Bernoulli polynomials

§1 In his work *Ars conjectandi*, published posthomously in 1713, J. Bernoulli gave formulas for calculating the sum of k-th powers of consecutive integers. Namely,

$$1 + \cdots + 1 = n$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\vdots$$

The expressions on the right involve the numbers now known as Bernoulli numbers.

How do we get these formulas?

We can ask the following question: given $f \in \mathbb{Q}[X]$ is there an $F \in \mathbb{Q}[X]$ such that

(1) $$F(x+1) - F(x) = f(x) \quad ?$$

If so, we have

(2) $$\sum_{k=0}^{n-1} f(k) = F(n) - F(0)$$

We will show later that such an F always exists.
Note for now that F is defined only up to a constant (it is, after all, a sort of primitive of f) and

$\deg F = \deg f + 1$

$$\text{leading coeff } f = \text{leading coeff } F \cdot \deg F;$$

consequently, if f is monic $\frac{d}{dx}F$ is also monic, ~~[illegible]~~ of the same degree as f, and ~~[illegible]~~ unambiguously determined by f.

We define $B_n(x)$, the n^{th} Bernoulli polynomial, as $\frac{dF}{dx}$ for $f(x) = x^n$.

It will be convenient for us to use the language of operators. A linear operator on $\mathbb{Q}[x]$ will be a linear map $L : \mathbb{Q}[x] \to \mathbb{Q}[x]$ of $\mathbb{Q}$ vector spaces.

For example,

$D = \frac{d}{dx}$ differentiation, I = identity operator,

S translation by 1 : $Sf(x) = f(x+1)$.

Notice that any power series $\sum_{n \geq 0} a_n D^n$ where $D^0 = I$ with rational a_n's gives a meaningful operator on $\mathbb{Q}[x]$ since the sum

$$\sum_{n \geq 0} a_n D^n f = \sum_{n=0}^{\deg f} a_n D^n f$$

is a finite sum for each $f \in \mathbb{Q}[x]$.

For example, we have the operator

$$e^D = \sum_{n\geq 0} \frac{D^n}{n!} \quad \text{and} \quad e^{rD} = \sum_{n\geq 0} \frac{r^n D^n}{n!}$$

for any $r \in \mathbb{Q}$

In this context Taylor's theorem is equivalent to the equality of operators on $\mathbb{Q}[X]$

(3) $$e^D = S;$$

indeed

$$e^D f(x) = \sum_{k=0}^{\deg f} \frac{f^{(k)}(x)}{k!} \cdot 1^k = f(x+1)$$

(Taylor expansion of f about x).

So now equation (1) may be rewritten as

$$(S - I)F = f$$

and by (3) as

$$(e^D - I)F = f$$

We can now solve for DF

(4) $$DF = \frac{D}{e^D - I} f$$

The right hand side does in fact make sense since $\frac{D}{e^D - I}$ is a power series $a_0 + a_1 D + \dots$ with rational coefficients; we conclude that DF (and hence also F) exists as we had claimed.

The expansion

(5) $$\frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$$

where t is a variable defines rational numbers B_n; these are the Bernoulli numbers.

Note that $B_0 = 1$ and in general $B_n \in \mathbb{Q}$.

In particular, if $f(x) = x^n$ we find

$$B_n(x) = DF = \sum_{k\geq 0} B_k \frac{D^k}{k!} x^n$$

(6) $$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\, X^{n-k}$$

and hence

(7) $$B_n(0) = B_n$$

We find that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{-1}{30}$

It is not hard to see from (5) that $B_n = 0$ for n odd $n > 1$.

We will prove later (in (11) p. 7) that $DB_n = nB_{n-1}$ and therefore

$$\sum_{k=0}^{m-1} k^n = \frac{1}{n+1}\left(B_{n+1}(m) - B_{n+1}\right),$$

which are the formulas of J. Bernoulli.

§2 Multiplication theorem

The Bernoulli polynomials satisfy many properties ~~and~~ ~~[illegible]~~ some of which can be used as alternative definitions for them. We will concentrate on the following, sometimes known as multiplication theorem, due to Raabe (1851).

(9) For any integers $m \geq 1$ and $n \geq 0$

$$\frac{1}{m} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) = m^{-n} B_n(mx)$$

Again we will use the language of operators. For $m \geq 1$ let

$$R_m f(x) = \frac{1}{m} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)$$

We may reformulate (9) as saying that B_n is an eigenvector of R_m with eigenvalue m^{-n}; i.e.:

(10) $$R_m B_n = m^{-n} B_n$$

Notice that R_m preserves degrees and in the basis $1, x, \ldots, x^n$, of polynomials of degree at most n, it has a matrix which is upper triangular and with entries $1, m^{-1}, m^{-2}, \ldots, m^{-n}$ along the diagonal.

Hence if we fix $m > 1$ we deduce that there is a sequence of monic polynomials $A_0, A_1, \ldots$ with $\deg A_n = n$ such that

$$R_m A_n = m^{-n} A_n$$

i.e: the A_n's are eigenvectors of R_m and diagonalize the action of R_m on $\mathbb{Q}[x]$.
The multiplication theorem for the Bernoulli polynomials (10) then says that $A_n = B_n$ for all n. (indep of m)
a fact which will prove later.

Many properties of B_n can be deduced from the equality $A_n = B_n$ and the following lemma

<u>Lemma</u> Let E be a nonzero operator on $\mathbb{Q}[x]$ such that

$$E R_m = e R_m E$$

for some $e \in \mathbb{Q}^\times$ then

$e = m^{-k}$ for some $k \in \mathbb{N}$ and ~~[illegible]~~ for some $c_n \in \mathbb{Q}$ not all zero

(and conversely) $$E A_n = c_n A_{n-k} \quad \text{for all } n \geq 0$$ where we set $A_\ell = 0$ if $\ell < 0$

<u>Pf</u>: Pick A_n such that $E A_n \neq 0$ (E is nonzero and the A_n's form a basis of $\mathbb{Q}[x]$)

then

$$(E R_m) A_n = m^{-n} E A_n$$

on the other hand

$$(E R_m) A_n = e (R_m E) A_n$$

So $\quad R_m (E A_n) = \frac{m^{-n}}{e} E A_n$

hence $\quad e^{-1} = m^{k}$ for some $k \in \mathbb{N}$, $k \le n$ and

$$E A_n = c_m A_{n-k} \quad \text{for some } c_m \in \mathbb{Q} \quad \square$$

For example, consider $E = D = \frac{d}{dx}$ then

$\frac{1}{m} R_m D = D R_m$ by the chain rule

hence by comparing leading coefficients

(11) $$D A_n = n A_{n-1}$$

Proposition

For all $m, l \in \mathbb{N}$ we have

$$R_m R_l = R_l R_m = R_{ml}$$

PF:
$$\begin{aligned} R_m (R_l f)(x) &= \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{l} \sum_{j=0}^{l-1} f\left(\frac{\frac{x+k}{m} + j}{l} \right) \\ &= \frac{1}{ml} \sum_{k=0}^{m-1} \sum_{j=0}^{l-1} f\left(\frac{x+k+jm}{lm} \right) \\ &= \frac{1}{ml} \cdot \sum_{r=0}^{ml-1} f\left(\frac{x+r}{ml} \right) \\ &= R_{ml} f(x) \quad \square \end{aligned}$$

Using the lemma with $E = R_\ell$ we find

$$R_\ell A_n = \ell^{-n} A_n \qquad \text{for all } \ell \in \mathbb{N}$$

by comparing leading coefficients.

We may extend the definition of R_ℓ to all nonzero integers by setting

$$R_{-1} f(x) = f(1-x)$$

and $\quad R_{-\ell} = R_{-1} R_\ell \qquad \ell \in \mathbb{N}$

It is then easy to check that the proposition holds in general and by the lemma we deduce

(12) $$A_n(1-x) = (-1)^n A_n(x)$$

We see then that the A_n's are eigenvectors for all operators $R_{\pm\ell}$ $\ell \in \mathbb{N}$.

<u>Proposition</u> $A_n = B_n$ for all $n \geq 0$

<u>Pf:</u> We assume $n \geq 1$ since $A_0 = B_0 = 1$.

We may proceed in two different ways

1) ~~Starting~~ We start with the identity

$$R_m A_n = m^{-n} A_n$$

and replace x by mx; i.e:

$$\frac{1}{m} \sum_{k=0}^{m-1} A_n\left(x + \frac{k}{m}\right) = m^{-n} A_n(mx)$$

Now we let $m \to \infty$. The left hand side converges to

$$\int_x^{x+1} A_n(x)\, dx$$

since it is a Riemann sum for this integral. The right hand side converges to x^n since A_n is a monic ~~poly~~nomial of degree n.

Hence if $F_n \in \mathbb{Q}[X]$ is a primitive for A_n ~~for~~

~~polynomial~~

we find

$$F_n(x+1) - F_n(x) = x^n$$

and this gives by definition $B_n = A_n$

2) We ~~can~~ prove that B_n is an eigenfunction for some R_m with $m > 1$. By definition

$$B_n(x) = \frac{D}{e^D - 1} x^n$$

So

$$m^n B_n(x) = \frac{D}{e^D - I} (mx)^n$$

Applying R_m

$$R_m(m^n B_n(x)) = R_m\left(\frac{D}{e^D - I}\right)(mx)^n$$

$$(*) \quad = \frac{mD}{e^{mD} - I} R_m (mx)^n$$

$$= \frac{mD}{e^{mD} - I} \cdot \frac{1}{m} \sum_{k=0}^{m-1} S^k x^n$$

$$= \frac{mD}{e^{mD} - I} \frac{1}{m} \frac{e^{mD} - 1}{e^D - 1} x^n$$

$$= \frac{D}{e^D - I} x^n = B_n(x) \quad \square$$

(*) since $R_m D = mDR_m$ as we noted earlier it follows that for any power series $a_0 + a_1 D + \cdots$ $R_m(a_0 + a_1 D + \cdots) = (a_0 + a_1 mD + \cdots) R_m$, and in particular

$$R_m \frac{D}{e^D - I} = \frac{mD}{e^{mD} - I} R_m$$

By (12) $B_n(1) = (-1)^n B(0) = (-1)^n B_n$ by (7)

By (6) $(-1)^n B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$

giving recursion for the Bernoulli numbers.

§3 Fourier expansion

The following proposition gives another way of looking at the operators R_m. We consider the space of continuous functions on the interval $[0,1]$, which we extend to periodic functions on all of $\mathbb{R}$. Notice that the operators R_m make sense on this larger space.

<u>Proposition</u> For f, g continuous on $[0,1]$ extended periodically to all of $\mathbb{R}$ we have

$$\int_0^1 R_m f(x) \cdot g(x)\, dx = \int_0^1 f(x)\, g(mx)\, dx$$

for all $m \geq 1$.

<u>Pf</u>: The left hand side equals

$$\frac{1}{m} \sum_{k=0}^{m-1} \int_0^1 f\left(\frac{x+k}{m}\right) g(x)\, dx$$

doing the change of variables $y = \frac{x+k}{m}$ on each integral we get

$$= \sum_{k=0}^{m-1} \int_{k/m}^{\frac{k+1}{m}} f(y)\, g(my+k)\, dy$$

now g is periodic so $g(my+k) = g(my)$ and reassembling the integral we get

$$\int_0^1 f(y) g(my)\, dy$$

$\square$

This it is easily seen to extend to R_m for $m \leq -1$

We can reformulate this result as saying that the operator $R_m^*: g(x) \mapsto g(mx)$ is the adjoint to R_m under the inner product given by $(f,g) = \int_0^1 f(x) g(x)\, dx$.

From this fact it is not difficult to see again that the R_m's commute with each other since the R_m^*'s clearly do.

Let us now consider $g(x) = e^{2\pi i n x}$ in the proposition. We define the n^{th} Fourier coefficient of a function as above by the formula

$$\hat{f}(n) = \int_0^1 f(x) e^{2\pi i n x}\, dx$$

Then

$$\begin{aligned}(\widehat{R_m f})(n) &= \int_0^1 R_m f(x) e^{2\pi i n x}\, dx \\ &= \int_0^1 f(x) e^{2\pi i n m x}\, dx \\ &= \hat{f}(nm)\end{aligned}$$

Hence if f is the restriction of the polynomial B_n for $n \geq 1$ to the interval $[0,1]$ we have

$$\hat{B}_n(m) = (\widehat{R_m B_n})(1)$$

but $\quad R_m B_n = m^{-n} B_n$

hence

$$\hat{B}_n(m) = m^{-n}\,\hat{B}_n(1)$$

and in order to compute all Fourier coefficients of B_n we only need to compute $\hat{B}_n(1)$.

Once again let us introduce an operator.
For f continuous on $[0,1]$ let

$$\delta f = f(1) - f(0^+) \qquad \text{(the jump at } 0\text{)}$$

It is now a simple matter to check that

$$\delta R_m = \frac{1}{m} R_m\, \delta \qquad \text{for all } m \geq 1.$$

So by the lemma in p. 6

$$\delta B_n = c_n B_{n-1} \qquad \text{for some } c_n \in \mathbb{Q},$$

but δB_n is a constant and hence $c_n = 0$ for all $n > 1$.
$B_0 = 1$ and $B_1 = x - 1/2$ hence

$$\delta B_0 = 0 \quad \text{and} \quad \delta B_1 = 1$$

(This also follows from (12) and the vanishing of the numbers B_n for n odd $n > 1$)

Back to our calculation of $\hat{B}_n(1)$, if we integrate by parts we get

$$\hat{B}_n(1) = \int_0^1 B_n(x)\, e^{2\pi i x}\, dx \qquad \text{(integrating by parts)}$$

$$= \frac{1}{2\pi i} \delta B_n - \frac{1}{2\pi i} \int_0^1 B_n'(x)\, e^{2\pi i x}\, dx$$

by (11) $$= \frac{1}{2\pi i} \delta B_n - \frac{n}{2\pi i} \hat{B}_{n-1}(1)$$

Now $\hat{B}_0(1) = 0$. and $\delta B_1 = 1$ so by induction

$$\hat{B}_n(1) = \frac{n!}{(2\pi i)^n} (-1)^{n-1}$$

Therefore, we have proved

(13) $$\hat{B}_n(m) = m^{-n} \frac{n!}{(2\pi i)^n} (-1)^{n-1}$$ for all $n \geq 1$ and all $m \in \mathbb{Z} \setminus 0$

It is not hard to check that $\hat{B}_n(0) = 0$ unless $n = 0$

By the theory of Fourier series we find

(14) $$B_n(x) = \frac{(-1)^{n-1}\, n!}{(2\pi i)^n} \sum_{m \in \mathbb{Z} \setminus \{0\}} m^{-n} e^{2\pi i m x}$$

for $n > 1$, $0 \leq x \leq 1$

(A similar formula holds for $n = 1$ and $0 < x < 1$ and the sum interpreted as $\lim\limits_{N \to \infty} \sum\limits_{m=-N}^{N} m^{-1} e^{2\pi i m x}$)

In particular, if $x = 0$ and n is even then

$$B_{2k} = \frac{2\,(2k)!}{(2\pi)^{2k}} (-1)^{k-1} \sum_{m=1}^{\infty} m^{-2k}$$

or

(15) $$\zeta(2k) = \frac{(2\pi)^{2k}}{2\,(2k)!} (-1)^{k-1} \cdot B_{2k}$$