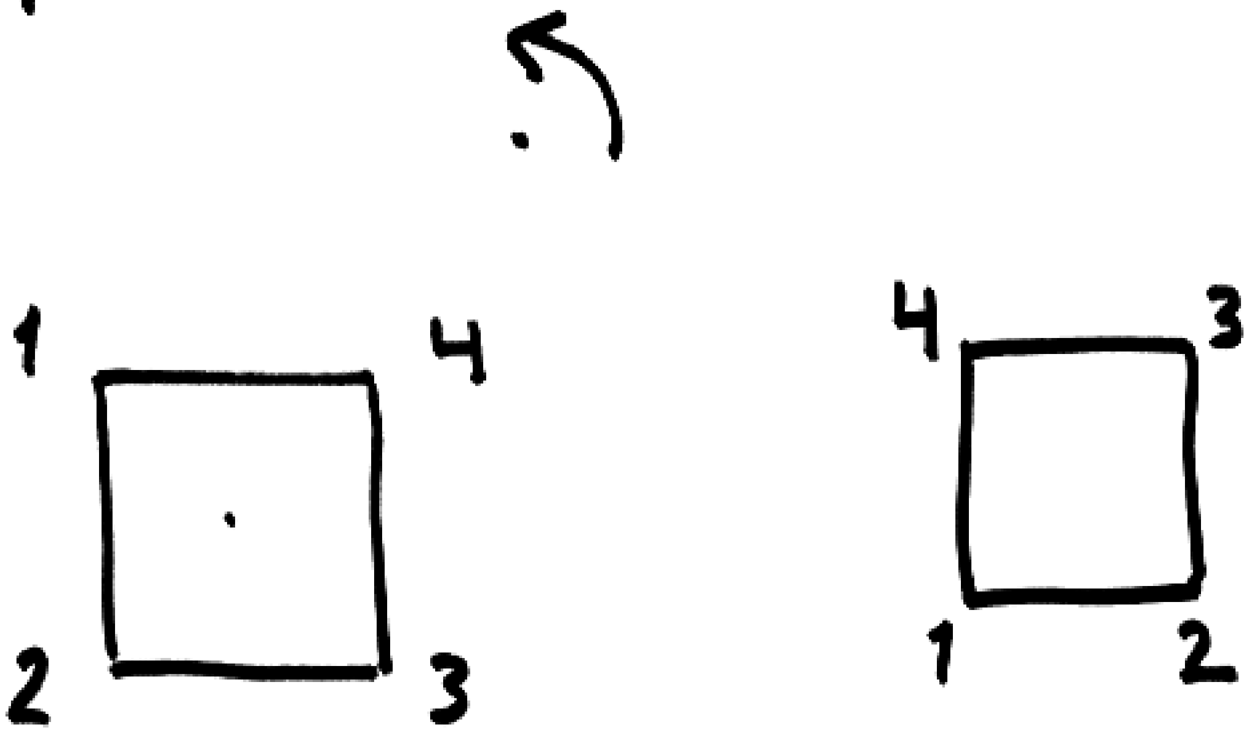
$$\rho^5 = (1\,4)\,(2\,3\,5) \qquad = \rho \cdot \rho^4$$

$$\rho^6 = (1)(4)\,(2)(3)(5) \qquad = \rho \cdot \rho^5$$

<u>order 6</u>.

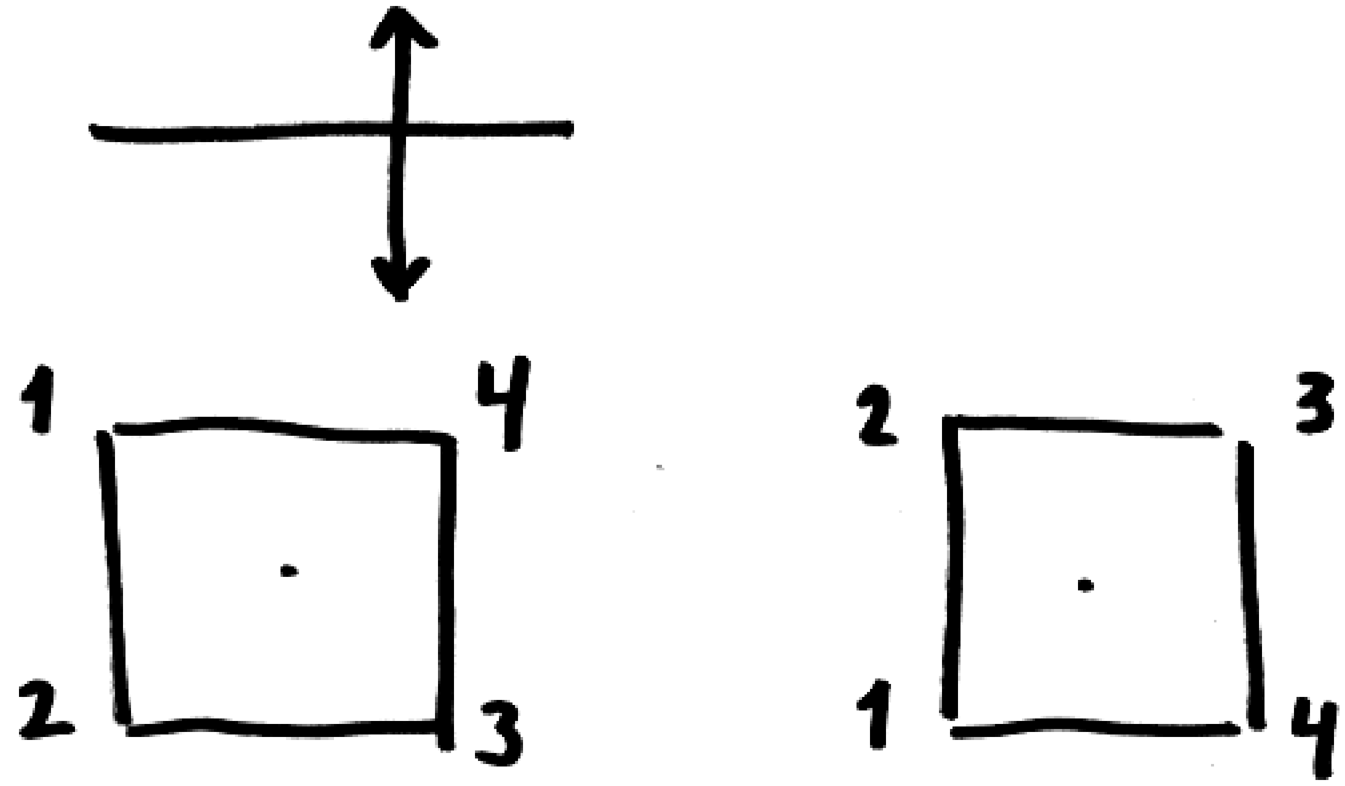
	1	2	3	4	5
ρ	4	5	2	1	3
ρ^2	1	3	5	4	2
ρ^3	4	2	3	1	5
ρ^4	1	5	2	4	3
ρ^5	4	3	5	1	2
ρ^6	1	2	3	4	5

r



$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

s

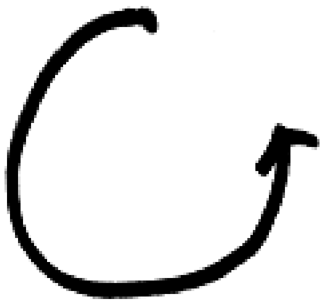


$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\boxed{s r s = r^{-1}}$$

$r^{-1} =$

$= r^3$



$$r^4 = 1$$

$$r \cdot r^3 = 1$$

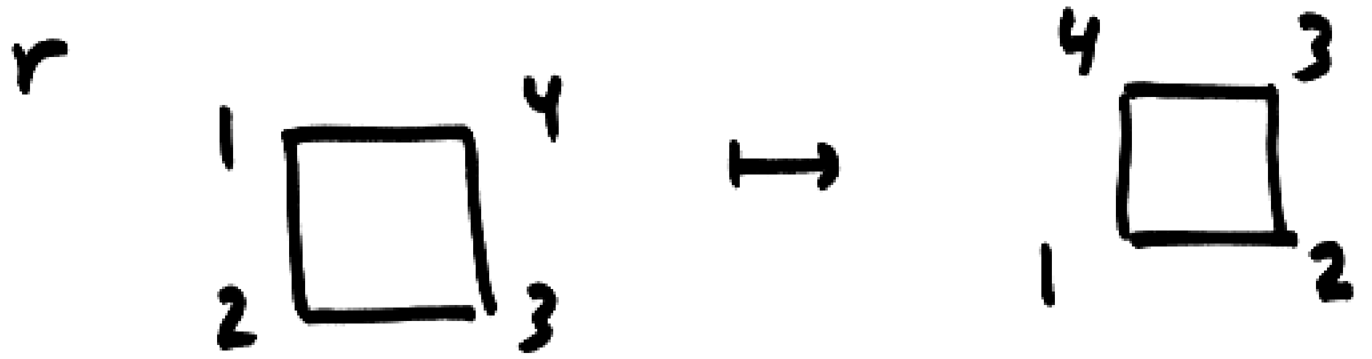
$$r^{-1}(r \cdot r^3) = r^{-1} \cdot 1 = r^{-1}$$

$$(r^{-1} \cdot r)\, r^3 = r^{-1}$$

$$1 \cdot r^3 = r^{-1}$$

$$r^3 = r^{-1}$$

March 1, 2007



$r = (1\,2\,3\,4)$

$r^{-1} = (1\,4\,3\,2)$

$\quad = (4\,3\,2\,1)$

$$\sigma = \underset{\substack{\uparrow\\ a}}{(1\,2)}\,\underset{\substack{\uparrow\\ b}}{(3\,4)}$$

$\sigma = ab$

<u>disjoint</u> cycles $\qquad ab = ba$

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$

In general $\underset{\text{2nd}\nearrow}{a} \cdot \underset{\nwarrow\text{1st}}{b}$

$$(a \cdot b \cdot c)^{-1} = c^{-1} \cdot b^{-1} \cdot a^{-1}$$

$$\sigma = (1\,2)(3\,4)$$

$$\sigma^{-1} = (3\,4)^{-1}(1\,2)^{-1}$$

$$= (3\,4)\,(1\,2)$$

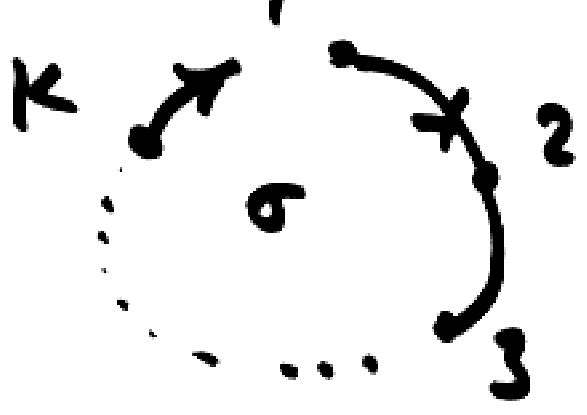
$$= (1\,2)(3\,4)$$

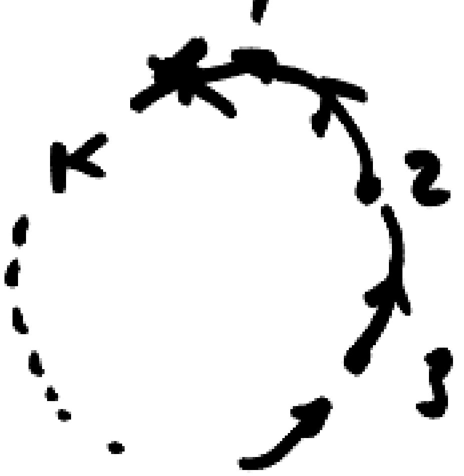
$$\sigma = (1\,2\,3)(4\,5)(6\,8\,7)$$

$$\sigma^{-1} = (6\,8\,7)^{-1}(4\,5)^{-1}(1\,2\,3)^{-1}$$

$$= (7\,8\,6)\,(5\,4)\,(3\,2\,1)$$

$$(1\,2\,3 \cdots k)^{-1} = (k \cdots 3\,2\,1)$$





$$(1\,2\,3\,4\,5)^{-1} = (5\;4\,3\,2\,1) = (3\,2\,1\,5\,4)$$

Permutation Puzzle

15-puzzle, Rubik's cube, ...

Basic Moves: $U_1, U_2, \ldots, U_N$

(include the inverses)

What are all the possible sequences of moves?

Look at all possible products of these permutations. They form a group G.

- How big is G?

- Differently: S_n permutations on n things, what is a set of basic permutations that will give all.

[illegible] → 12345

2 3 4 5 1

2 3 4 1 5

2 3 1 4 5

2 1 3 4 5

1 2 3 4 5

Bubbling algorithm for sorting

$(1\,2\,3\,4\,5) = (15)(14)(13)(12)$

<u>not</u> disjoint

Any cycle is a product of transpositions

$(5\,7\,8) = (58)(57)$

Any permutation is a product of cycles

→ every permutation is a product of 2-cycles (transpositions)

How many 2-cycles in S_4?

(12) (13) (14)

(23) (24)

(34)

6 transpositions

In S_n ? $\binom{n}{2}$

$(ij) = (ji)$

$|S_n| = n!$

$\sigma \in S_4$

$\sigma, \ \sigma \cdot \sigma, \ \sigma \cdot \sigma . \sigma, \ \sigma \cdot \sigma \cdot \sigma \cdot \sigma, \ldots$

$\sigma, \sigma^2, \sigma^3, \sigma^4, \ldots$

$\sigma = (1234)$

$1, \ (1234), \ (1234)^2 = (13)(24)$

$$(1234)^3 = (1432)$$

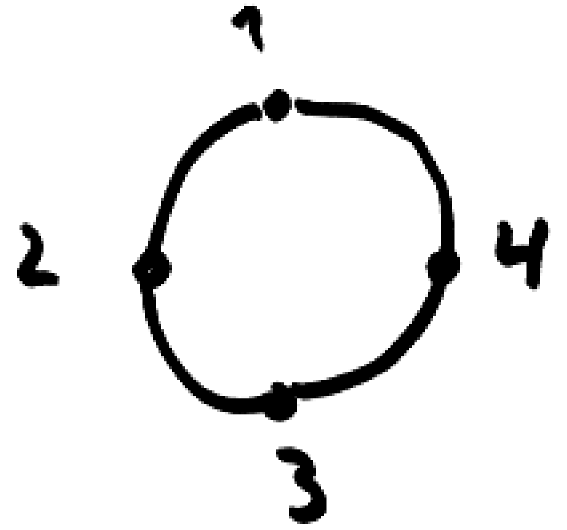
$$(1234)^4 = 1$$

$$(1234)^5 = (1234)\cdot(1234)^4$$

$$= (1234)$$

$1, \sigma, \sigma^2, \sigma^3.$

total of 4 possible moves.



No one $\sigma \in S_n$ is not sufficient.
(if $n > 2$)

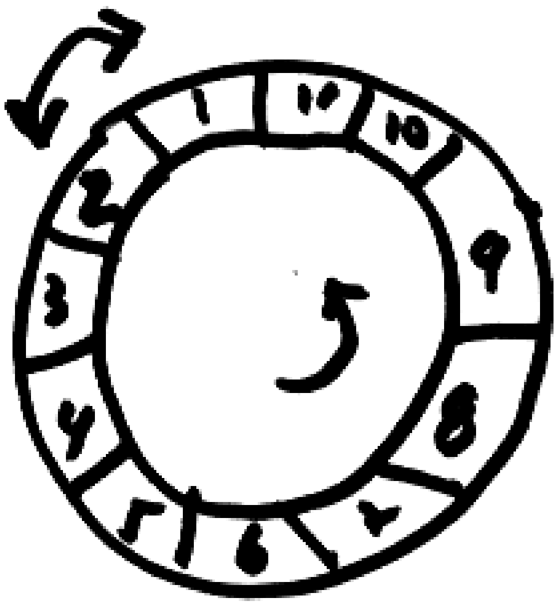
order of σ = least $\overset{>0}{k}$ such that

$$\sigma^k = \underbrace{\sigma \cdots\cdots \sigma}_{k} = 1.$$

Already two permutations σ, τ can generate very large groups.

E.g. σ, τ can generate all of S_n

S_4 ? $\sigma = (12)$, $\tau = (1234)$



- all swaps (transpositions) can be obtained from σ, τ

$$
\begin{aligned}
\tau \sigma \tau^{-1} &= (1234)(12)(1234)^{-1} \\
&= (1234)(12)(4321) \\
&= (1)(23)(4) \\
&= (23)
\end{aligned}
$$

$$\tau^2 \sigma \tau^{-2} = (34)$$

$$\tau^3 \sigma \tau^{-3} = (41)$$

We have: $(12), (23), (34), (41)$

$(1\,3)$?

$(2\,4)$?

$$(12)(23) = (1\,2\,3)$$

$$(23)(12) = (1\,3\,2)$$

$$(23)(12)(23)^{-1} = (23)(12)(23)$$

$$= (13)$$

$$(24) = (14)(12)(14)$$

All transpositions arise from σ and τ. $\rightsquigarrow$ All permutations arise from σ and τ.

For any n

$\sigma = (12) \quad \tau = (12 \cdots n)$

they generate all of S_n.

March 8, 2007

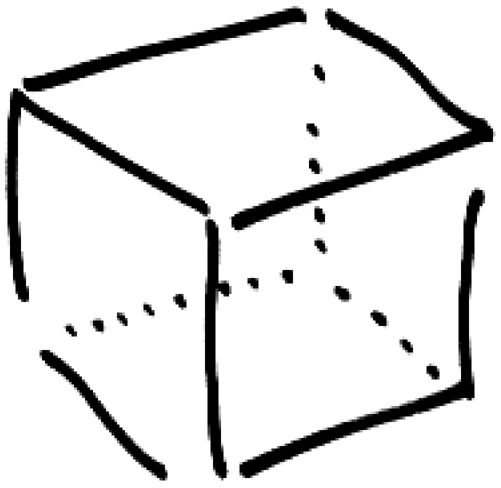
S_n = { permutations of n things }

$|S_n| = n!$

↑

# of elements.

n	1	2	3	4	5	6	
$n!$	1	2	6	24	120	720	. . .



group
Rotations of cube
has 24 elements

Is it S_4 in disguise?

4 diagonals

S_n

transposition $(i\,j)$

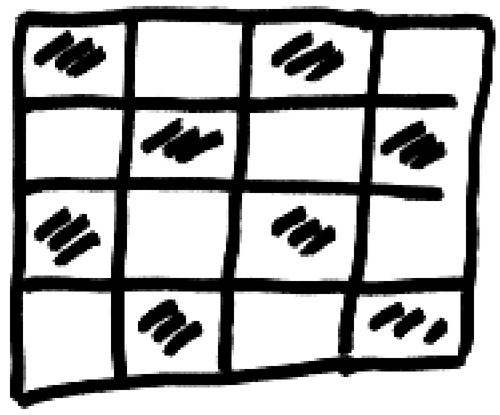
$i \longleftrightarrow j$

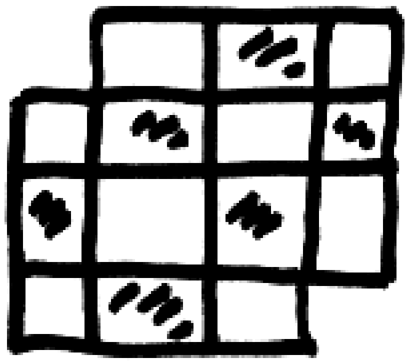
- All transposition generate S_n.

Any permutation can be obtained as a sequence of swaps (transpositions).

Even/odd permutations

Parity.





Dominos 

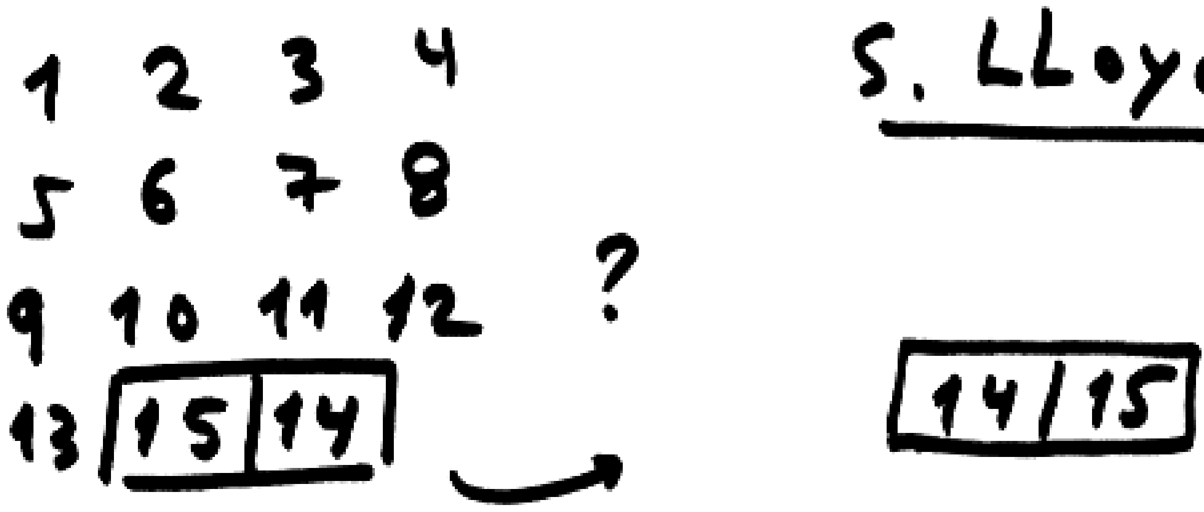
can you tile with dominos?

Each tile covers 1 □
1 ■

but there is a different number of these.

15-puzzle

S. Lloyd



(14 15)

can't be done.

All permutations that can be achieved in this puzzle are **even**. But (14 15) is **odd**.

A **transposition** is odd.

product of odd → even permutation

odd · odd = even
even · even = even
odd · even = odd

⊕	0	1
0	0	1
1	1	0

·	+1	−1
+1	+1	−1
−1	−1	+1

$$(1\,2\,3) = \underset{-1\,\cdot\,-1}{(13)(12)} \;\to \text{even} = +1$$

$$(1\,2 \cdots k) = (1k)\cdots(14)(13)(12)$$

$$(1\,2\,3\,4) = \underset{-1\,\cdot\,-1\,\cdot\,-1}{(14)(13)(12)} \;\to \text{odd.} = -1$$

Any permutation is a ~~production~~ product of transpositions

$$\sigma = \tau_1 \cdot \tau_2 \cdot \cdots \cdot \tau_N$$

$$\mathrm{sgn}(\sigma) = \underbrace{(-1)\cdot(-1)\cdot\cdots(-1)}_{N}$$

$$\boxed{\mathrm{sgn}(\sigma) := (-1)^N}$$

$$(1\,2\,3) = (21)(23)$$

$$(1\,2\,3) = (21)(23)(12)(12)$$

Potentially the trouble is that writing $\sigma = \tau_1 \cdots \tau_N$ is not unique and hence $(-1)^N$ may depend on how we do it.

As it happens $(-1)^N$ is always the same

$$(1\,2\,3) = \tau_1 \cdots \tau_N$$

necessarily N is even.

$\rightsquigarrow$ Even/odd permutations

$$\sigma = \underset{+1}{(1\,2\,3)} \cdot \underset{-1}{(4\,5)} \cdot \underset{-1}{(6\,7\,8\,9)}$$

$$= +1$$

$$= (1\,3)(1\,2)\,(4\,5)\,(6\,9)(6\,8)(6\,7)$$

($$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1\cdot5\cdot9 - 2\cdot6\cdot7 \quad + \cdots$$)

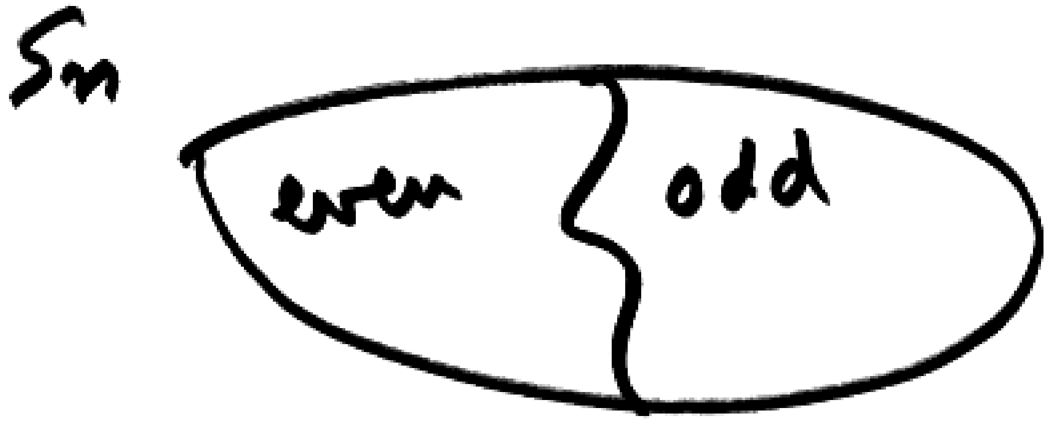
even · even = even

All even permutations forms a subgroup of S_n.

$\text{even}^{-1} = \text{even}$.

A_n = alternating group.

$|A_n| = \frac{1}{2} n!$ $(n>1)$



~~[illegible]~~

$$\overset{\sigma}{\text{even}} \longmapsto \text{~~[illegible]~~}\ \underbrace{(12)\,\sigma}_{\text{odd}}$$

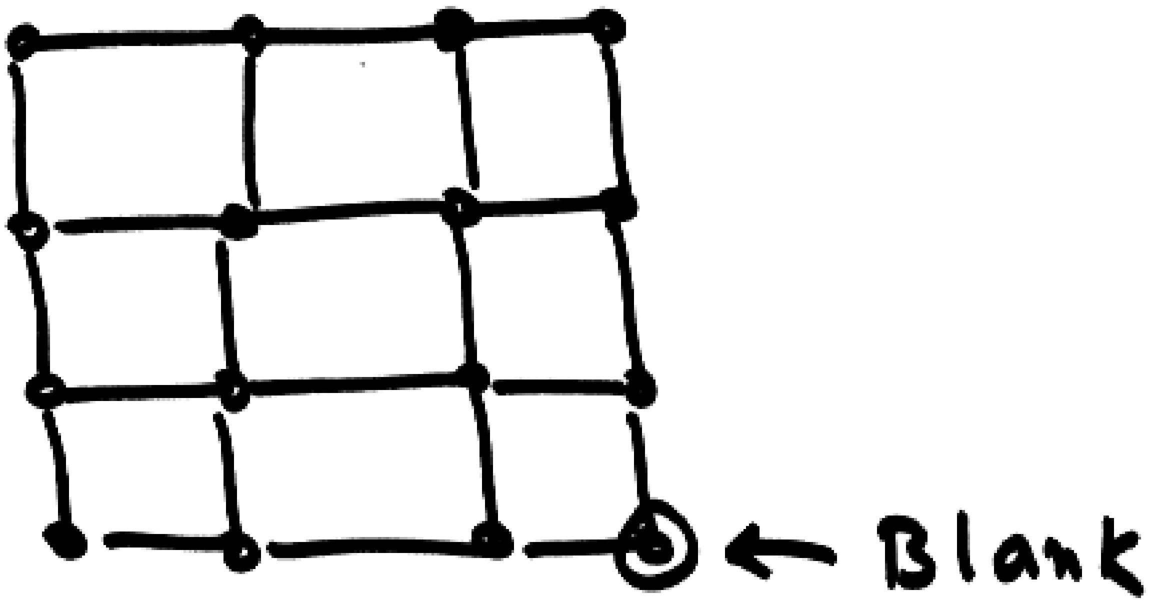
$$\underset{\text{even}}{(12)\,\eta} \longleftarrow\!\!\!\longmapsfrom \underset{\text{odd}}{\eta}$$

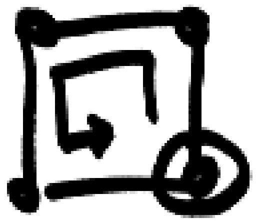
1-1 correspondence between even and odd permutations

$\#$ even $= \#$ odd

$\Rightarrow \#$ even $= \frac{1}{2}$ total.

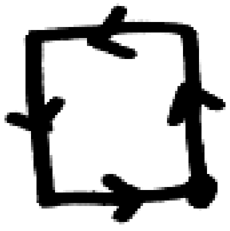
4. The permutations we can get from moves in the 15-puzzle are all even.



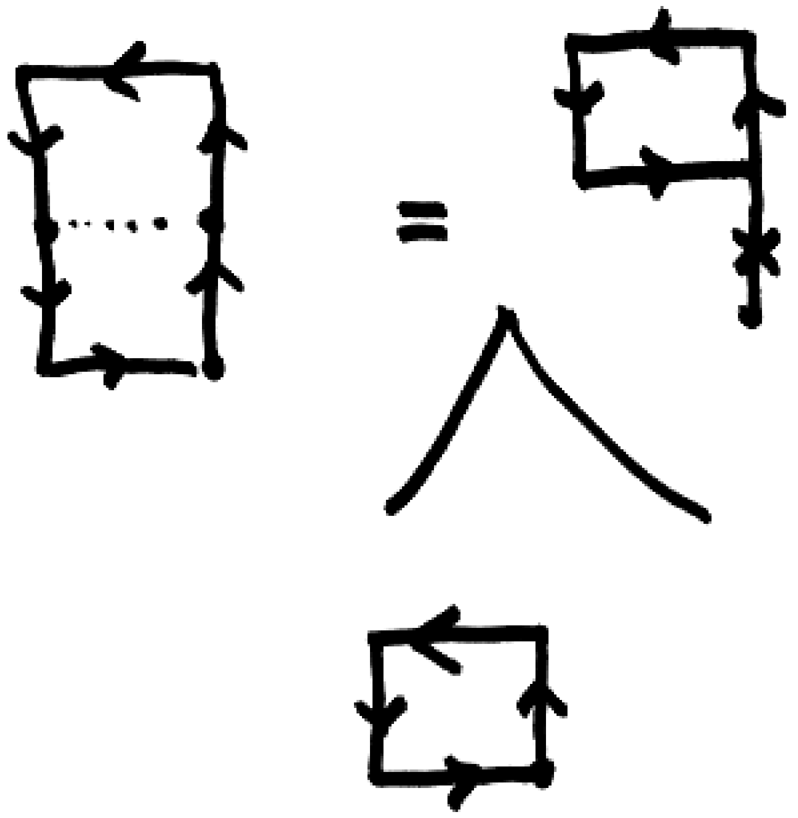


$$\begin{array}{cc|cc|cc|} 2 & 1 & 2 & 0 & 0 & 2 \\ 3 & 0 & 3 & 1 & 3 & 1 \end{array}$$

$$\begin{array}{cc|cc|} 3 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}$$

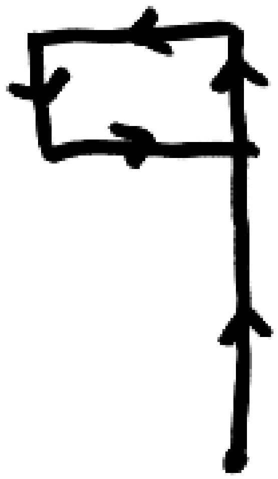


$\rightsquigarrow (1\ 3\ 2)$
even



Any path can be decomposed as a sequence of little squares moves.

Each little square is 3-cyle



All moves are even.

RATE
YOUR
MIND
PAL

RATE
YOUR
MIND
PLA

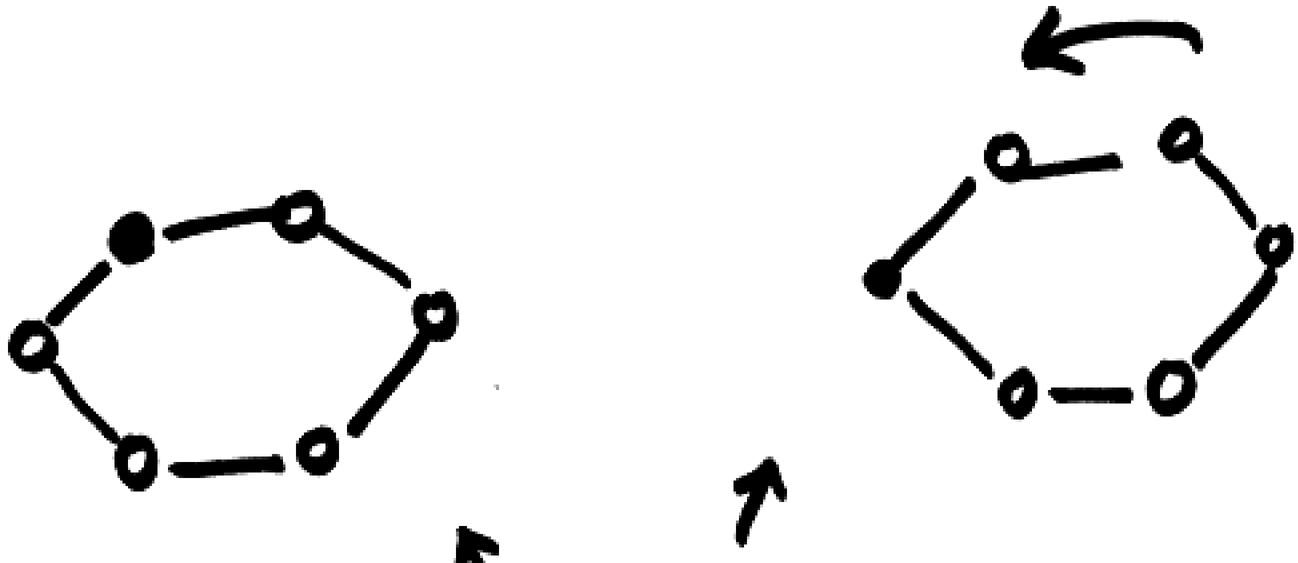
you can solve this by swapping
LA and AA

March 20, 2007

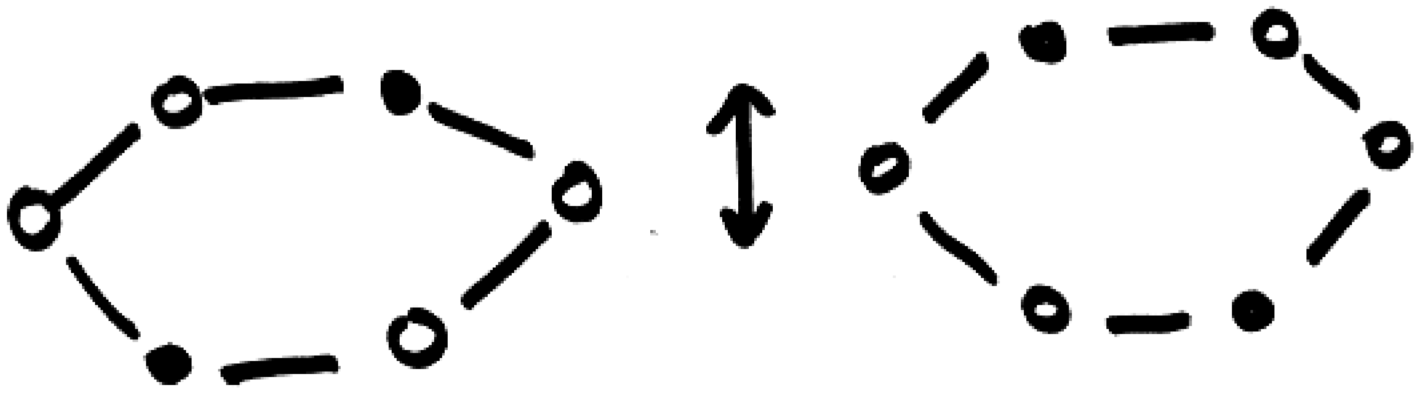
Count necklaces

$n = 6$ beads

$m = 2$ colors

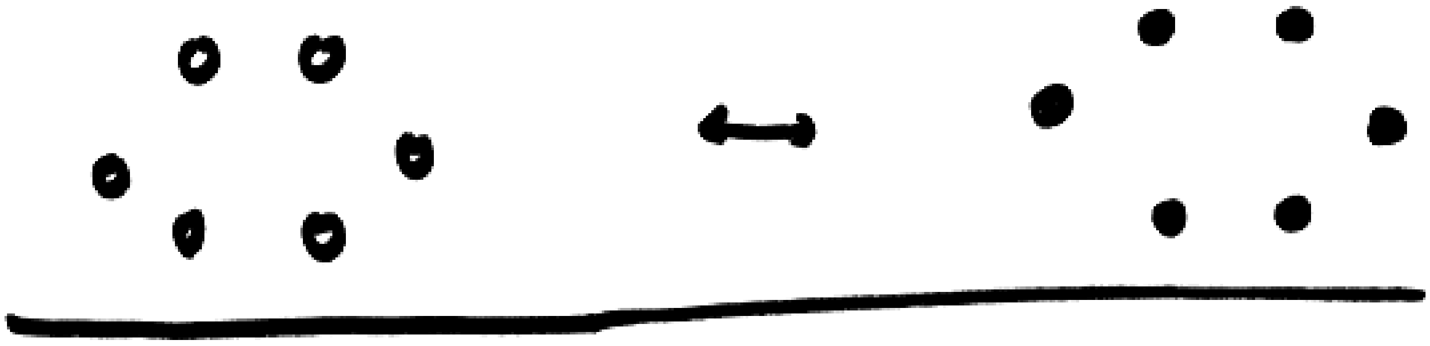


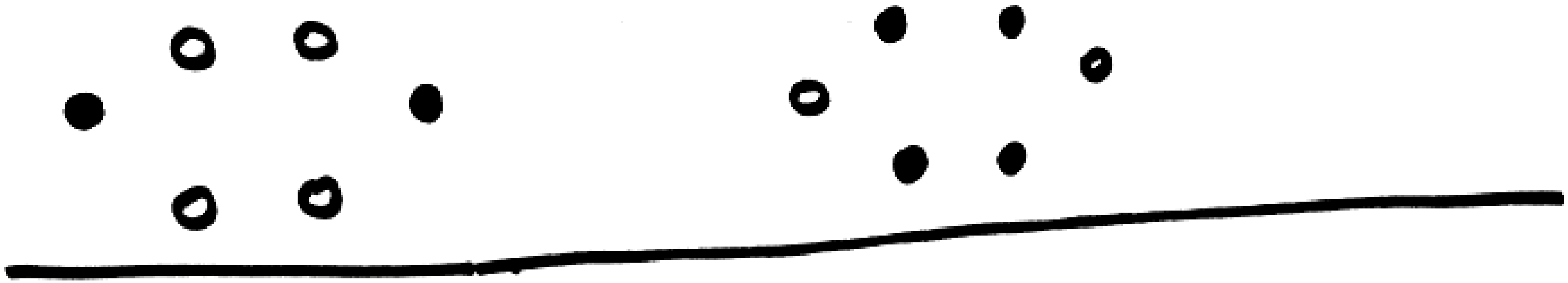
same necklace

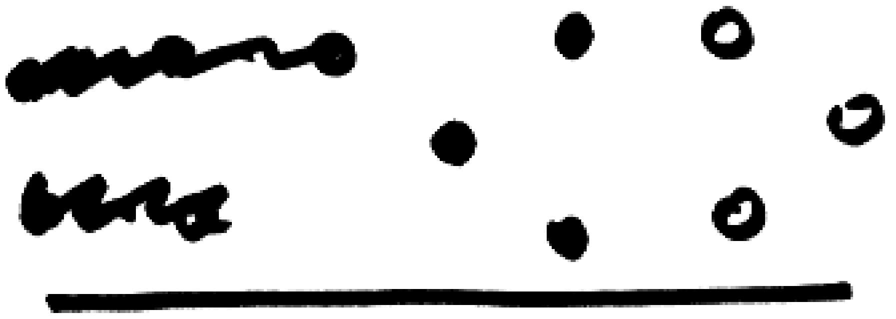


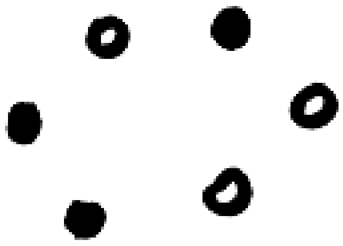
How many (different) necklaces can we get?

Polya's theory of counting.

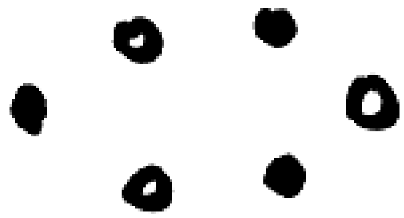








TOTAL = 13



We can give the number of necklaces with 6 beads and any # of colors with one formula

Group

G := Symmetries of the hexagon

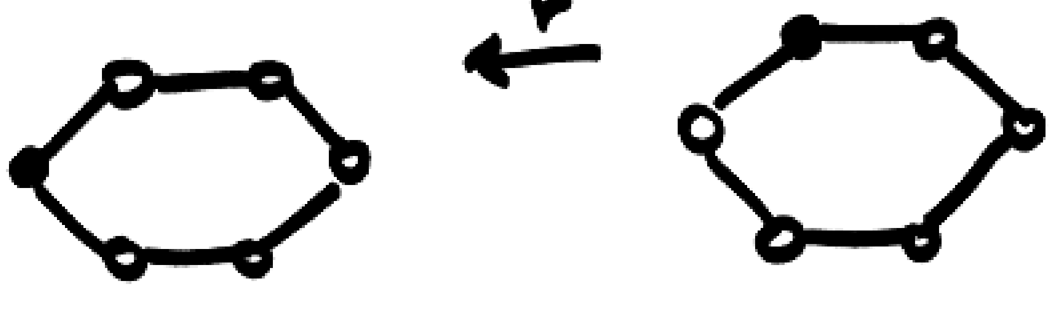
Total symmetries = 12

$1, r, r^2, r^3, r^4, r^5$

$s_0, s_1, s_2, s_3, s_4, s_5$

Dihedral group D_6 of order 12

Think of pictures of necklaces



Two pictures are the same necklace if we can take one to the other by some $g \in G$.

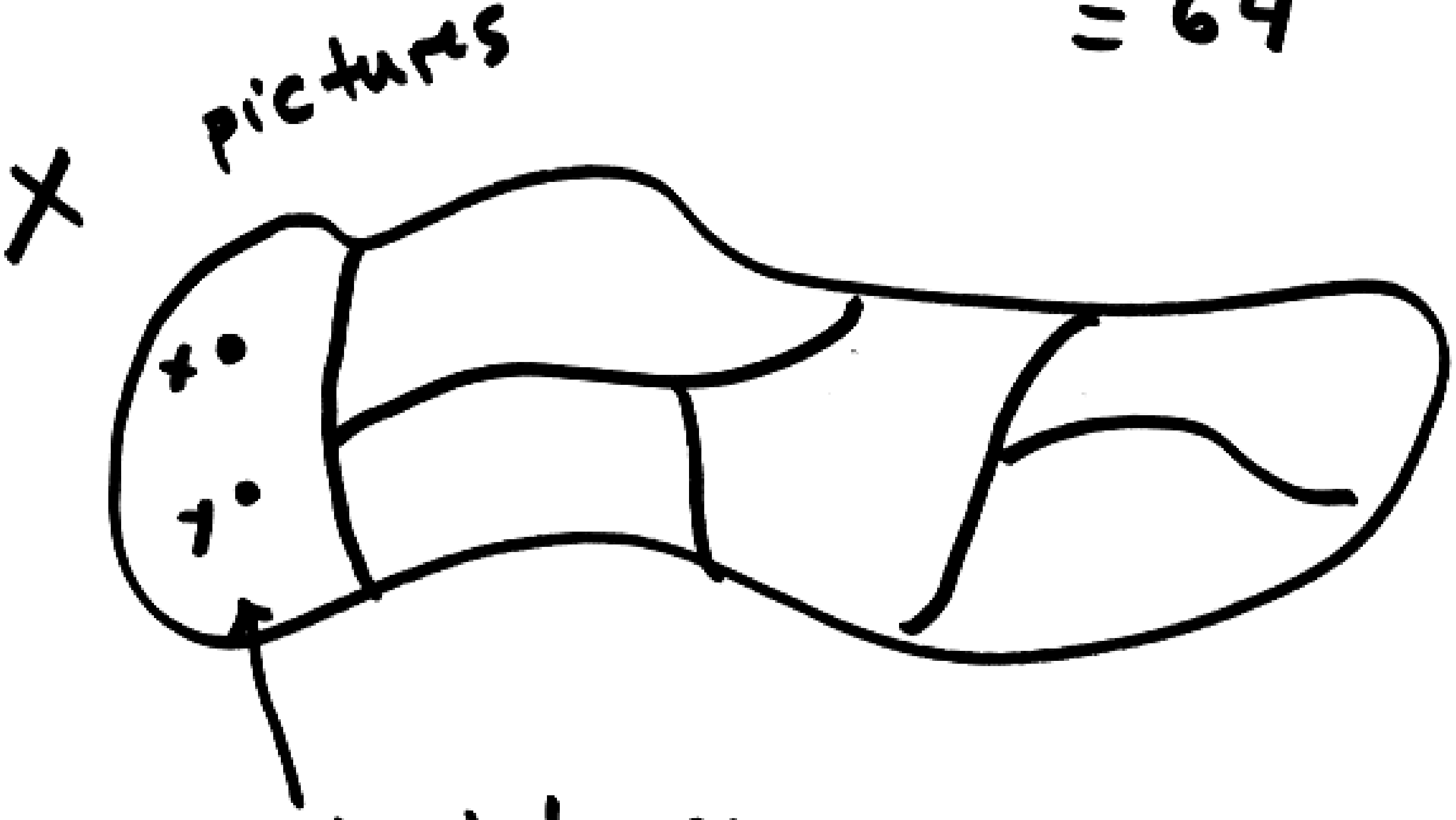
How many pictures?

Each spot in the hexagon can be one of two possibilities

The total number $= \underbrace{2 \times 2 \times \cdots \times 2}_{6}$

$= 2^6$

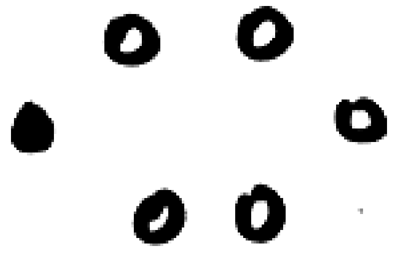
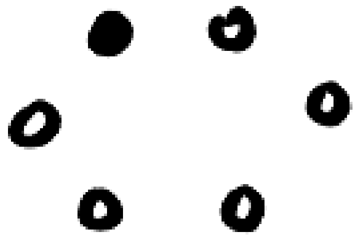
$= 64$



Say two pictures $x, y \in X$ are *equivalent* (i.e. represent same necklace) if

$$y = gx \quad , \qquad g \in G$$

E.g.

$x =$  $y =$ 

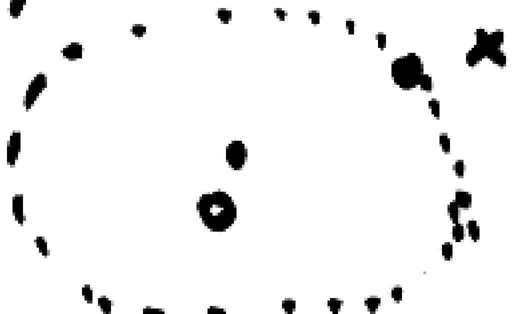
x equivalent to y because

$$y = r^{-1} x$$

The set of All pictures equivalent to x is called the *orbit* of x

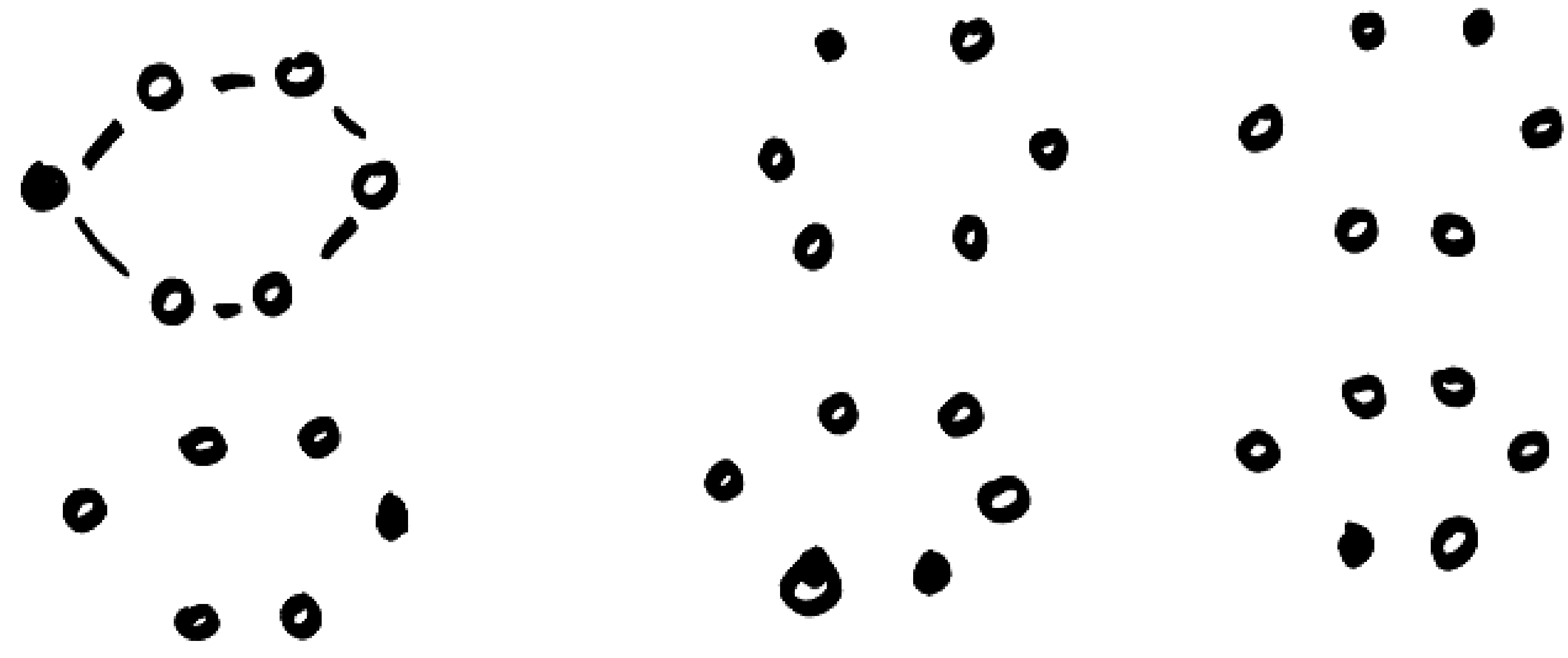
$$Gx = \{ y \mid \ y = gx \ \text{for some } g \}$$

Say G is all rotations in the plane



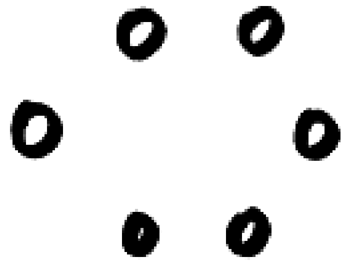
E.g.

1)



6 elements in this orbit.

2)

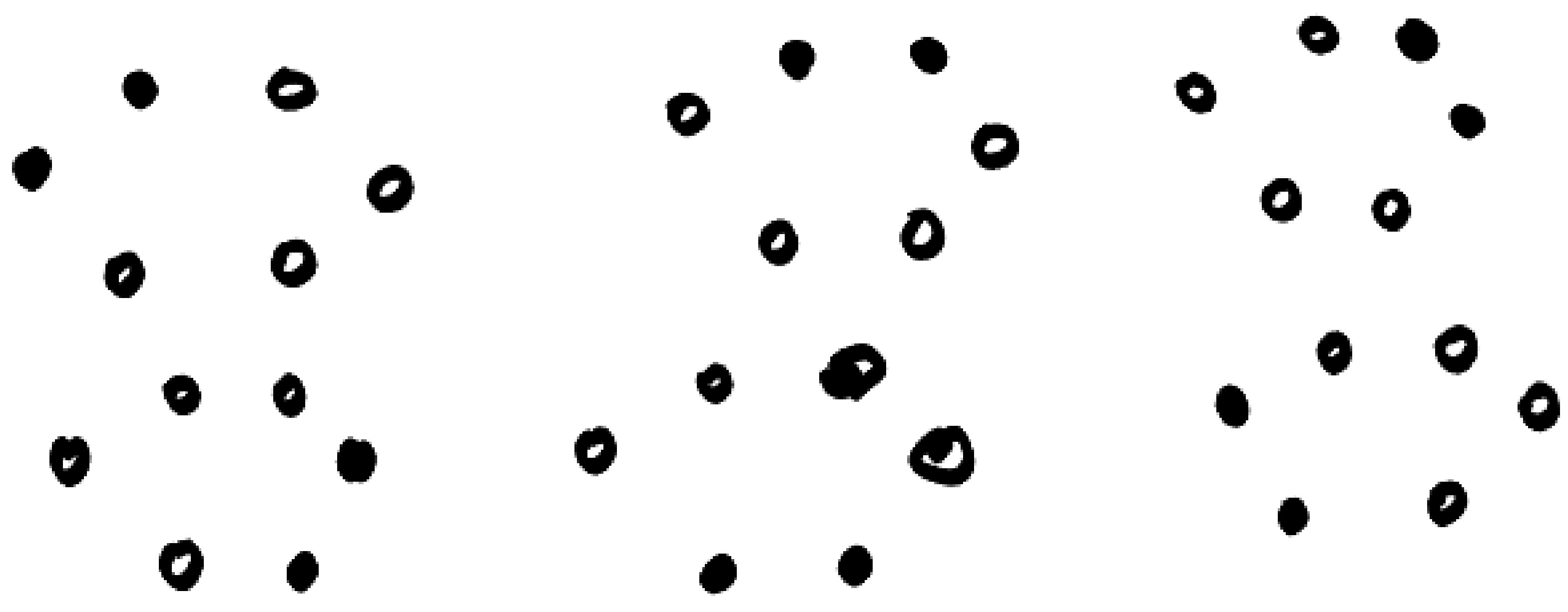


orbit has 1 element.

3)

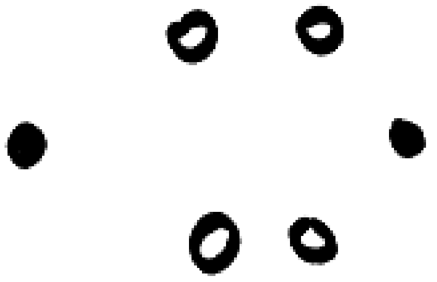
orbit has 2 elements.

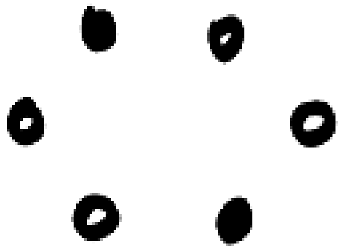
4)

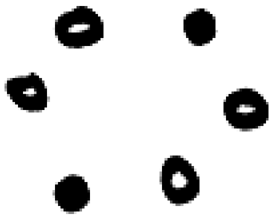


6 elements.

5)







3 elements

Fact size of an orbit divides the order of G

Z A factor of $|G|$ *may* not be the size of an actual orbit

$x \in X$ **Stabilizer** of x in G

$$G_x := \{ g \in G \mid gx = x \}$$

is a subgroup of G.

$$g_1, g_2 \in G_x$$

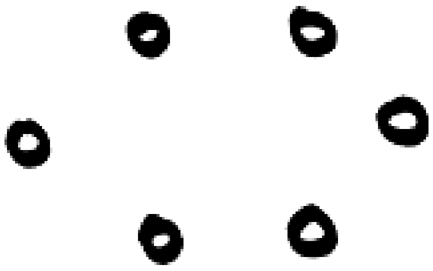
$$\Rightarrow \quad g_1 g_2 \in G_x$$

$$g_2 x = x$$

$$(g_1 g_2) x = g_1 (g_2 x) = g_1 x = x$$

Examples

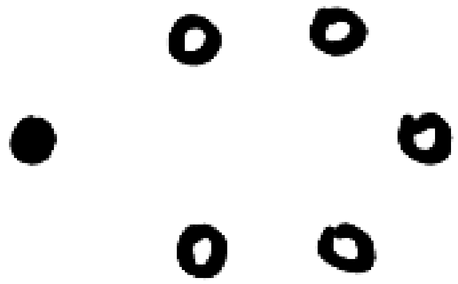
1)



$G_x = G$

$|G_x| = 12$

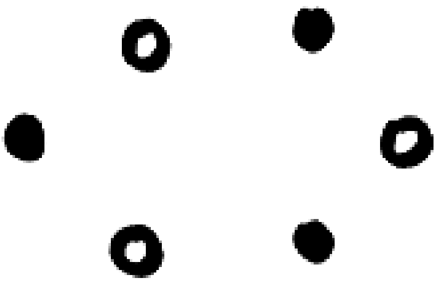
2)



$G_x = \{1, s_0\}$

$|G_x| = 2$

3)



$G_x = \{1, r^2, r^{-2}, s_0, s_2, s_4\}$

$r^6 = 1 \quad r^4 \cdot r^2 = 1$

$r^4 = r^{-2}$

$|G_x| = 6$

$$\boxed{\underset{\substack{\uparrow \\ \text{orbit}}}{\#Gx} \cdot \underset{\substack{\uparrow \\ \text{stabilizer}}}{|G_x|} = |G|}$$

March 22, 2007

Count Necklaces

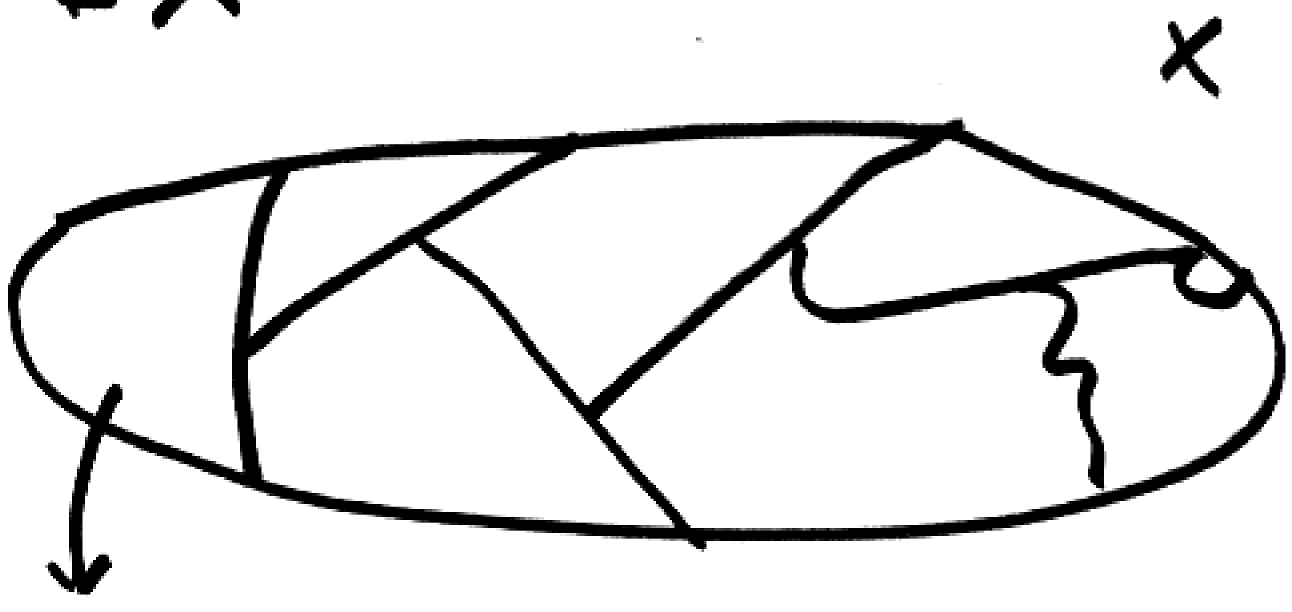
X = pictures

$n = 6$ # beads

$m = 2$ # colors

Two pictures of same necklace.

$\# X = 2^6 = 64$



all pictures of <u>same</u> necklace

G = group of symmetries of the regular hexagon

= D_6 (dihedral group of order 12)

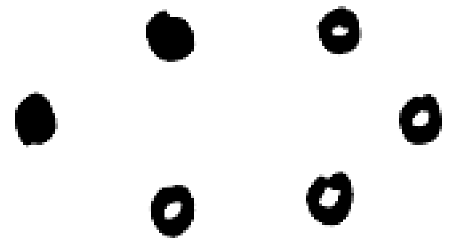
Group G acts on the set X

$x \in X$, $g \in G$

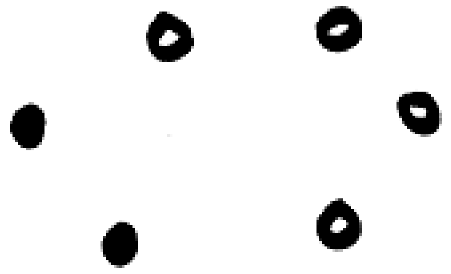
$g \cdot x \in X$

E.g. if $g = r$ ↰

$x =$



$g \cdot x =$



The orbit of $x \in X$ under G

$Gx = \{ y \mid gx , g \in G \}$

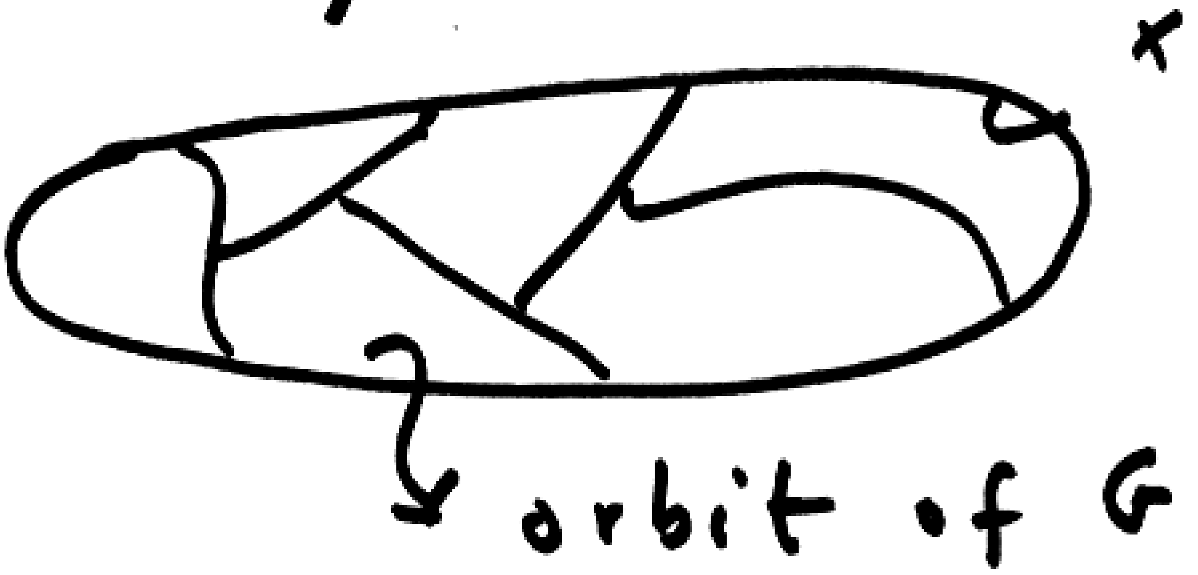
In terms of necklaces two pictures $x, y \in X$ are pictures of the same necklace

$$y = gx$$

For some $g \in G$.

I.e. y is in the orbit of x

or x and y are in the same orbit.



Alternatively: $x, y \in X$

$$x \sim y$$

if $y = gx$ for some $g \in G$.

Defines <u>equivalence relation</u> in X

- $x \sim x$

$$x = 1 \cdot x$$

- $x \sim y \Rightarrow y \sim x$

$y = gx$ for some $g \in X$

$x = h \cdot y \qquad h\,?$

$x = g^{-1} \cdot y$

- $x \sim y \,,\; y \sim z \Rightarrow x \sim z$

$y = gx \,,\quad z = hy$

$$\Rightarrow \quad z = h(gx) = (h \cdot g)\, x$$

Equivalence classes $\longleftrightarrow$ orbits

$$Gx = \{ y \mid y \sim x \}$$

Typically orbits have different sizes complicates counting them.

Stabilizer

$$\mathrm{Stab}_G(x) := \{ g \in G \mid gx = x \}$$

$$\mathrm{Stab}_G(x) \subseteq G$$

Is a **subgroup**

- $g_1, g_2 \in \mathrm{Stab}_G(x)$

$$g_1 x = x$$
$$g_2 x = x$$

$$(g_1 g_2) x = g_1 (g_2 x) = g_1 x = x$$

$$\Rightarrow g_1 g_2 \in \mathrm{Stab}_G(x)$$

- $g \in$ [illegible] $\mathrm{Stab}_G(x)$

$$gx = x$$

$$g^{-1}(gx) = g^{-1}x$$

$$x = (g^{-1} \cdot g) x = g^{-1} x \quad \Rightarrow g^{-1} \in \mathrm{Stab}_G(x)$$

$$\boxed{\# Gx \cdot |\mathrm{Stab}_G(x)| = |G|}$$

In particular, the size of an orbit always divides the order of the group.

2 If d divides $|G|$ there may not be an orbit of size d.

Burnside's Lemma

If $g \in G$ let

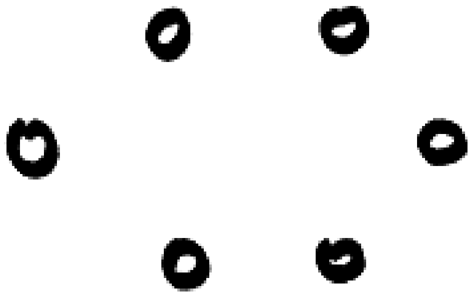
$$F(g) := \# \{ x \in X \mid gx = x \}.$$

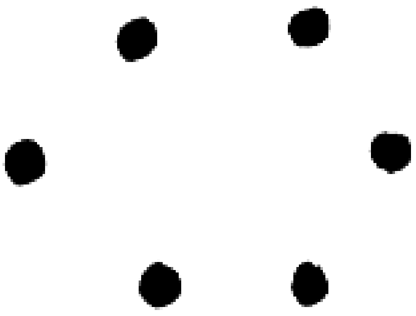
$$\boxed{\# \text{orbits} = \frac{1}{|G|} \sum_{g \in G} F(g)}$$

"average of # of fixed points"

$G = D_6$

$F(r) = 2 = 2^1$

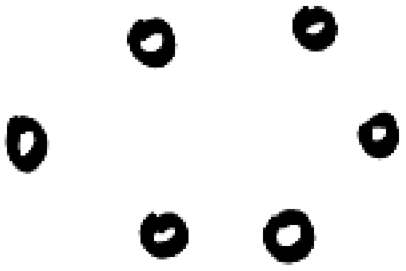


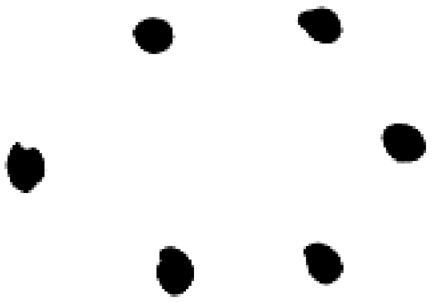


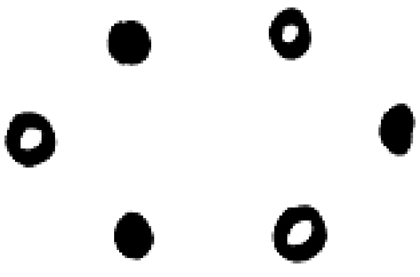
If $rx = x$ then $r^2x = x$

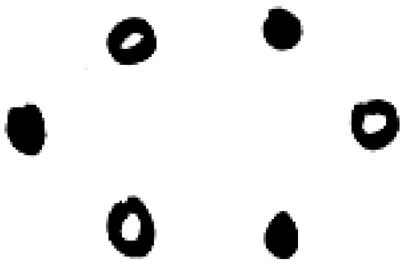
$r^3x = x \ldots$

$F(r^2) = 4 = 2^2$

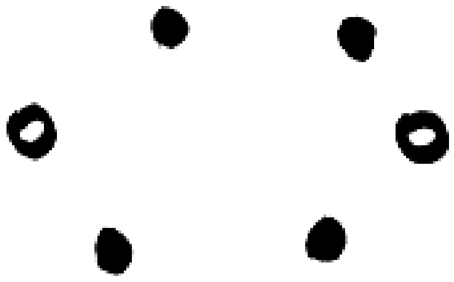


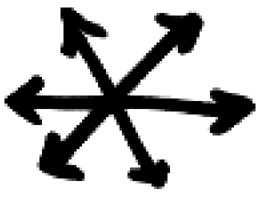






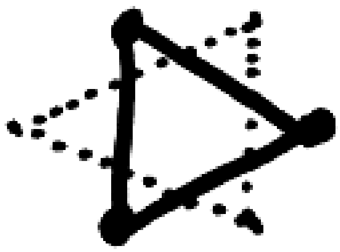
$F(r^3) = 8 = 2^3$ ~~previous one~~



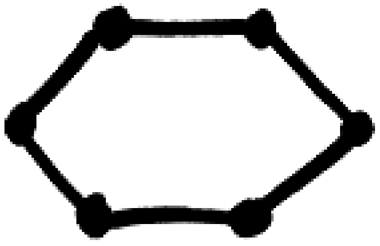


Ⓑ

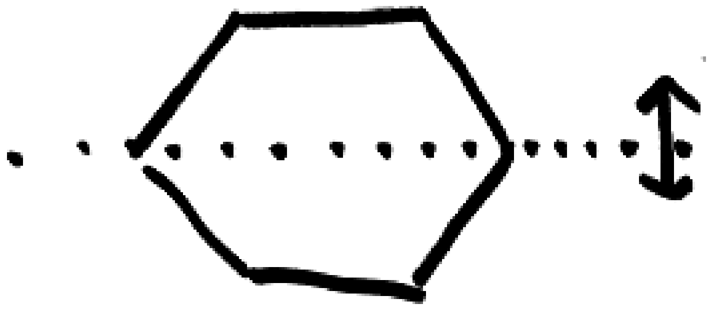
$F(r^4) = 2^2$

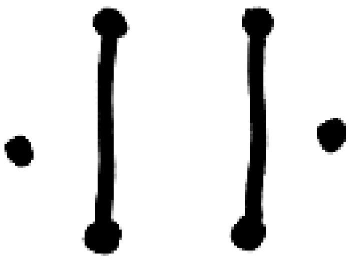


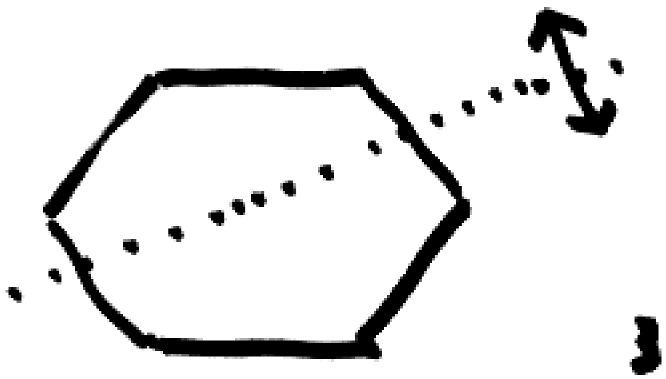
$F(r^5) = 2^1$

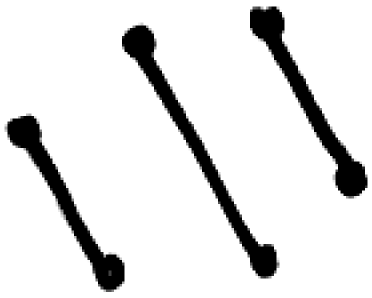


$F(s) = 2^4$









$F(s_1) = 2^3$

<u>Burnside</u>

$$\#\text{ necklaces} = \frac{1}{12}\Big(\underset{\underset{1}{\uparrow}}{2^6} + \underset{\underset{r}{\uparrow}}{2^1} + \underset{\underset{r^2}{\uparrow}}{2^2} + \underset{\underset{r^3}{\uparrow}}{2^3} +$$

$$\underset{r^4}{\underset{\uparrow}{2^2}} + \underset{r5}{\underset{\uparrow}{2}} + \underset{s_0,\ s_2,\ s_4}{2^4+2^4+2^4} + \underset{s_1,\ s_3,\ s_5}{2^3+2^3+2^3})$$

$$= \frac{1}{12}\Big(2^6 + 2 \times 2^1 + 2 \times 2^2 + 2^3 + 3 \times 2^4 + 3 \times 2^3 \Big)$$

$$\begin{array}{r} \overset{1}{6}4 \\ 20 \\ 48 \\ \underline{24} \\ 156 \end{array} = 12 \times 13$$

What if we had m colors?

$F(r) = m$

r

$$F(r^2) = m^2$$

$$\vdots$$

$$\#\text{necklaces} = \frac{1}{12}\Big(\underset{\substack{\uparrow \\ 1}}{m^6} + 2 \times \underset{\substack{\uparrow \\ r,\, r^{-1}}}{m} + 2 \times \underset{\substack{\uparrow \\ r^2,\, r^{-2}}}{m^2} + \underset{\substack{\uparrow \\ r^3}}{m^3}$$

$$+ \ 3 \times \underset{\substack{\uparrow \\ s_0,\, s_2,\, s_4}}{m^4} + 3 \times \underset{\substack{\uparrow \\ s_1,\, s_3,\, s_5}}{m^3} \Big)$$

$$\boxed{\#\text{necklaces} = \frac{1}{12}\left(m^6 + 3m^4 + 4m^3 + 2m^2 + 2m \right)}$$

$\underline{m = 1}$ ⇝ $\#$necklaces $= 1$

Note: In particular for any $m = 1, 2, \ldots$ we must have

$$m^6 + 3m^4 + 4m^3 + 2m^2 + 2m$$

is divisible by 12.

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}(n^2 - n)$$

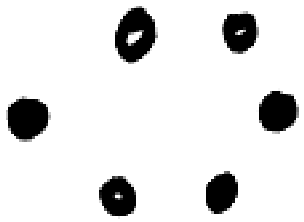
March 27, 2007

$$\underset{\substack{\uparrow \\ \text{group}}}{G} \circlearrowright \underset{\substack{\nwarrow \\ \text{set}}}{X}$$

(dihedral group order 12 symmetries of hexagon)

("pictures")

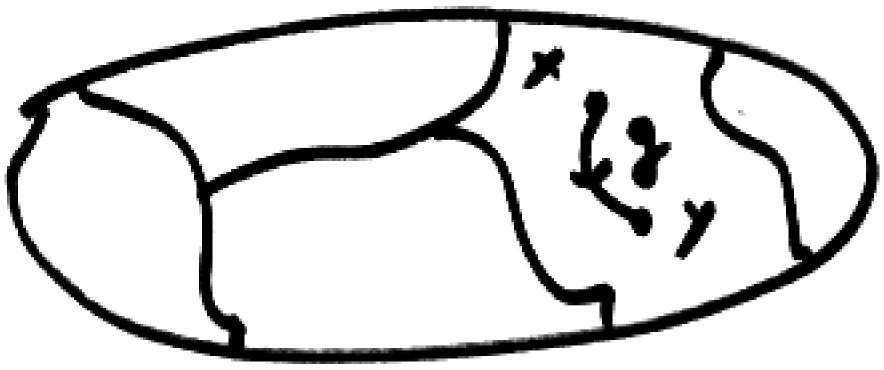
E.g.



$g \in G$ $\qquad g \cdot x \in X$

- $G \cdot x := \{ y \in X \mid y = gx \}$

<u>orbit</u> of x



("necklace")

- <u>Stabilizer</u> of x

$\text{Stab}_G(x) := \{ g \in G \mid gx = x \}$

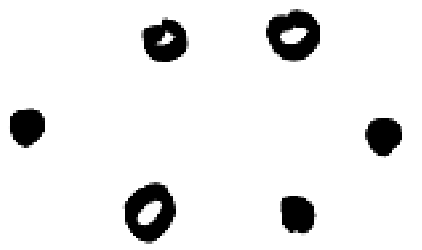
subgroup of G

Fact. $\# G\cdot x \cdot |\mathrm{Stab}_G(x)| = |G|$

In particular, the size of an orbit is a factor of $|G|$.

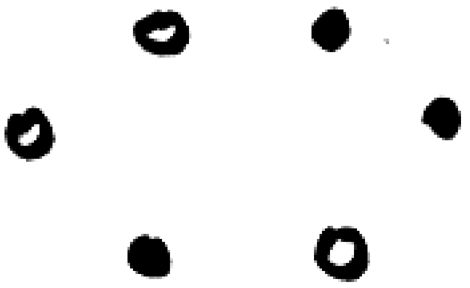
Eg.

$x =$

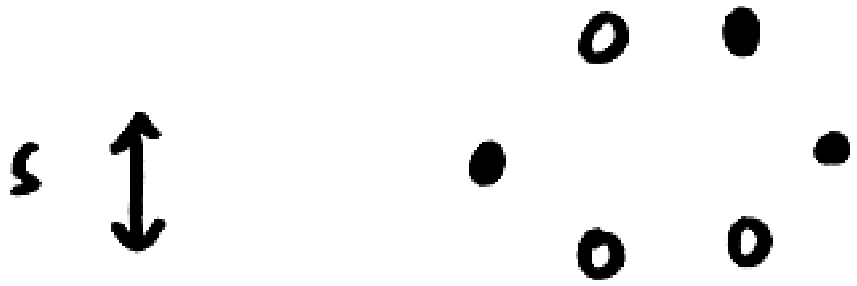


what is the size of Gx ?

r

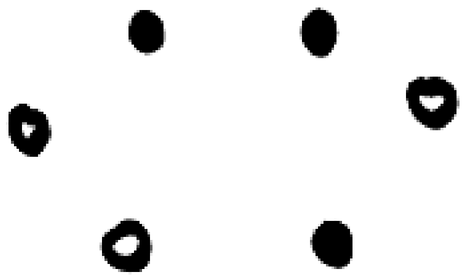


s



⋮

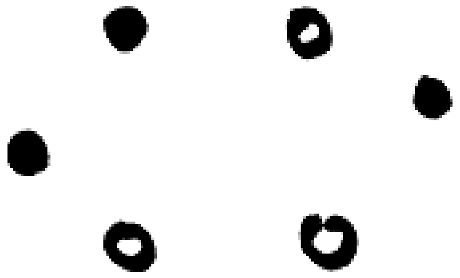
r^2



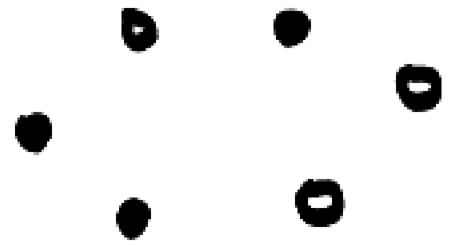
$\# Gx \geq 7$

$\Longrightarrow \# Gx = 12$

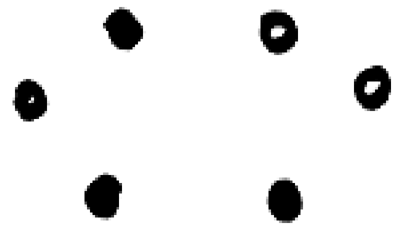
r^3



r^4

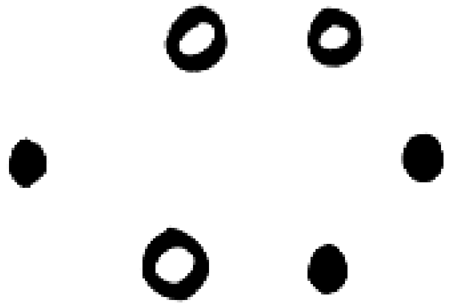


r^5



$\mathrm{Stab}_G(x) = \{1\}$

$x =$



r, r^2, r^3, r^4, r^5 do not fix x

$s_0, s_1, s_2, s_3, s_4, s_5$ "

• $\Rightarrow$ $\#Gx = 12$

How many orbits are there?

Burnside's Lemma

$$\boxed{\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G} F(g)}$$

$F(g) = \#\{x \in X \mid gx = x\}$

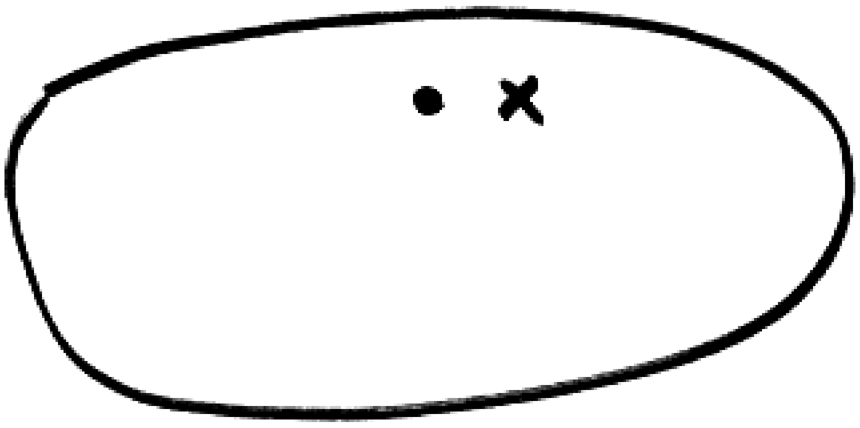
Proof

$$\sum_{g\in G} F(g)$$

$x \in X$

How many times does it get counted in this sum?

X



It will be counted in $F(g)$ if $\boxed{gx = x}$

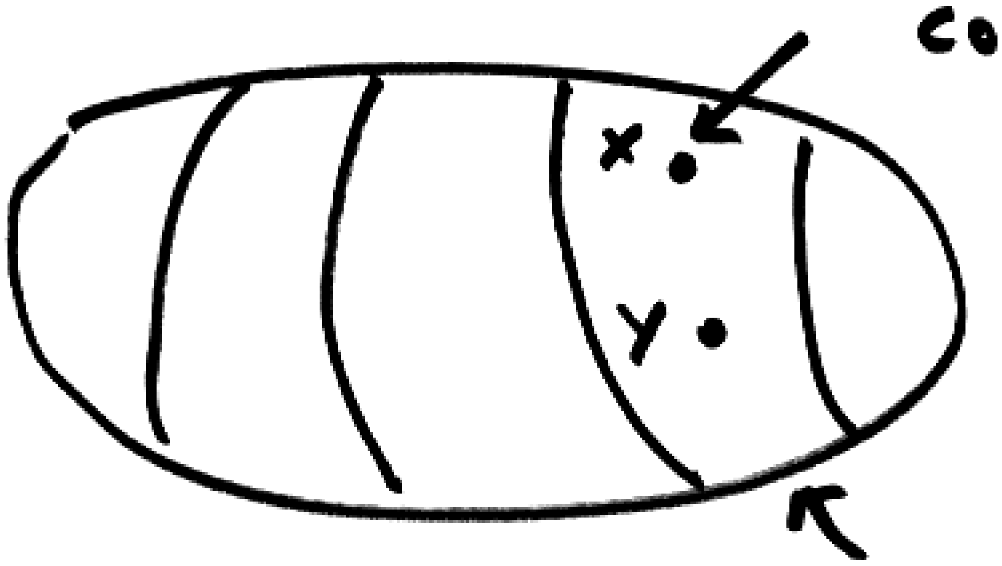
Total contribution of x to

$$\sum_{g \in G} F(g)$$

is $\{ g \in G \mid gx = x \} = \mathrm{Stab}_G(x)$

Each $x \in X$ contributes $|\mathrm{Stab}_G(x)|$ to the sum.

$$\sum_{g \in G} F(g) = \#\text{orbits} \cdot |G|$$



contributes $|\mathrm{Stab}_G(x)|$

all y in the orbit of x contribute the <u>same</u> amount

namely $|\mathrm{Stab}_G(y)| = \dfrac{|G|}{\# Gy}$

$$Gy = Gx$$

Total contribution of orbit is

$$|\mathrm{Stab}_G(x)| \cdot \# Gx = |G|$$

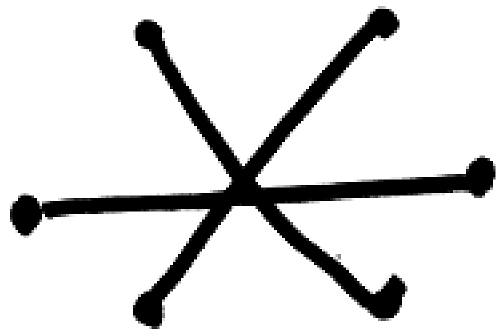
Total sum $|G| \cdot \#\text{orbits}$

$$= \sum_{g \in G} F(g) \qquad \square$$

Cycle indicator

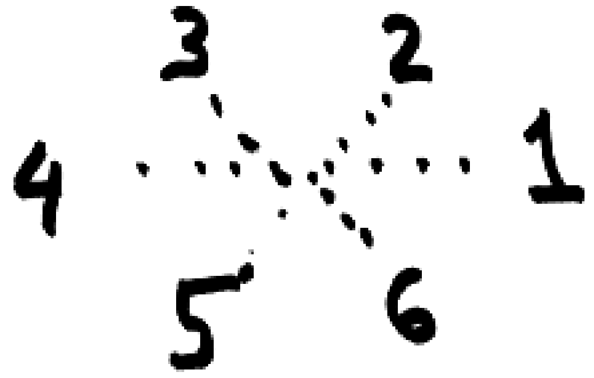
How do we compute $F(g)$?

Say $g = r^3$ what is $F(g)$?



If we have m colors then

$$F(r^3) = m \cdot m \cdot m$$
$$= m^3$$



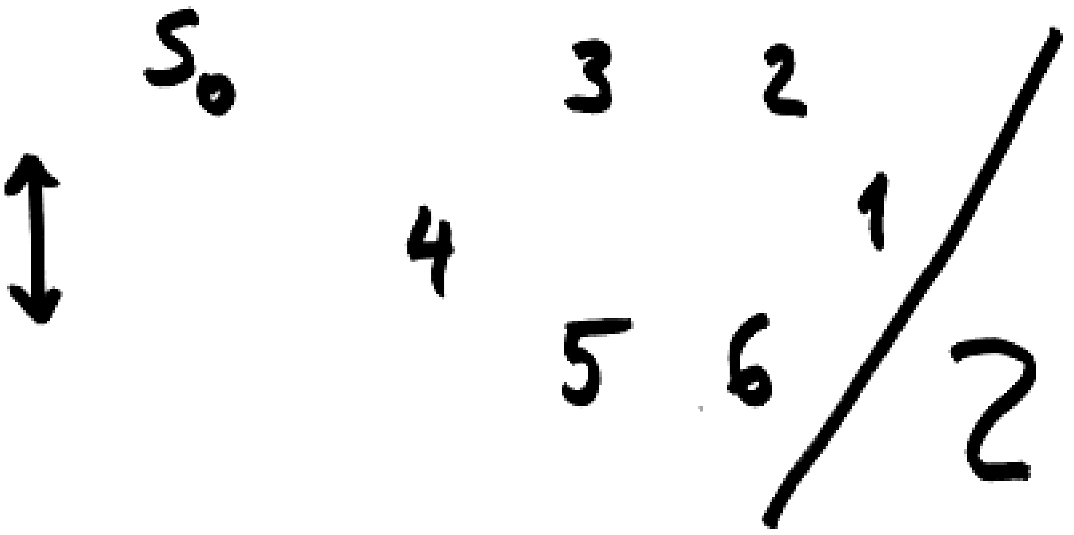
$$r^3 = \underbrace{(14)(25)(36)}_{3 \text{ cycles}}$$

In general $F(g) = m^{\ell(g)}$

$\ell(g) := \#$ cycles in g

length of g

$$F(g) = m^{\ell(g)}$$



$s_0 = (35)(26)$

$\ell(g)$ includes the 1-cycles.

$$\ell(s_0) = 4$$

$$F(s_0) = m^4$$

cycle indicator

$$Z_G(x_1, x_2, \dots) = \frac{1}{|G|} \sum_{g \in G} x_1^{k_1(g)} \cdot x_2^{k_2(g)} \cdots$$

$K_i(g) :=$ # cycles in g of length i

For $g = s_0$ $\qquad s_0 = (35)(26)$

$$K_1(s_0) = 2$$
$$K_2(s_0) = 2$$
$$K_3(s_0) = 0$$
$$\vdots$$

Contribution to Z_G : $\qquad x_1^2 x_2^2$

Formal way to keep track of the cycle decomposition of all $g \in G$.

For us counting orbits with m colors

$$Z_G(m, m, m, \ldots)$$

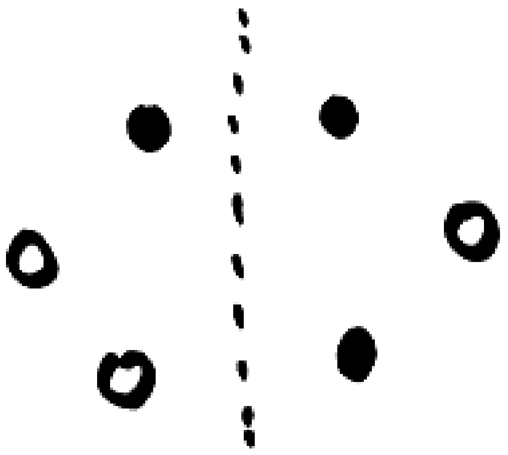
If $x_i = m$ then

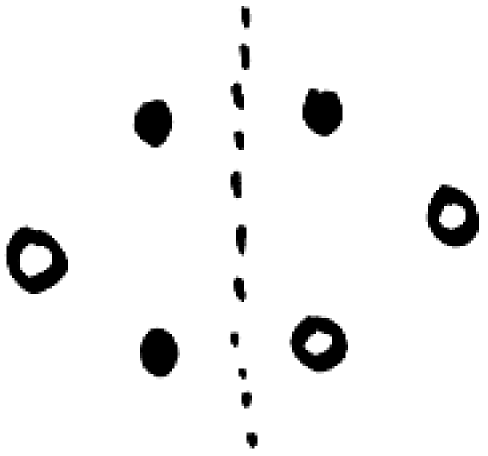
$$x_1^{k_1(g)} x_2^{k_2(g)} \cdots\cdots = m^{k_1(g)+k_2(g)+\cdots}$$

$$= m^{\ell(g)}$$

$$Z_G(m, m, m, \ldots) = \frac{1}{|G|} \sum_{g \in G} m^{\ell(g)}$$

$$= \# \text{ orbits.}$$

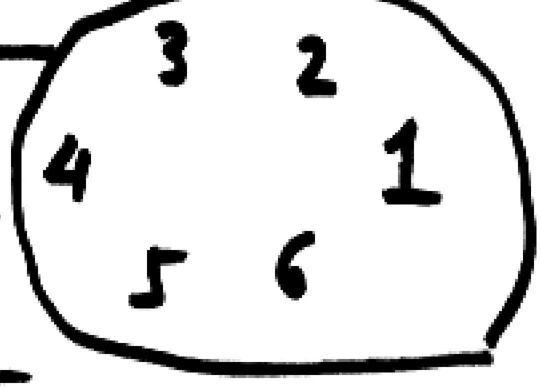




<u>Cycle indicator</u> for rotations of hexagons (no flips allowed)

$G = \{1, r, r^2, r^3, r^4, r^5\}$

		k_1	k_2	k_3	k_4	k_5	k_6
1	1	6					
r	(123456)	0	0	0	0	0	1
r^2	(135)(246)			2			
r^3	(14)(25)(36)		3				
r^4	([illegible])			2			[illegible]
r^5	(654321)						1



(642)(531)

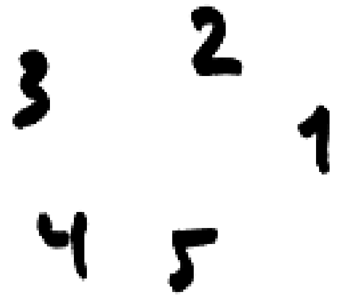
$$Z = \frac{1}{6}\left(x_1^6 + x_6 + x_3^2 + x_2^3 + x_3^2 + x_6 \right)$$

$$\boxed{Z = \frac{1}{6}\left(x_1^6 + 2x_6 + 2x_3^2 + x_2^3 \right)}$$

$m=2$

$$\frac{1}{6}\left(2^6 + 2\cdot 2 + 2\cdot 2^2 + 2^3 \right)$$

$$= \frac{1}{6}(64 + 4 + 8 + 8) = \frac{84}{6} = 14$$



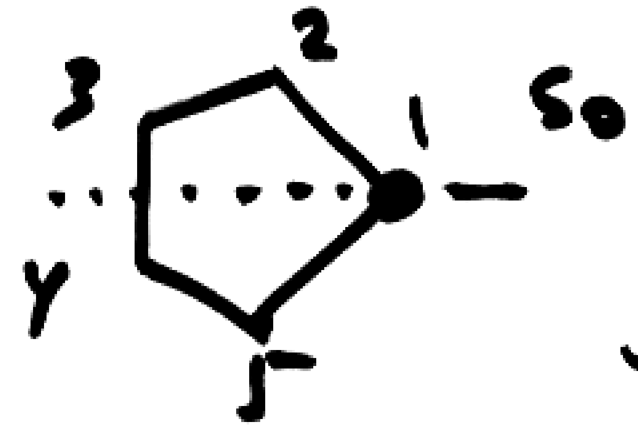
$r = (12345)$

1	1	x_1^5
r	(12345)	x_5
r^2	(13524)	x_5
r^3	$(14\cdots)$	x_5
r^4	(54321)	x_5

$$Z = \frac{1}{5}\left(x_1^5 + 4 x_5 \right)$$

March 29, 2007

flips



$$\overbrace{(1)\ (25)(34)}^{l=3}$$

cycle index $x_1^1 \cdot x_2^2$

color counting $x_i = m$

$$m \cdot m^2 = m^3$$

$$\frac{1}{10}\left(m^5 + 4m + 5m^3\right)$$

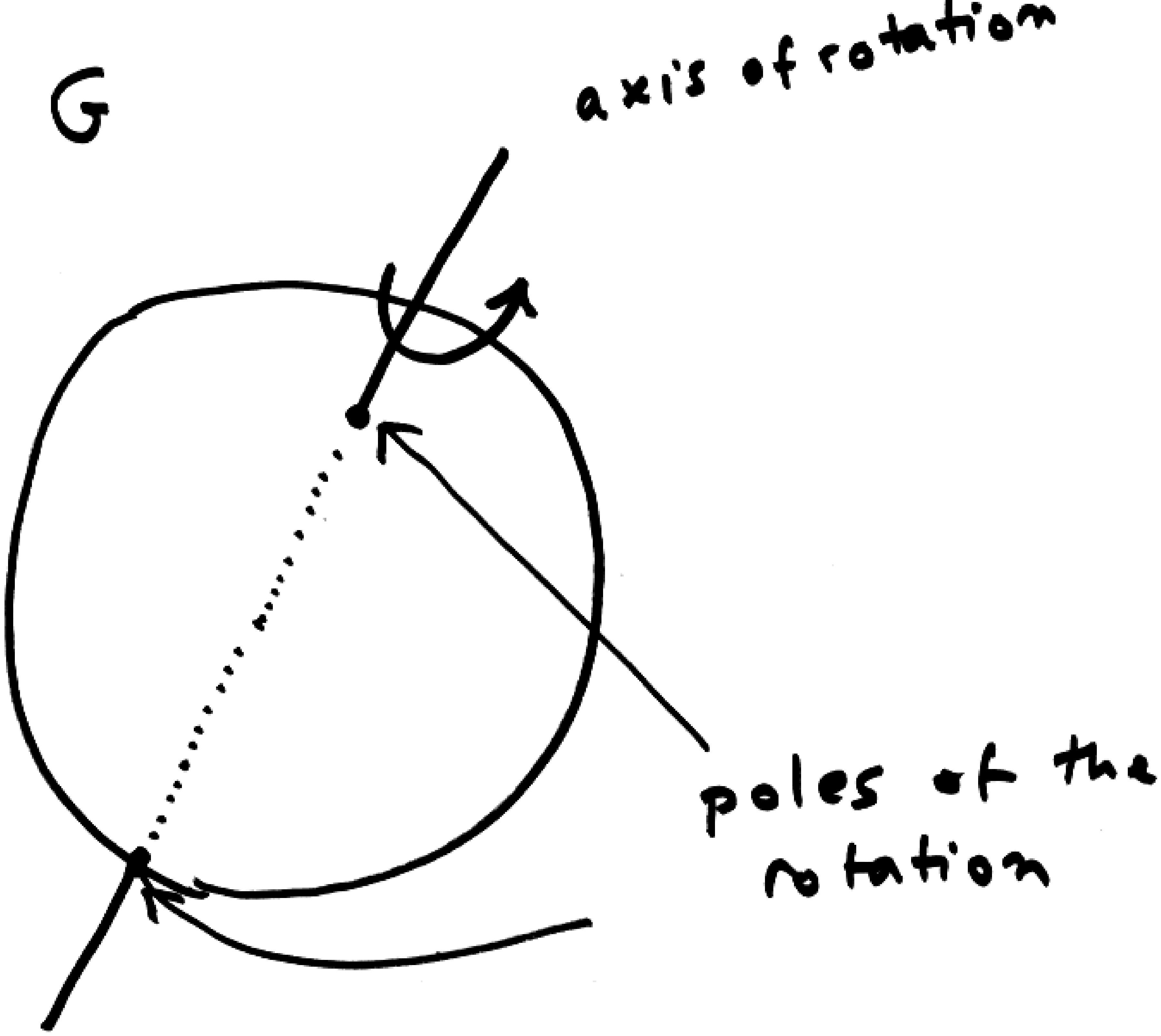
$m=3$, $3^5 + 4 \times 3 + 5 \times 3^3$

$$= \frac{1}{10}(243 + 12 + 135)$$

$$= \frac{390}{10} = 39$$

Finite group of rotations in $\mathbb{R}^3$

G



Let $\mathcal{P}$ = { poles of rotations of G }

Finite number of poles in $\mathcal{P}$.

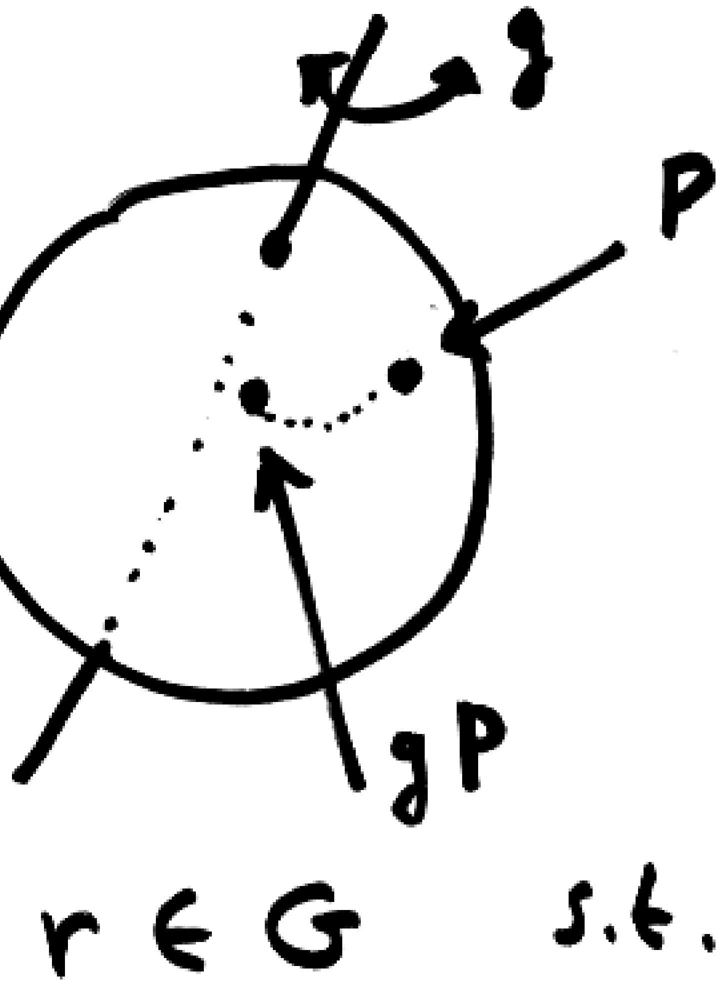
G acts on $\mathcal{P}$

- If $P \in \mathcal{P}$, $g \in G$

then $g \cdot P$

$\boxed{hP = P}$ for some $h \in G$
$h \neq 1$.

To see that gP is a pole I need to find a rotation in G that fixes gP.



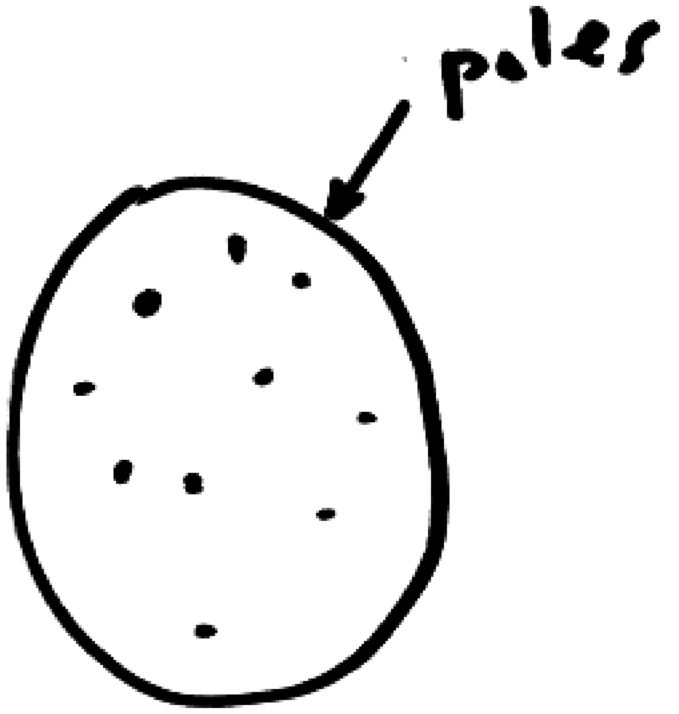
I.e. $r \in G$ s.t. $r(gP) = gP$

$$g^{-1}(gP) = P$$

$$hg^{-1}(gP) = hP = P$$

$$\underbrace{ghg^{-1}}_{r}(gP) = gP$$

Found $r \in G$ $\rightsquigarrow$ gP is a pole

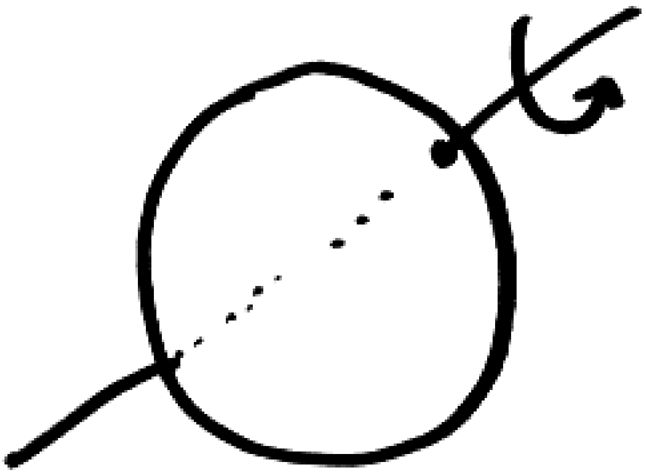


$N :=$ # orbits of G acting on $\mathcal{P}$.

$$N = \frac{1}{|G|} \sum_{g \in G} F(g)$$

by Burnside.

$$F(g) = \begin{cases} \#\mathcal{P} & g = 1 \\ 2 & g \neq 1 \end{cases}$$



$$N = \frac{1}{|G|}\left(\#\mathcal{P} + 2\overbrace{(|G|-1)}^{\text{\# of non-identity rotations}}\right)$$

$$|G|N = \#\mathcal{P} + 2|G| - 2$$

$$\boxed{|G|(N-2) = \#\mathcal{P} - 2}$$

Assume $|G|>1$, I.e. there is some non-identity rotation $g \in G$. It fixes two point. Hence $\# \mathcal{P} \geqslant 2$.

rhs $\geqslant 0 \Rightarrow N \geqslant 2$.

If $N=2$ then $\# \mathcal{P} = 2$

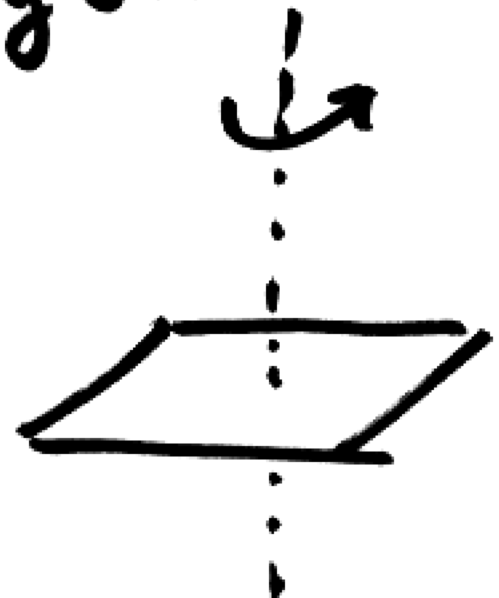
I.e. we have only have two poles all rotations share same axis

$\rightarrow \quad 1, r, r^2, \ldots, r^{n-1}$

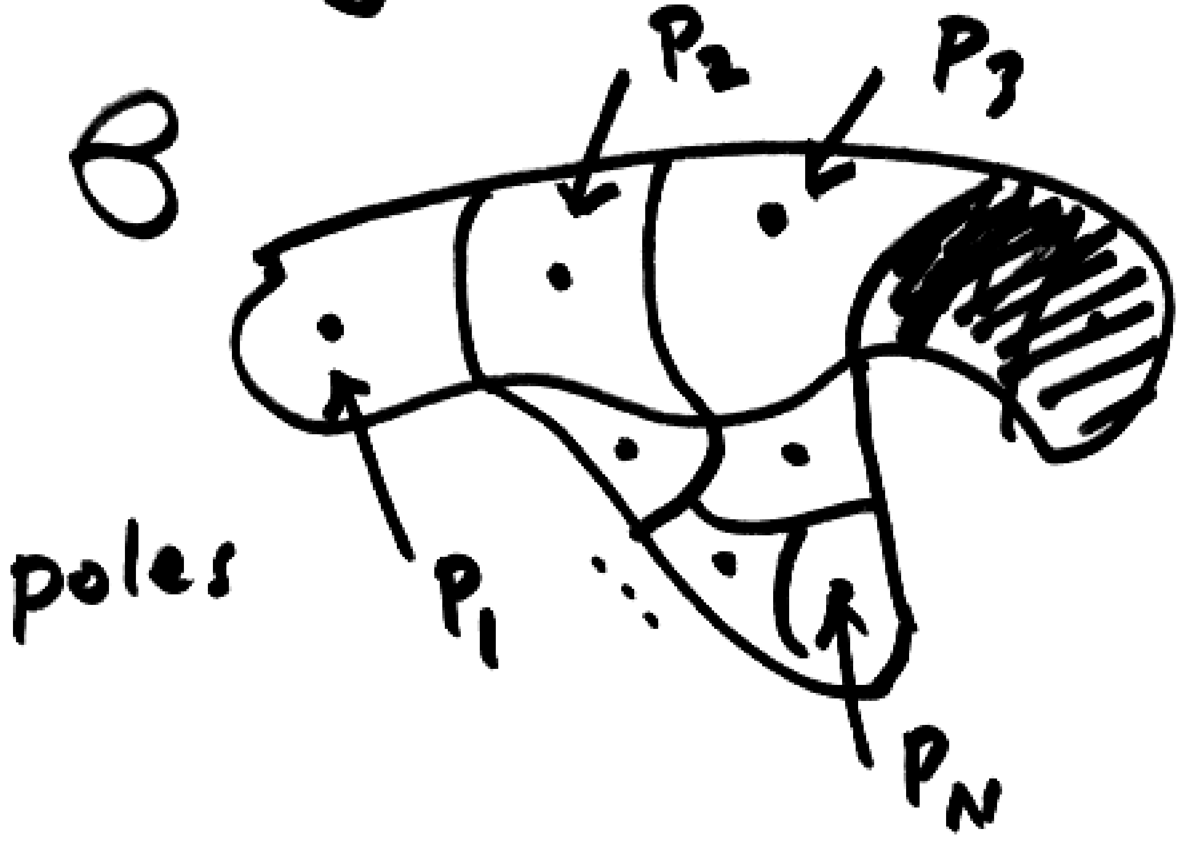
some n

I.e. ~~[illegible]~~ rotations ~~of~~ fixing a n-gon

E.g. $\underline{n=4}$



Say $N \geq 3$. $\# \mathcal{P} > 2$

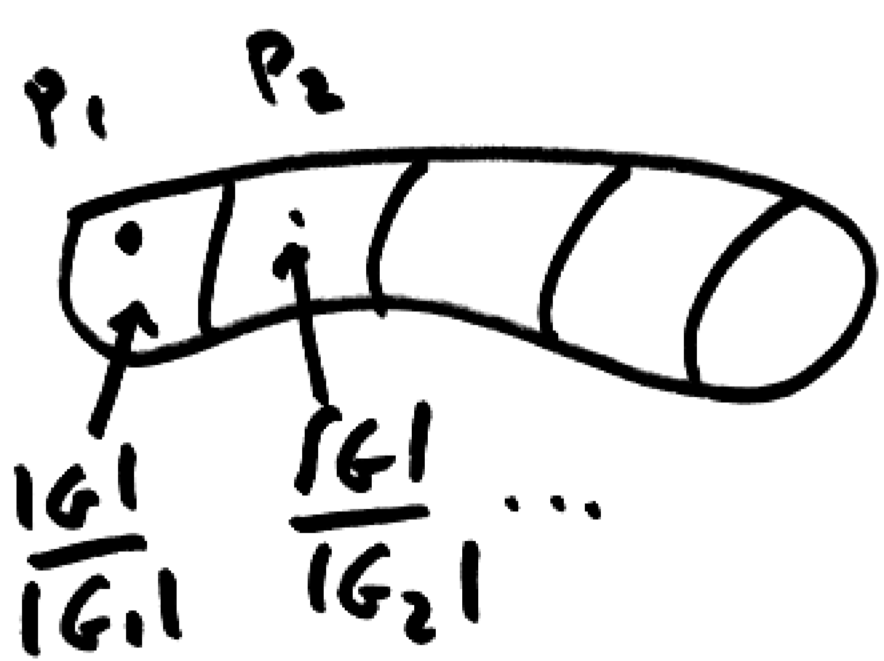


$p_1, p_2, p_3, \dots, p_N$ are poles one per orbit.

$$G_i := \mathrm{Stab}_G \, p_i$$

$$\# G p_i \cdot | G_i | = | G |$$

$$\# G p_i = \frac{|G|}{|G_i|}$$



$$\# \mathcal{P} = \frac{|G|}{|G_1|} + \frac{|G|}{|G_2|} + \cdots + \frac{|G|}{|G_N|}$$

(II) $$\boxed{\frac{\# \mathcal{P}}{|G|} = \frac{1}{|G_1|} + \frac{1}{|G_2|} + \cdots + \frac{1}{|G_N|}}$$

$$N = \frac{\# \mathcal{P}}{|G|} + 2\left(1 - \frac{1}{|G|}\right)$$

(from before).

(I) $$N = \underbrace{1 + 1 + \cdots\cdots + 1}_{N \text{ times}}$$

(I) − (II)

$$N - \frac{\# \mathcal{P}}{|G|} = \left(1 - \frac{1}{|G_1|}\right) + \left(1 - \frac{1}{|G_2|}\right) + \cdots + \left(1 - \frac{1}{|G_N|}\right)$$

$$= 2\left(1 - \frac{1}{|G|}\right)$$

Finally:

$$\boxed{\sum_{i=1}^{N}\left(1-\frac{1}{|G_i|}\right) = 2\left(1-\frac{1}{|G|}\right)}$$

$(|G| > 1 \quad \rightarrow \quad |G| \geq 2)$

~~[illegible]~~

rhs < 2

$G_i = \mathrm{Stab}_G P_i$

$|G_i| \geq 2$ since P_i is the pole of <u>some</u> non-trivial rotation $g_i \in G_i$.

$$1 - \frac{1}{|G_i|} \geq \frac{1}{2}$$

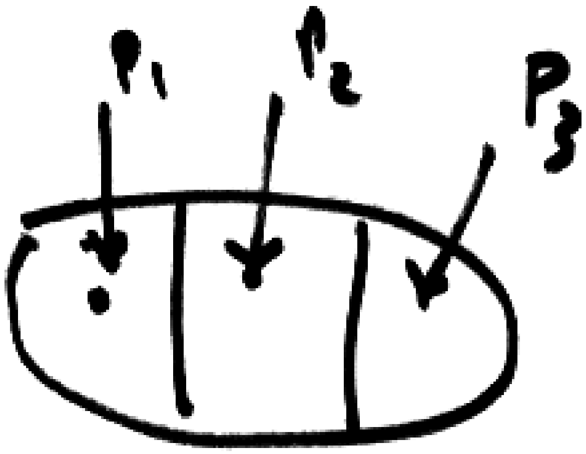
$\Rightarrow \quad N < 4$

we know $N \geq 2$

$N=2$ dealt with already

$\Rightarrow$ $N=3$

$G_1, \; G_2, \; G_3$



$n_i = |G_i|$

$n_1 \le n_2 \le n_3$

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$n_i \geqslant 2$, $|G| \geqslant 2$

We can't have $n_i \geqslant 3$
Otherwise

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \le \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

lhs ≤ 1 but   rhs > 1

$\Rightarrow$ $\boxed{n_1 = 2}$

$$\frac{1}{2} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{|G|}$$

$$\frac{1}{n_2} + \frac{1}{n_3} = \frac{1}{2} + \frac{2}{|G|}$$

~~[illegible]~~

~~[illegible]~~

If $n_2 = 2$ n_3 could be anything. $n_3 = n$

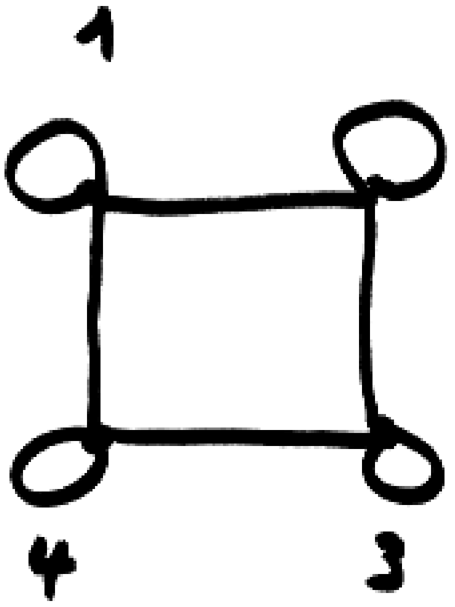
$$\frac{1}{2} + \frac{1}{n} = \frac{1}{2} + \frac{2}{|G|}$$

$\Rightarrow$ $|G| = 2n$

$\rightsquigarrow$ $G = D_n$ dihedral group

G	n_1	n_2	n_3	
D_n	2	2	n	Dihedral n-gon
A_4	2	3	3	Tetrahedron
S_4	2	3	4	Cube/octahedron
A_5	2	3	5	Icosahedron/ Dodecahedron

April 12, 2007



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

(Adjacency matrix of the graph)

ij entry of $A = 1$ if $i — j$

0 otherwise

Initial state : $s_I = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$

Move : $t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}$

$s_F = s_I + At$ final state

goal $S_F = 0$

$$S_I + At = 0$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 + t_4 \\ t_1 + t_2 + t_3 \\ t_2 + t_3 + t_4 \\ t_1 + t_3 + t_4 \end{pmatrix}$$

$$\boxed{S_I = -At}$$

$$\begin{cases} t_1 + t_2 + t_4 = -s_1 \\ t_1 + t_2 + t_3 = -s_2 \\ t_2 + t_3 + t_4 = -s_3 \\ t_1 + t_3 + t_4 = -s_4 \end{cases}$$

Linear system of equations

Solving puzzle ⟷ solving system of equations.

$n \times n$ matrix A

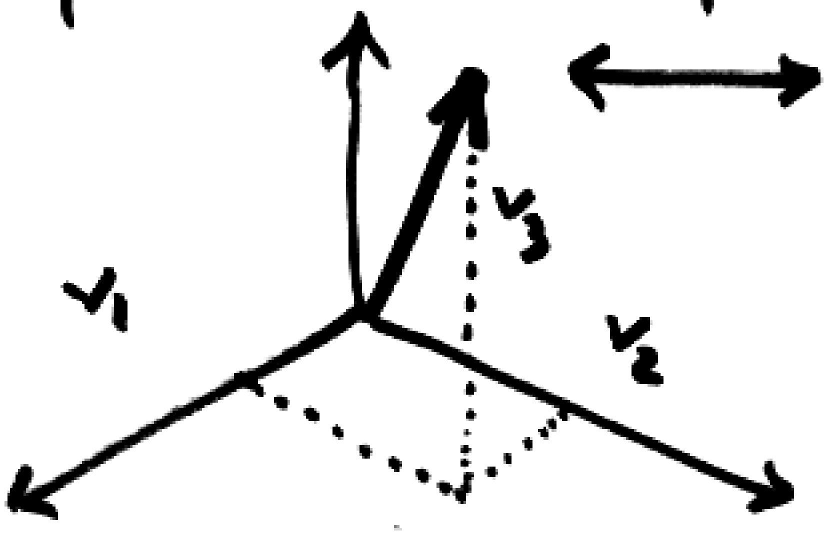
$A = (a_{ij})$ $\qquad a_{ij}$ numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

<u>vectors</u>: $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

<u>$n=3$</u> $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\rightsquigarrow$ point in space



$\longleftrightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$V = \{$ all n-dim'l vectors $\}$

A $n \times n$ matrix

$v \in V$ vector

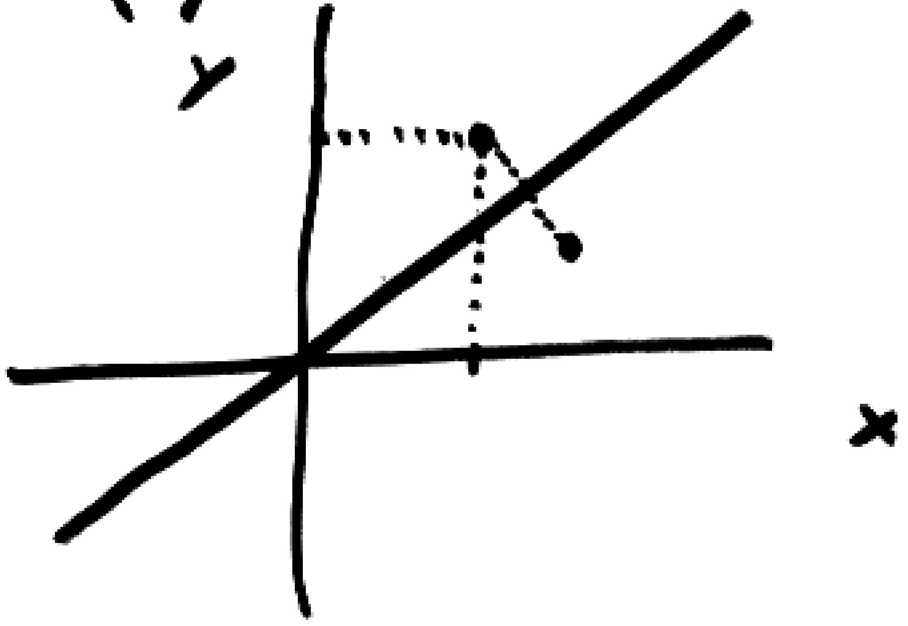
$\rightsquigarrow A \cdot v \in V$

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \underline{n=2}$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Effect of multiplication by A?

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$$



I.e. A is the reflection through the $y=x$ line.

- Identity $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$I_m \cdot v = v$$

- $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2 I_2$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

scaling by two.

<u>Scalars</u> a, $\quad av = \begin{pmatrix} av_1 \\ av_2 \\ \vdots \\ av_n \end{pmatrix}$

- Rotations can also be written in terms of matrices.

Features of transformation

$$v \longmapsto Av$$

= <u>linear transformation</u>

- $A \cdot (av) = a(Av)$
- $A \cdot (u+v) = Au + Av$

- $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$A\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x+y\end{pmatrix}$$

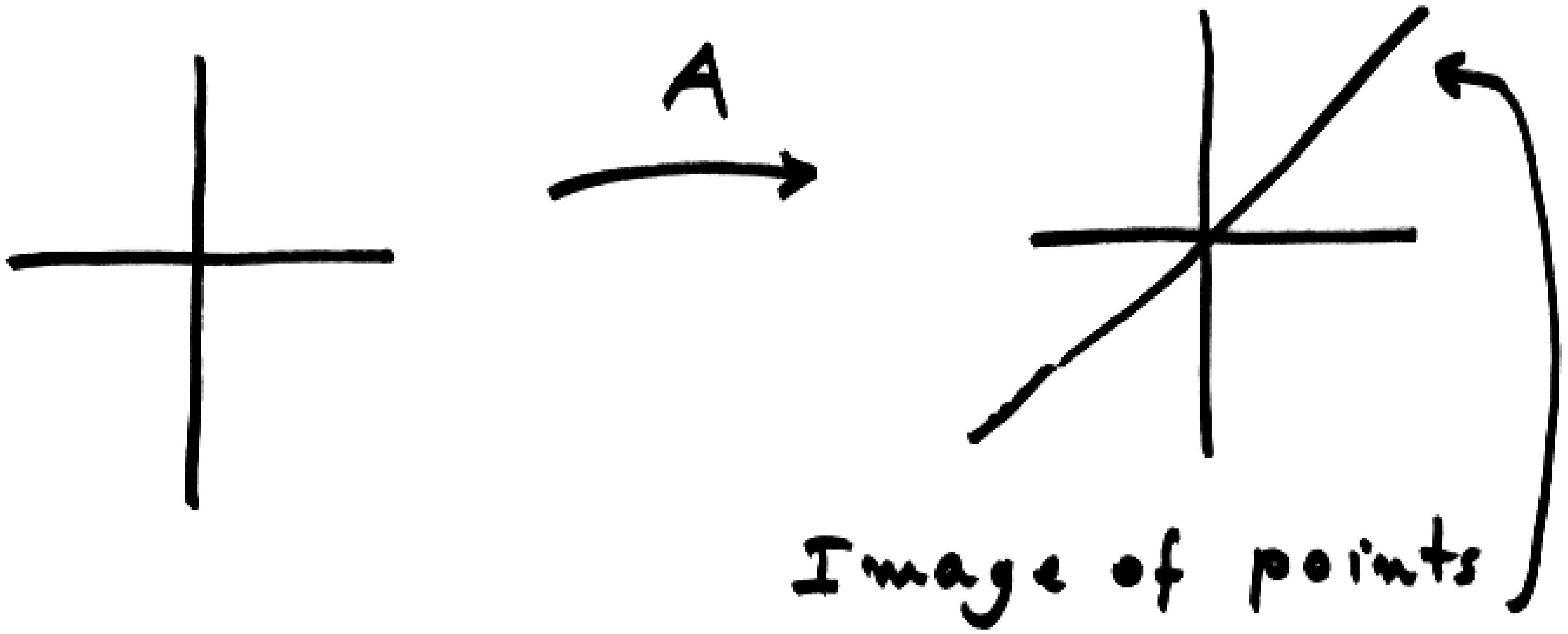


Image of points is on this line

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

this system of equations not always has a solution. If $a \neq b$ we have no solution. If $a=b$ we can solve it

$$A\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x+y\end{pmatrix}=\begin{pmatrix}a\\a\end{pmatrix}$$

$$\left.\begin{matrix}x+y=a\\x+y=a\end{matrix}\right\}\longrightarrow\boxed{x+y=a}$$

To solve: Pick any x and set

$$y=a-x$$

system $\boxed{Av = u}$ $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$u = \begin{pmatrix} a \\ b \end{pmatrix}$

- <u>NO</u> solution unless $u = \begin{pmatrix} a \\ a \end{pmatrix}$

- If $u = \begin{pmatrix} a \\ a \end{pmatrix}$ then it has a lot of solutions.

all solutions : $\begin{pmatrix} x \\ a-x \end{pmatrix}$

For $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ situation is different.

$$\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$y = a$$
$$x = b$$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$

Every $\begin{pmatrix} a \\ b \end{pmatrix}$ has a <u>unique</u> solution.

Say A is <u>invertible</u> if it has an inverse

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n$$

In this case

$$A v = u$$

can be solved by multiplying by A^{-1}.

$$A^{-1}(Av) = A^{-1} \cdot u$$

$$(A^{-1}A)v = A^{-1}u$$

$$\boxed{v = A^{-1}u}$$

Unique solution for each choice of u.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^{-1} = A$$

- How do we determine if a matrix A has an inverse?

There is a number called the **determinant** of A $\det(A)$

A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{= A^*}{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \det(A) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A A^* = \det(A) \cdot I_2$$

- If $\det(A) = 0$ then $A A^* = 0$

if A had an inverse

$$A^{-1}(A A^*) = 0$$

$$(A^{-1}A) A^* = A^*$$ not the case

April 17, 2007

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \text{3-dim vectors} \right\}$$

3×3 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$v \in V$

$$A \cdot v \in V$$

$$\begin{tikzcd} V \arrow[r] & V \\ v \arrow[r, mapsto] & A \cdot v \end{tikzcd}$$

- $$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$$

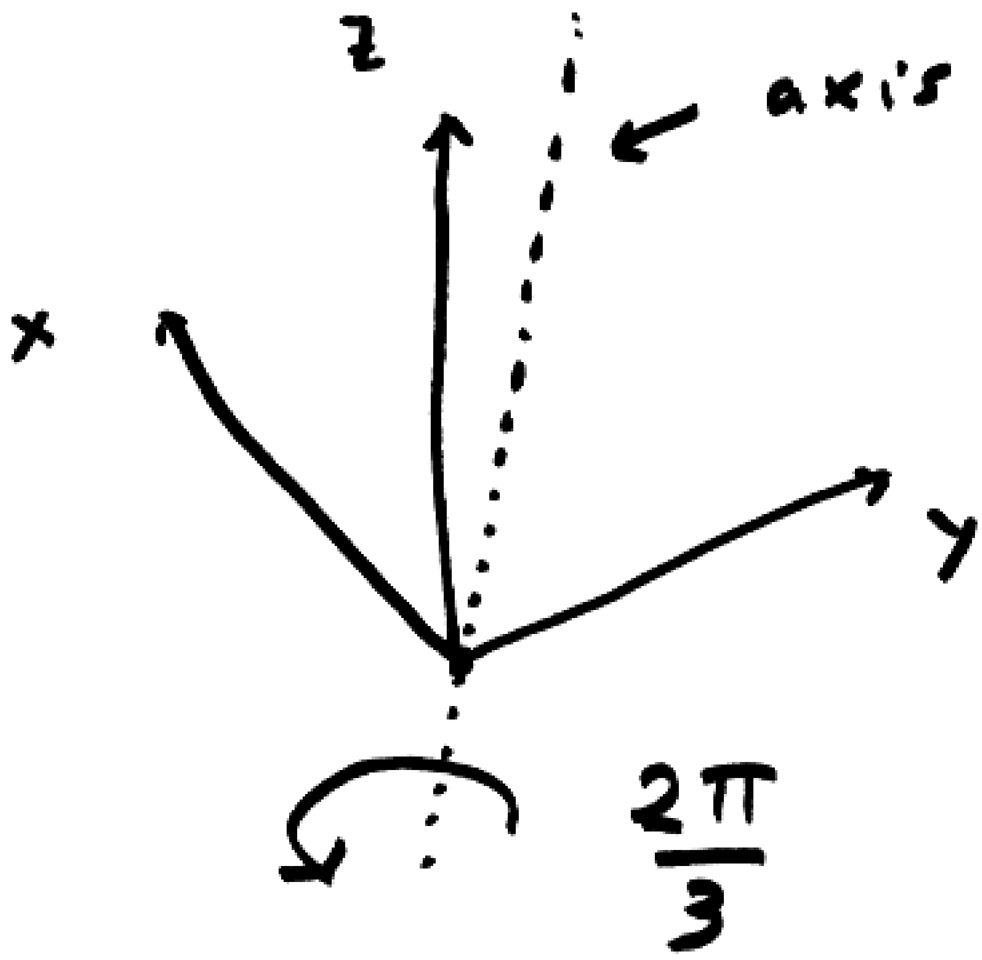
$$(x, y, z) \mapsto (z, x, y)$$

Axis of rotation?

$$(a, a, a) = a(1, 1, 1)$$

fixed by A.

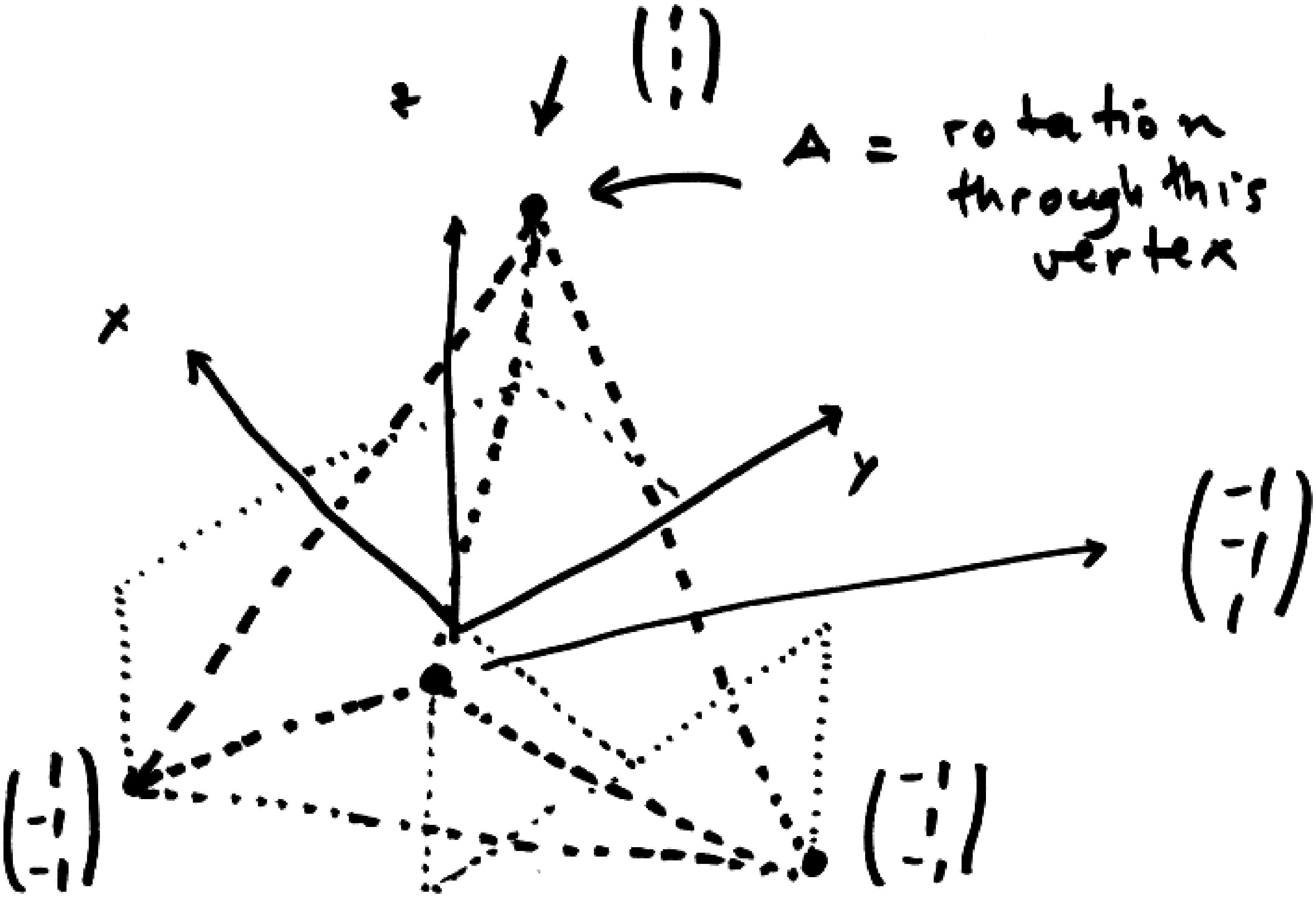
$x = y$
$y = z$



$$\begin{tikzcd}(x, y, z) \arrow[d] \\ (z, x, y) \arrow[d] \\ (y, z, x) \arrow[uu, bend right] \end{tikzcd}$$

Four points

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\-1\\-1\end{pmatrix} \begin{pmatrix}-1\\-1\\1\end{pmatrix} \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$



A = rotation through this vertex

$$\frac{1}{4}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\\1\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right)$$

= center of mass.

$$= \begin{pmatrix}0\\0\\0\end{pmatrix}$$

R_V $(= R_F)$ rotations assoc. to vertices (faces)

R_E rotations assoc. to edges

- $$R_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R_E \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}$$

$$\begin{tikzcd}
\begin{pmatrix}1\\1\\1\end{pmatrix} \arrow[r, mapsto] \arrow[d, bend right, leftrightarrow] & \begin{pmatrix}1\\-1\\-1\end{pmatrix} \\
\begin{pmatrix}1\\-1\\-1\end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\begin{pmatrix}-1\\-1\\1\end{pmatrix} \arrow[r, mapsto] \arrow[d, bend right, leftrightarrow] & \begin{pmatrix}-1\\1\\-1\end{pmatrix} \\
\begin{pmatrix}-1\\1\\-1\end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix}-1\\-1\\1\end{pmatrix}
\end{tikzcd}$$

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

We can describe the action of the rotation group of (this) tetrahedron as follows:

cyclically permute (x, y, z) and change two signs.

E.g.

$$(x, y, z) \longmapsto (-y, -z, x)$$

(<u>signed permutation</u>)

(x, y, z) $(-x, -y, z)$ $(-x, y, -z)$ $(x, -y, -z$

(z, x, y) $(-z, -x, y)$ $(-z, x, -y)$ $(z, -x, -y$

(y, z, x) $(-y, -z, x)$ $(-y, z, -x)$ $(y, -z, -$

Total of 12.

group of rotations of tetrahedron $\simeq A_4$ (alternating group)

Rotations of cube (octahedron)

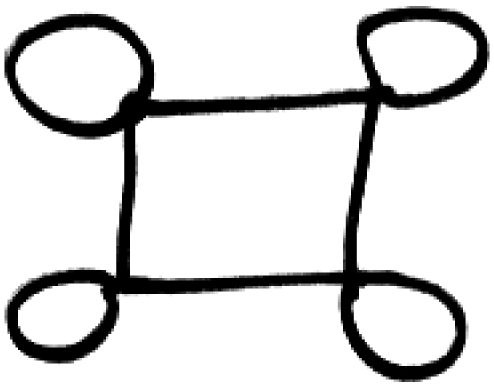
we can think of two inscribed tetrahedra inside cube.

Rotations of tetrahedron & swapping both tetrahedra.

get all signed permutations of 3-dim determinant = 1

group of rotations of cube $\simeq S_4$

April 19, 2007



$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Equation to solve

$$\boxed{S_I + At = 0}$$

$$t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} \qquad t_i = 0, 1$$

$\updownarrow$

dont / press button i

If A has an inverse A^{-1} then we can solve for t

$$\boxed{-A^{-1} S_I = t}$$

Row reduction algorithm.

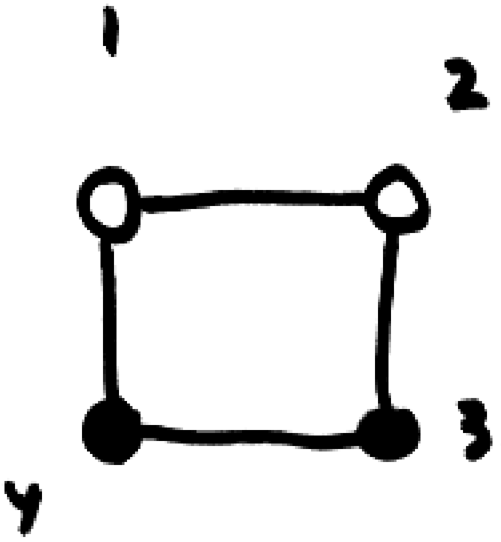
$$A^{-1} = A$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

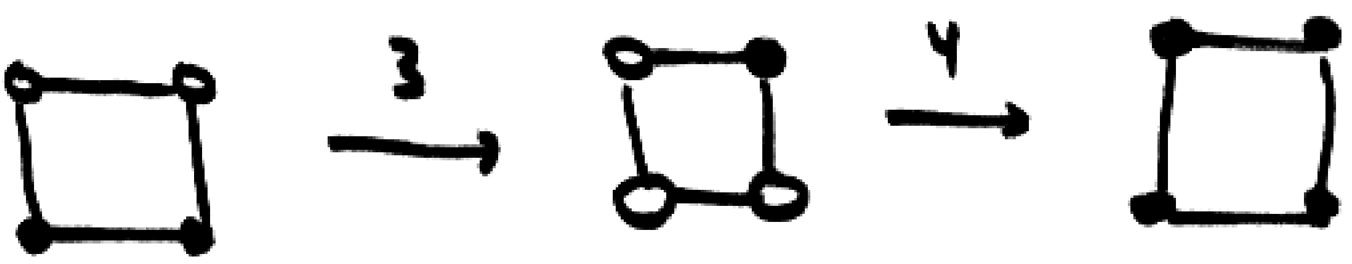
(mod 2
binary
sense)

$$\boxed{t = A s_I}$$

E.g.



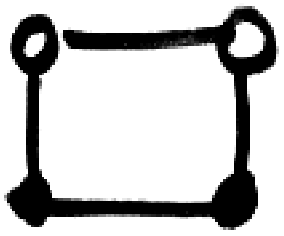
$$t = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$



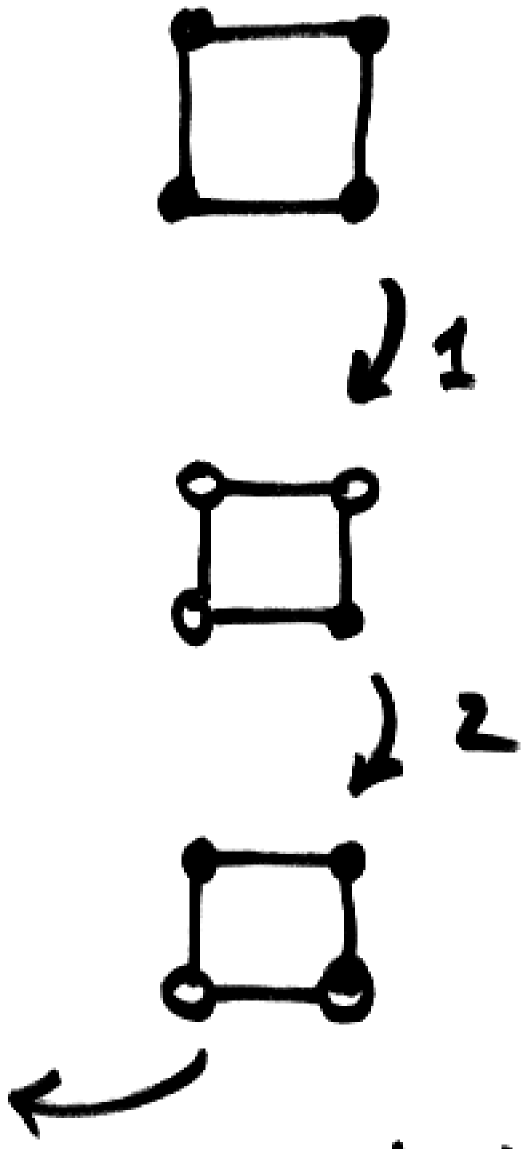
$-A^{-1} \rightsquigarrow$ cheat puzzle

In our case cheat puzzle = original puzzle b/c $-A^{-1} = A$.

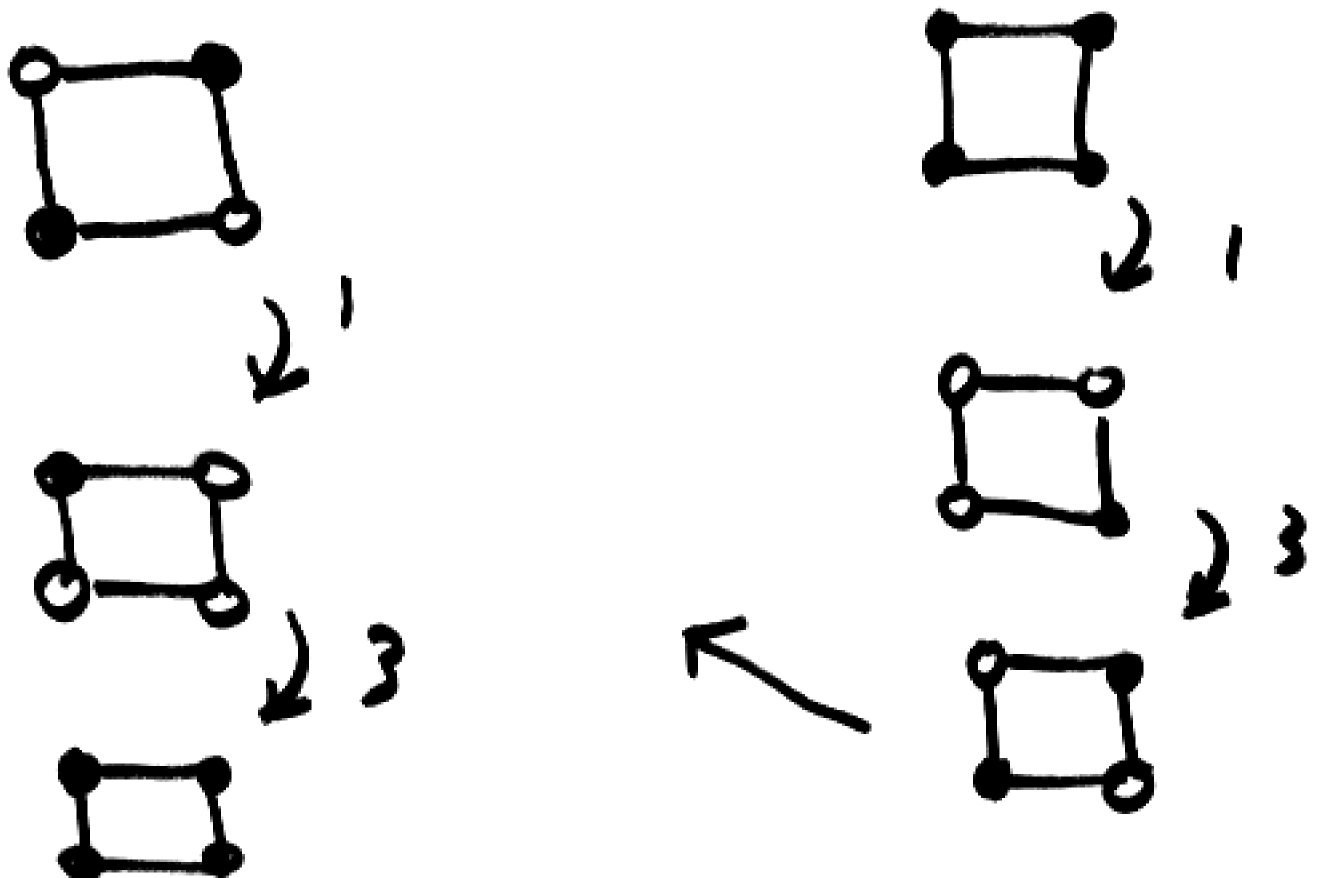
orig



cheat



Solution: press buttons 3 & 4.



- If A has an inverse then every initial state can be solved in only one way.

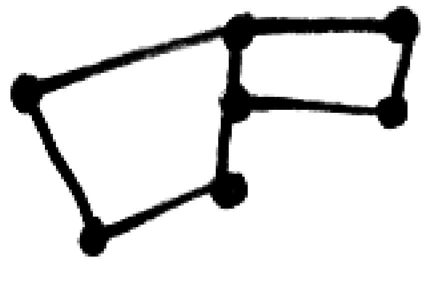
- If A has no inverse then some initial states will not be solvable. And when solvable there will be more than one way to do it.

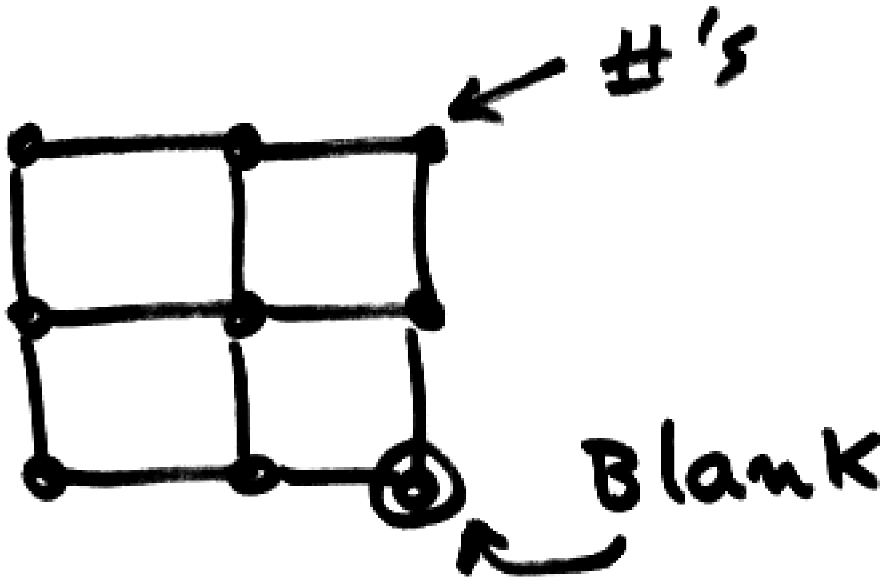
This dicothomy on A depends on how many states each light could be in.

general 15 puzzle

Can play it on a simple graph

↑
it has
no loops
or multiple
edges





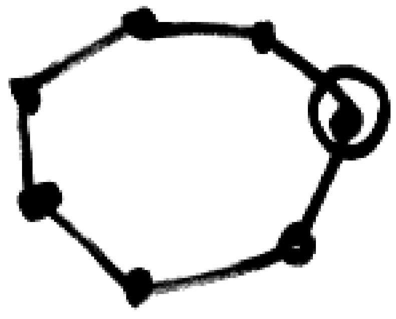
__Move__: Exchange Blank w/ nbh. #.

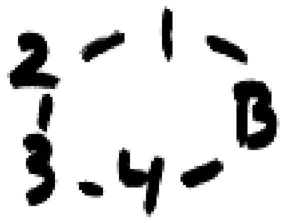
Move will take a blank on a grand tour of graph.

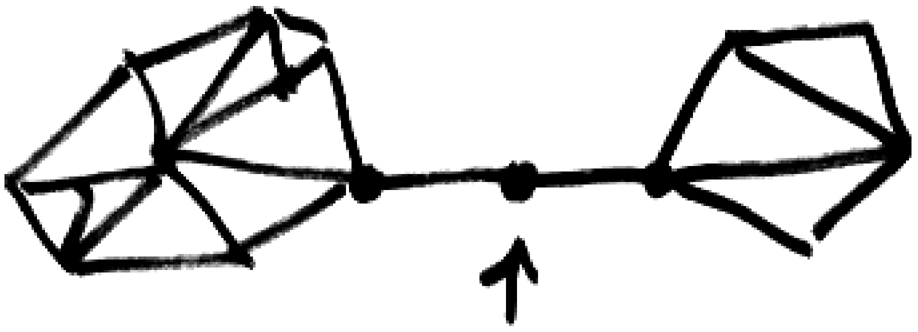
Each such path gives a permutation of the #'s.

__Question__: What permutations do we get?

Theorem of Wilson gives answer.

-  ⤳ cycle permutation



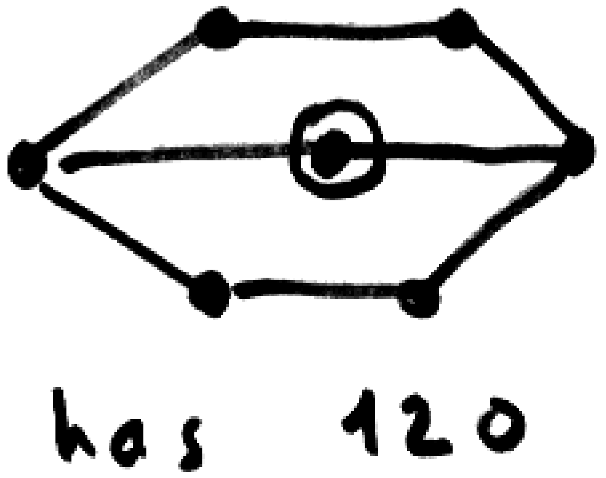
- 

take away this vertex graph disconnects

two separate puzzle one per component

In all other cases the group is either the symmetric group OR the alternating group (even permutation)

Except for this graph(!)



group has 120 elements.

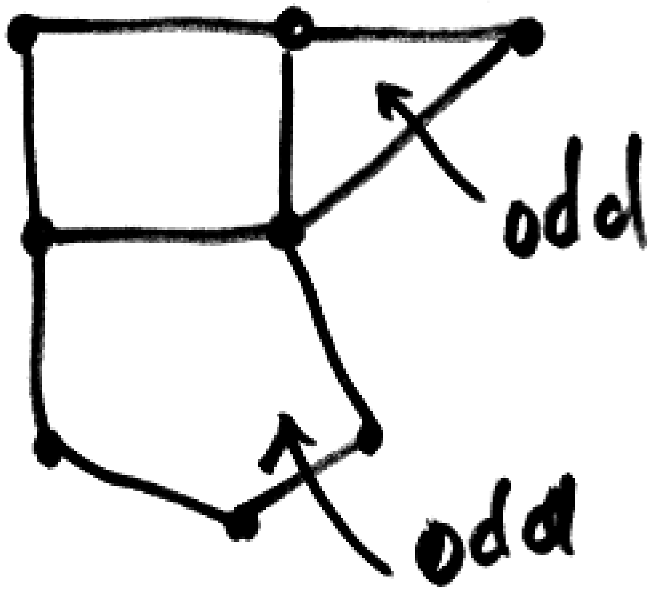
A priori our group could be as large as 6! = 720.

How can we tell symmetric/alternating apart?

$\rightsquigarrow$ 3-cycle
(even)

$\rightsquigarrow$ 4-cycle
(odd)

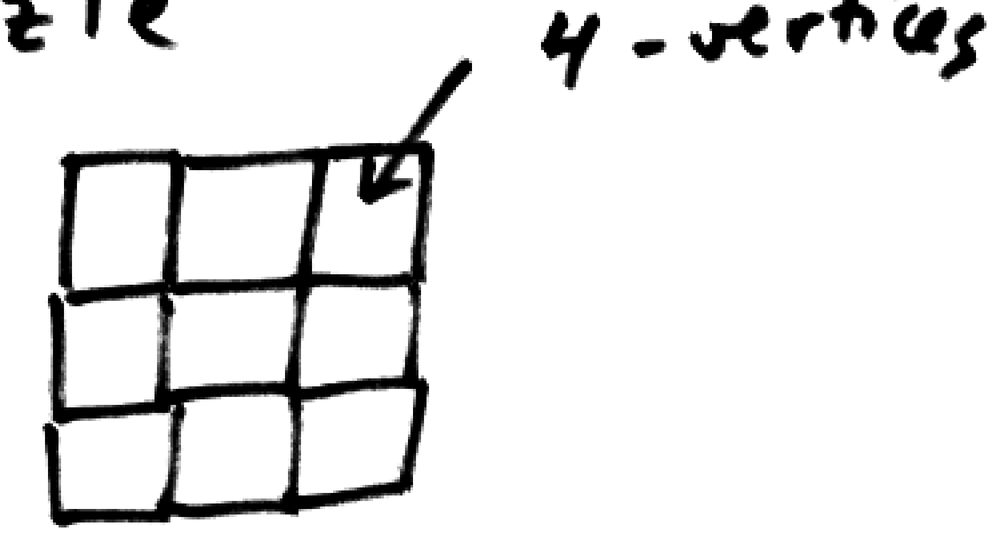
As soon as we have an k-cycle with k even the group is all permutations.



group $\simeq S_7$

A_n (even permutations only) $\longleftrightarrow$ all cycles in graph are even

E.g. 15-puzzle



$\rightsquigarrow$ group $\simeq A_8$

$\leftrightarrow$ graph bipartite

((color vertices ●/○ such that no two $\overset{\text{nbhs.}}{\text{vertices}}$ have same color