CENTRAL VALUES OF HECKE L-FUNCTIONS OF CM NUMBER FIELDS

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0. Introduction. It is well known that the zeta function of CM (complex multiplication) abelian varieties can be given in terms of L-functions of associated Hecke characters. In this paper, we prove a formula expressing the central special value of the L-function of certain Hecke characters in terms of theta functions. The formula easily implies that the central value is nonnegative and yields a criterion for its positivity. Combining this criterion with the work of Arthaud and Rubin, we show that certain CM elliptic curves have Mordell-Weil rank zero over their field of definition.

Let F be a totally real number field of degree t and let μ be a quadratic Hecke character of F of conductor $\mathfrak{f}$ such that $(2\mathcal{O}_F, \mathfrak{f}) = 1$. Given a CM extension E of F, we consider the twist $\chi = \chi_{\mathrm{can}}\tilde{\mu}$ of μ by a "canonical" Hecke character χ_{can} of E (see Section 2), where $\tilde{\mu} = \mu \circ N_{E/F}$, as well as its odd powers χ^{2k+1}, $k \in \mathbb{Z}_{\geq 0}$. Consider the following condition:

$(*)$ All units of E are real and every prime of F dividing $2\mathfrak{f}$ is split in E/F.

Our main result is the following.

THEOREM 0.1 (Sketch of Theorem 2.5). *Assume that F has ideal class number* 1 *and $(-1)^{kt}\mu_\infty(-1) = 1$, where $\mu_\infty : (F \otimes \mathbb{R})^* \longrightarrow \mathbb{C}^*$ is the infinite part of μ. Then there is an explicit theta function $\theta_{\mu,k}$ over F, depending only on μ and k, such that for every CM quadratic extension E of F satisfying the condition $(*)$, the central L-value*

$$L\bigl(k+1, \chi^{2k+1}\bigr) = \kappa \left| \sum_{C \in \mathrm{CL}(E)} \frac{\theta_{\mu,k}(\mathfrak{A})}{\chi^{2k+1}(\bar{\mathfrak{A}})} \right|^2. \tag{0.1}$$

Here, κ is an explicit positive number, $\mathfrak{A} \in C^{-1}$ is a primitive ideal relatively prime to $2\mathfrak{f}$, and $\theta_{\mu,k}(\mathfrak{A})$ is essentially the value of a theta function $\theta_{\mu,k}$ at a CM point in E associated to $\mathfrak{A}^2$.

We emphasize that $\theta_{\mu,k}$ is *independent* of the CM field E, which is one of the

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reasons we view χ as a twist of μ by a canonical Hecke character of E. We also remark that under the hypothesis of the theorem, the root number of χ^{2k+1} is 1.

As a corollary to the theorem, we deduce that under certain conditions on the ideal class group of E, the central L-value vanishes if and only if all the terms $\theta_{\mu,k}(\mathfrak{A})$ in the theorem vanish (Theorem 2.6). Using this criterion, we obtain, for example, the following result.

For a prime $p \equiv 7 \bmod 8$ and a squarefree integer $d \equiv 1 \bmod 4$, let $A(p)^d$ be the CM elliptic curve with CM by $E = \mathbb{Q}(\sqrt{-p})$, studied by Gross in his thesis (see [Gr]). Let h be the class number of E, and χ be one of the h Hecke characters of E associated to $A(p)^d$. Then we have the following theorem.

THEOREM 0.2. *Assume that every prime divisor of d splits in E. There is a constant M depending only on k such that if $p > Md^4$, $\operatorname{sign}(d) = (-1)^k$, and $(2k+1, h) = 1$, then the central L-value $L(k+1, \chi^{2k+1}) > 0$. For $k = 0, 1$, we may take $M = .081, .206$, respectively.*

Combining this with an unpublished result of Arthaud, later generalized by Rubin [Ru], we obtain the following corollary.

COROLLARY 0.3. *Assume $d \equiv 1 \bmod 4$, $d > 0$, and every prime factor of d is split in E. Then for all $p > .081d^4$, the elliptic curve $A(p)^d$ has Mordell-Weil rank zero over its field of definition.*

We made no effort to optimize the constant M in 0.2, and our result can surely be improved. The prime p can be replaced by a positive squarefree integer $D \equiv 7 \bmod 8$. It would certainly be interesting to lower the power of d in the statement. We should also point out that the hypothesis $(2k+1, h) = 1$ in 0.2 is crucial to our proof (it guarantees that there is only one Galois orbit of Hecke characters); and indeed, as pointed out in [RV2], $L(\chi^3, 2) = 0$ when $p = 59$ and $d = 1$.

The Hecke characters considered here and their associated CM abelian varieties were studied by Shimura (see [S, Section 7.8], [S2], and [S3]), who also discussed their special values. The terminology “canonical” is due to Rohrlich (see [Roh2], [Roh3], [Roh4], and Section 2).

The main formula of this paper has its root in [RV] (and its sequel [RV2], [RVZ], and [RVZ2]) and is a variation of [Y, Theorem 0.7].

Special cases of Theorem 0.2 have been previously proven by Montgomery and Rohrlich [MRoh], [Roh2], [Roh3] and the first author [RV2]. Corollary 0.3 was proved by Gross [Gr], using descent theory, in the case $d = 1$. The split condition in Theorem 0.2 and Corollary 0.3 is not essential and can be dropped. We refer to [Y4] for details.

By a result of Rogawski ([Ro]; see also [Y]), the nonvanishing of the central L-value is equivalent to the nonvanishing of certain global theta lifting.

The idea to derive the main formula (0.1) is as follows. We have the formula of [Y] expressing the central L-value as the absolute square of some theta integral on a unitary group of one variable. The task is simply to compute the theta integral. In

Section 1, we break the integral into local integrals and compute these in the split case and the infinite case. For a finite place of F that does not split in E, the calculation amounts to finding an eigenfunction of the Weil representation of the unitary group mentioned above. A general explicit construction of such eigenfunctions is given in [Y3], [Y4]. For the characters considered in this paper, the eigenfunctions we need turn out to be very simple (Lemmas 2.2 and 2.3). Putting things together, we find that the theta integral is the sum of the special values of certain theta series at CM points in E (Theorem 2.4). Now assume that F has ideal class number 1. Then the theta series involved become the same and, independent of the CM fields E, we get the main formula. To obtain the nonvanishing results, we use a theorem of Shimura concerning special values of L-functions of conjugate Hecke characters (the trick goes back to Rohrlich; see [Roh2], [Roh3], [MRoh], [RV2]). In Section 3, we specialize to the case $F = \mathbb{Q}$ and obtain a more explicit form of the main formula (Theorem 3.2) and the results mentioned above.

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Notation. In this paper, F is always a totally real number field, and E is always a "generic" quadratic CM extension of F in the sense of Rohrlich [Roh4, p. 519]; that is, $\mathbb{O}_E^* = \mathbb{O}_F^*$, and the natural map $\mathrm{CL}(F) \longrightarrow \mathrm{CL}(E)$ is injective. We further assume that every prime of F above 2 is split in E to avoid technical complications at 2. We fix a "canonical" additive character $\psi = \prod \psi_v$ of $F_{\mathbb{A}}/F$ as follows:

$$\psi_v(x) = \begin{cases} e^{2\pi i x} & \text{if } v \text{ is real,} \\ e^{-2\pi i \lambda(x)} & \text{if } v \mid p \text{ is finite,} \end{cases} \tag{0.2}$$

where $\lambda : F_v \xrightarrow{\mathrm{tr}} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Of course, every additive character of $F_{\mathbb{A}}/F$ has the form $a\psi : x \mapsto \psi(ax)$ for some $a \in F$. Let $\psi_E = \psi \circ \mathrm{tr}$ and let $n(\psi_v)$ be the local conductor of ψ_v. On F_v or E_v, we take the Haar measure self-dual with respect to ψ_v or ψ_{E_v}, respectively.

For the sake of simplicity, we assume that there is a $\delta \in E^*$ with $\Delta = \delta^2 \in F^*$ such that

$$\mathrm{ord}_v \Delta = \begin{cases} 1 & \text{if } v \text{ is ramified in } E/F, \\ 0 & \text{if } v \text{ is unramified in } E/F. \end{cases} \tag{0.3}$$

Such a δ always exists if F has ideal class number 1. (Recall that we have assumed that every prime factor of 2 in F splits in E.)

Let E^1 be the norm-1 subgroup of E. Given a character η of $[E^1] = E^1\backslash E^1_{\mathbb{A}}$, let $\tilde{\eta}$ be the Hecke character of E given by $\tilde{\eta}(z) = \eta(z/\bar{z})$. On E^1_v, we take the Haar measure such that $\operatorname{meas}(E^1_v) = 1$ if v is nonsplit and $\operatorname{meas}(\mathbb{O}^*_v) = 1$ if v is split in E/F. In the latter case, we have identified E^1_v with F^*_v via $(z, z^{-1}) \mapsto z$. On $E^1_{\mathbb{A}}$, we take the Haar measure such that $\operatorname{meas}([E^1]) = 1$. Then the Tamagawa number of the algebraic group E^1 is characterized by the identity

$$\int_{[E^1]} f(g)\,dg = \operatorname{Tam}\left(E^1\right) \prod \int_{E^1_v} f_v(g)\,dg \tag{0.4}$$

for every Schwartz function $f = \prod f_v \in S([E^1])$.

Let χ be a fixed Hecke character of E whose restriction to $F^*_{\mathbb{A}}$ is the quadratic Hecke character $\epsilon = \epsilon_{E/F}$ of $F^*_{\mathbb{A}}$ associated to E/F by class field theory. Then every such Hecke character of E is of the form $\chi\tilde{\eta}$ for some character η of $[E^1]$. If the global root number of $\chi\tilde{\eta}$ is 1, there is $\alpha \in F^*$ (unique up to norm from E^*) such that

$$\prod_{w|v} \epsilon\left(\frac{1}{2}, \left(\chi\tilde{\eta}\right)_w, \frac{1}{2}\psi_{E_w}\right) \left(\chi\tilde{\eta}\right)_w(\delta) = \epsilon_v(\alpha). \tag{0.5}$$

Here, the product takes over places w of E above v, and $\epsilon(1/2, (\chi\tilde{\eta})_w, (1/2)\psi_{E_w})$ are Tate's local root numbers. We write, in short,

$$\epsilon\left(\frac{1}{2}, \left(\chi\tilde{\eta}\right)_v, \frac{1}{2}\psi_{E_v}\right) = \prod_{w|v} \epsilon\left(\frac{1}{2}, \left(\chi\tilde{\eta}\right)_w, \frac{1}{2}\psi_{E_w}\right).$$

For a datum (χ, η, α) satisfying (0.5), we define

$$\begin{aligned} S_{\text{sp}} &= \{v : E/F \text{ is split at } v\}, \\ S_{\text{in}} &= \{v \nmid \infty : E/F \text{ is inert at } v\}, \\ S_{\text{ra}} &= \{v \nmid \infty : E/F \text{ is ramified at } v\}, \\ S_{\chi\tilde{\eta}} &= \{v \nmid \infty : \chi\tilde{\eta} \text{ is ramified at } v\}. \end{aligned}$$

When E/F is split at a place v of F, we choose a place w of E above v and write $\delta = (x_v, -x_v) \in E^*_v = E^*_w \oplus E^*_{\bar{w}} = (F^*_v)^2$. We define

$$n_v = \begin{cases} n\left(x_v^3\alpha\psi_v\right) & \text{if } v \in S_{\text{sp}}, \\ \left[\dfrac{n\left((\delta\alpha/4)\psi_{E_v}\right) + 1}{2}\right] & \text{if } v \in S_{\text{in}} \cup S_{\text{ra}}. \end{cases} \tag{0.6}$$

Here $[x]$ means the integral part of a real number x. For $v \in S_{\chi\tilde{\eta}}$, we define k_v to be the smallest integer k such that both χ_w and $\tilde{\eta}_w$ are trivial on $1 + \pi_w^k \mathbb{O}_w$, where w is a place of E above v. For other finite places v, we take $k_v = 0$. Let Φ be a CM type of E, and identify infinite places of F with Φ. For $v=\sigma\in\Phi$, there is an integer m_σ such that

$$\left(\chi\tilde{\eta}\right)_\sigma(z)=\left(\frac{|z|}{z}\right)^{2m_\sigma+1}.$$

We define

$$k_\sigma=m_\sigma \operatorname{sign}\left(\operatorname{Im}\left(\sigma(\delta\alpha)\right)\right)-\frac{1-\operatorname{sign}\left(\operatorname{Im}\left(\sigma(\delta\alpha)\right)\right)}{2}. \tag{0.7}$$

The condition (0.5) implies that $k_\sigma\geq 0$. In particular, if $m_\sigma\geq 0$, then $\operatorname{Im}(\sigma(\delta\alpha))>0$ and $k_\sigma=m_\sigma$. For technical reasons, we assume throughout this paper (one can always chose α to satisfy both (0.5) and (0.8)) that

$$n\left(\frac{x_v^3\alpha}{4}\psi_v\right)\leq 0 \quad \text{for every } v\in S_{\mathrm{sp}}\text{; that is, } n_v\leq -2\operatorname{ord}_v 2. \tag{0.8}$$

Finally, we normalize the classical Hermite functions as follows: Let

$$\phi^0=e^{-\pi x^2}\in S(\mathbb{R}),\qquad \phi^k=\frac{1}{2^k}\left(x-\frac{1}{2\pi}\frac{d}{dx}\right)^k\phi^0(x) \tag{0.9}$$

be the kth Hermite function for $k\geq 0$. Then

$$i^k\hat{\phi^k}=\phi^k, \tag{0.10}$$

where

$$\hat{\phi}(x)=\int_{\mathbb{R}}\phi(y)e^{-2\pi ixy}\,dy$$

is the Fourier transform of ϕ. One has

$$\left\langle\phi^k,\phi^k\right\rangle=\frac{1}{\sqrt{2}}\frac{k!}{(4\pi)^k}. \tag{0.11}$$

The following properties of ϕ^k are well known:

$$\phi^{k+1}=x\phi^k-\frac{k}{4\pi}\phi^{k-1},\qquad \frac{d}{dx}\phi^k=k\phi^{k-1}-2\pi x\phi^k. \tag{0.12}$$

Notice that there is a unique polynomial $H_k(x)$ of degree k (the Hermite polynomials) such that

$$\phi^k(x)=H_k(x)e^{-\pi x^2}. \tag{0.13}$$

It is easy to check that $H_0=1$ and $H_1=x$. In general, H_k has the same parity as k.

Given positive numbers a and $b>0$, we define for the purpose of this paper

$$\phi^k_{a,b}(x) = \phi^k\left(\sqrt{a^3 b x}\right). \tag{0.14}$$

Note

$$\left\langle \phi^k_{a,b}, \phi^k_{a,b} \right\rangle = \frac{1}{\sqrt{2a^3 b}} \frac{k!}{(4\pi)^k}. \tag{0.15}$$

1. General setting. For the first two lemmas, we change the notation temporarily. Let F be a p-adic field with the ring of integers $\mathbb{O}$, and let π be a uniformizer of F. Let ψ be an arbitrary nontrivial additive character of F, and let $n = n(\psi)$ be its conductor, by which we mean the smallest integer n such that $\psi|_{\pi^n\mathbb{O}} = 1$. So ψ is unramified if and only if $n \le 0$. Let χ be a character of F^* and let $m = n(\chi)$ be its conductor, that is, the smallest integer m such that χ is trivial on $1+\pi^m\mathbb{O}$. Define

$$I(\psi, x) = \text{meas}(\mathbb{O})^{-1} \int_{\mathbb{O}} \psi(xy)\psi\left(y^2\right) dy \tag{1.1}$$

to be the Fourier transform of the function $\psi(x^2)\,\text{char}(\mathbb{O})(x)$ with respect to ψ. We also define

$$I(\psi, \chi) = \text{meas}(\mathbb{O}^*)^{-1} \int_{\mathbb{O}^*} \chi(x)\psi(x)\, d^*x. \tag{1.2}$$

Both $I(\psi, x)$ and $I(\psi, \chi)$ are independent of the choice of the Haar measure on F or F^*, respectively.

LEMMA 1.1. *Let the notation be as above. Then the following hold.*

(1) *If $p \ne 2$, then*

$$I(\psi, x) = \begin{cases} \text{char}(\pi^n\mathbb{O})(x) & \text{if } n \le 0, \\ q^{-(n/2)}\psi\left(-\frac{1}{4}x^2\right)\gamma^n\,\text{char}(\mathbb{O})(x) & \text{if } n > 0. \end{cases}$$

Here

$$\gamma = \gamma\left(\pi^{n-1}\psi\right) = \frac{\sum_{a\in(\mathbb{O}/\pi\mathbb{O})} \psi\left(\pi^{n-1}a^2\right)}{\sqrt{q}}$$

is the Weil index of the character $\tilde{\psi}$ of $\mathbb{O}/\pi\mathbb{O}$ induced by $\pi^{n-1}\psi$ (see [We], [RR, Appendix]).

(2) *If $p = 2$ and $n(\psi) \le 0$, then*

$$I(\psi, x) = \text{char}(\pi^n\mathbb{O})(x).$$

LEMMA 1.2. *Assume that either χ or ψ is ramified. One has*

$$I(\psi, \chi) = \begin{cases} 0 & \text{if } n \ne m, \\ q^{-n}\left(1-q^{-1}\right)^{-1} G(\psi, \chi) & \text{if } n = m. \end{cases}$$

Here

$$G(\psi,\chi)=\sum_{a\in(\mathcal{O}/\pi^n\mathcal{O})^*}\psi(a)\chi(a)$$

is the Gauss sum of χ *and* ψ.

Now we switch back to the notation of the introduction. First we have the following proposition.

PROPOSITION 1.3. *Let* $f=\prod f_v$ *be an integrable function on* $[E^1]$ *such that* $f_v(-x)=f_v(x)$ *whenever* E/F *is ramified at* v. *Let* $\mathrm{CL}(E)$ *be the ideal class group. Let* $\mathrm{CL}^-(E)=\mathrm{CL}(E)/\mathrm{CL}(F)$ *and let* $h^-=\#\mathrm{CL}^-(E)$ *be the relative ideal class number of* E/F. *Then*

$$\int_{[E^1]}f(g)\,dg=\frac{1}{h^-}\sum_{C\in\mathrm{CL}^-(E)}I_C(f), \tag{1.3}$$

where

$$I_C(f)=\prod I_{v,C}(f),$$

and

$$I_{v,C}(f)=\begin{cases}\int_{E_v^1}f(g)\,dg & \text{if } v \text{ is nonsplit,}\\ \int_{\mathcal{O}_v^*}f\left(ga_wa_{\bar{w}}^{-1}\right)dg & \text{if } v=w\bar{w} \text{ is split.}\end{cases} \tag{1.4}$$

Here $a=(a_w)\in E_{\mathbb{A}}^*$ *corresponds to an ideal in* C^{-1}, *and the Haar measures are normalized as in the introduction. In particular, the Tamagawa number of* E^1, $\mathrm{Tam}(E^1)$, *equals* $2^s/h^-$, *where* s *is the number of finite places of* F *ramified in* E/F.

Notice that $I_{v,C}(f)$ is independent of C when v is nonsplit.

Proof. Let

$$U_E=\prod_{v\nmid\infty}\mathcal{O}_{E_v}^*\times\prod_{\sigma\in\Phi}\mathbb{C}^*,$$
$$U_F=\prod_{v\nmid\infty}\mathcal{O}_v^*\times\prod_{\sigma\in\Phi}\mathbb{R}^*,$$

and let $U^1=\lambda(U_E)$ be a subgroup of $E^1(\mathbb{A})$, where $\lambda(z)=z/\bar{z}$. Since E is generic, the natural map $\mathrm{CL}(F)\longrightarrow\mathrm{CL}(E)$ is injective and $E^*\cap F_{\mathbb{A}}^*=F^*$. So one has a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_F & \longrightarrow & U_E & \longrightarrow & U^1 & \longrightarrow & 1\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & [F^*] & \longrightarrow & [E^*] & \longrightarrow & [E^1] & \longrightarrow & 1. \end{array}$$

By the snake lemma, one obtains the exact sequence

$$1 \to \mathbb{O}_F^* \to \mathbb{O}_E^* \to E^1 \cap U^1 \to F^* U_F \backslash F_{\mathbb{A}}^* \to E^* U_E \backslash E_{\mathbb{A}}^* \to E^1 U^1 \backslash E_{\mathbb{A}}^1 \to 1.$$

Since E is generic, $\mathbb{O}_F^* = \mathbb{O}_E^*$ and $F^* U_F \backslash F_{\mathbb{A}}^* \hookrightarrow E^* U_E \backslash E_{\mathbb{A}}^*$. So $E^1 U^1 \backslash E_{\mathbb{A}}^1 \cong \mathrm{CL}^-(E)$, and $E^1 \cap U^1 = 1$. Therefore $\coprod_{C \in \mathrm{CL}^-(E)} U^1 a_C^{-1} \bar{a}_C$ is a fundamental domain for $[E^1]$, where $a_C \in E_{\mathbb{A}}^*$ corresponds to an ideal in C^{-1}. Notice that

$$U^1 = \prod_{v \in S_{\mathrm{ra}}} \lambda\big(\mathbb{O}_{E_v}^*\big) \times \prod_{v \in S_{\mathrm{in}} \cup \Phi} E_v^1 \times \prod_{v \in S_{\mathrm{sp}}} \mathbb{O}_v^*,$$

and $E_v^1 = \{\pm 1\} \times \lambda(\mathbb{O}_{E_v}^*)$ when $v \in S_{\mathrm{ra}}$. Since $f_v(-x) = f_v(x)$ for $v \in S_{\mathrm{ra}}$, one has by (0.4),

$$\begin{aligned}\int_{[E^1]} f(g)\,dg &= \sum_{C \in \mathrm{CL}^-(E)} \int_{U^1 a_C^{-1} \bar{a}_C} f(g)\,dg \\ &= \sum \mathrm{Tam}\big(E^1\big) 2^{-s} \prod_{v \notin S_{\mathrm{sp}}} \int_{E_v^1} f_v(g)\,dg \prod_{v \in S_{\mathrm{sp}}} \int_{\mathbb{O}_v^* a_w^{-1} a_{\bar{w}}} f_v(g)\,dg \\ &= \frac{\mathrm{Tam}\big(E^1\big)}{2^s} \sum I_C(f).\end{aligned}$$

Plugging in $f = 1$, one gets

$$1 = \frac{\mathrm{Tam}\big(E^1\big)}{2^s} h^-.$$

Recall that given a datum (χ, α, ψ), where χ is a Hecke character of E whose restriction on $F_{\mathbb{A}}^*$ is $\epsilon_{E/F}$, ψ is a nontrivial additive character of $F_{\mathbb{A}}$, and $\alpha \in F^*$, one has a Weil representation $\omega_{\alpha,\chi} = \otimes \omega_{\alpha,\chi,v}$ of $G(\mathbb{A})$ on the space $S(F_{\mathbb{A}})$ of Schwartz functions, where $G = U(1) = E^1$ is the norm-1 group of E/F (refer to [HKS]; see also [Y]). It depends on the choice of χ, α, and ψ. When a character η of $[E^1] = E^1 \backslash E_{\mathbb{A}}^1$ satisfies (0.5), there is a function $\phi_{\bar{\eta}} = \prod \phi_{\bar{\eta}_v} \in S(F_{\mathbb{A}})$, given in [Y, Theorem 2.15] ($\phi_{\bar{\eta}_v}$ is an eigenfunction of $(G_v, \omega_{v,\chi})$ with eigencharacter $\bar{\eta}_v$ for a nonsplit place v), such that

$$c\,\mathrm{Tam}\big(E^1\big) \frac{L\Big(\frac{1}{2}, \chi\tilde{\eta}\Big)}{L\big(1, \epsilon_{E/F}\big)} = 2\big|\theta_{\phi_{\bar{\eta}}}(\eta)(1)\big|^2. \tag{1.5}$$

Here

$$c = \prod_{v \in S_{\mathrm{in}} \cap S_{\chi\tilde{\eta}}} \big(1 + q_v^{-1}\big)^{-1} \prod_{v \in S_{\mathrm{sp}} \cap S_{\chi\tilde{\eta}}} \big(1 - q_v^{-1}\big)^{-2} q_v^{-k_v} \tag{1.6}$$

and

$$\theta_\phi(\eta)(1) = \int_{[G]} \sum_{x \in F} \omega_{\alpha,\chi}(g)\phi(x)\eta(g)\,dg. \tag{1.7}$$

Applying Proposition 1.3 and the analytic formula for $L(1, \epsilon_{E/F})$, one has the following corollary.

COROLLARY 1.4. *Let the notation be as above. Let $t = [F : \mathbb{Q}]$, and let s be the number of prime ideals of F ramified in E/F. Let D_F be the absolute discriminant of F. Then*

$$L\left(\frac{1}{2}, \chi\tilde{\eta}\right) = \frac{2^{1-s}\pi^t\sqrt{D_F}}{c\sqrt{D_E}} \left| \sum_{C \in \mathrm{CL}^-(E)} I_C(\eta) \right|^2. \tag{1.8}$$

Here

$$I_C(\eta) = \sum_{x \in F} \prod_v I_{v,C}(x) \tag{1.9}$$

and

$$I_{v,C}(x) = \begin{cases} \int_{E_v^1} \omega_{\alpha,\chi,v}(g)\phi_{\bar{\eta}}(x)\eta_v(g)\,dg & \text{if } v \text{ is nonsplit,} \\ \int_{\mathbb{O}_v^*} \omega_{\alpha,\chi,v}(g a_w a_{\bar{w}}^{-1})\phi_{\bar{\eta}}(x)\eta_v(g)\,dg & \text{if } v = w\bar{w} \text{ is split.} \end{cases} \tag{1.10}$$

Here $a = (a_w) \in E_{\mathbb{A}}^*$ corresponds to an ideal $\mathfrak{A} \in C^{-1}$.

Let $\mathrm{CL}_{\mathrm{ra}} = I_{\mathrm{ra}}/P_{\mathrm{ra}}$ be the “ramified” ideal class group of E, where I_{ra} is the group of fractional ideals of E whose prime factors are all ramified in E/F, and P_{ra} is the subgroup of I_{ra} of principal ideals. It is easy to check that $\mathrm{CL}_{\mathrm{ra}}$ is a subgroup of $\mathrm{CL}(E)$. Let $\mathrm{CL}_{\mathrm{ra}}^-$ be its image in CL^-. The following corollary is an immediate consequence of Proposition 1.3.

COROLLARY 1.5. *Let the notation and assumptions be as in Proposition 1.3. For $C \in \mathrm{CL}^-(E)$ and $C' \in \mathrm{CL}_{\mathrm{ra}}^-$, one has*

$$I_{CC'}(f) = I_C(f).$$

COROLLARY 1.6. *Let the notation and assumptions be as in Corollary 1.4. Let $\tilde{\xi}$ be an ideal class character of E trivial on $\mathrm{CL}(F)\,\mathrm{CL}_{\mathrm{ra}}$, so that there is a character ξ of $[E^1]$ such that $\tilde{\xi}(z) = \xi(z/\bar{z})$. Then $(\chi, \xi\eta, \alpha)$ also satisfies (0.5), and*

$$I_C(\xi\eta) = \tilde{\xi}(C)^{-1} I_C(\eta).$$

Proof. We first verify that $\tilde{\xi}_v = 1$ for every nonsplit place v of F. It is true at infinite places since $\tilde{\xi}$ is of finite order. When v is finite and nonsplit, $\tilde{\xi}_v$ is unramified. By assumption, $\tilde{\xi}_v(\pi_{E_v}) = \tilde{\xi}(\mathfrak{P}_v) = 1$, where π_{E_v} is a uniformizer of E_v and $\mathfrak{P}_v$ is the corresponding prime ideal of E. So $\tilde{\xi_v} = 1$. This proves the first claim and

$$I_{v,C}(\xi\eta, x) = I_{v,C}(\eta, x)$$

for a nonsplit place v of F. Here we write $I_{v,C}(\eta, x)$ for $I_{v,C}(x)$ to indicate its dependence on η. When v is split in E/F, one still has $\phi_{\overline{\xi\eta}} = \phi_{\bar{\eta}}$ since ξ is unramified at v (see [Y, Section 2.2]). Applying (1.10), one gets

$$I_{v,C}(\xi\eta, x) = \xi_v\left(a_w a_{\bar{w}}^{-1}\right) I_{v,C}(\eta, x).$$

Taking the product over v and using the assumption that $\tilde{\xi}$ is trivial on $\mathrm{CL}(F)\,\mathrm{CL}_{\mathrm{ra}}$, one proves the corollary.

For each ideal class $C \in \mathrm{CL}^-(E)$, we choose a primitive ideal $\mathfrak{A}$ (no factor from F) of E in C^{-1} relatively prime to $2\alpha\delta\mathfrak{F}$, where $\mathfrak{F}$ is the conductor of $\chi\tilde{\eta}$. Let

$$\mathfrak{A} = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_r^{e_r}$$

be the prime decomposition of $\mathfrak{A}$; then $\mathfrak{p}_i = N_{E/F}\mathfrak{P}_i$ is a prime of F split in E/F. Let $v_i = w_i\bar{w}_i$ be the place of F associated to $\mathfrak{p}_i$. We may assume $\bar{w}_i$ corresponds to $\mathfrak{P}_i$. Let π_i be a uniformizer of F_{v_i}, and define $a = (a_w) \in E_{\mathbb{A}}^*$ via

$$a_w = \begin{cases} \pi_i^{e_i} & \text{if} w = \bar{w}_i, \\ 1 & \text{otherwise.} \end{cases} \tag{1.11}$$

PROPOSITION 1.7. *With the notation and assumptions as above, one has*

(1.12)

$$I_{v,C}(x) = \begin{cases} \phi_{\bar{\eta}_v}(x) & \textit{if } v \in S_{\mathrm{ra}} \cup S_{\mathrm{in}}, \\ \left|2\sigma\left(\delta^3\alpha\right)\right|^{1/4}\sqrt{\dfrac{(4\pi)^{k_\sigma}}{k_\sigma!}}\,\phi^{k_\sigma}_{|\sigma(\delta)|,|\sigma(\alpha)|}\left(\sigma(x)\right) & \textit{if } v = \sigma \in \Phi, \\ \dfrac{1}{\chi'\tilde{\eta}(\bar{\mathfrak{P}}_i^{\,e_i})}\psi_v\left(\dfrac{1}{2}x_v^3\alpha x^2\right)\mathrm{char}\left(\pi_i^{-e_i}\mathcal{O}_v\right)(x) & \textit{if } v = v_i, \\ q_v^{-(k_v/2)+(n_v/2)-(n(\psi_v)/4)}\psi_v\left(\dfrac{1}{2}x_v^3\alpha x^2\right) & \\ \qquad\qquad \times I\left(x_v^3\alpha x\psi_v, (\chi\eta)_v^{-1}\right) & \textit{if } v \in S_{\mathrm{sp}} \cap S_{\chi\tilde{\eta}}, \\ q_v^{(n_v/2)-(n(\psi_v)/4)}\psi_v\left(\dfrac{1}{2}x_v^3\alpha x^2\right) I\left(\dfrac{x_v^3\alpha}{4}\psi_v, 4x\right) & \textit{otherwise,} \end{cases}$$

where $\chi' = \chi|\,|_{\mathbb{A}}^{-(1/2)}$, and $\phi_{\bar{\eta}_v}$ is a unitary eigenfunction of $(G_v, \omega_{\alpha,\chi,v})$ with eigencharacter $\bar{\eta}_v$. The numbers n_v, k_v, and k_σ were defined in the introduction.

Proof. When v is nonsplit, $\phi_{\bar{\eta}_v}$ is an eigenfunction of $\omega_{\alpha,\chi,v}$ with eigencharacter $\bar{\eta}_v$ by construction (see [Y, Theorem 2.15]). So $I_{v,C}(x) = \phi_{\bar{\eta}_v}(x)$. Explicit formulae for $\phi_{\bar{\eta}_v}$ are given in [Y4, Section 1] for finite v and for $\sigma \in \Phi$ with $\mathrm{Im}(\sigma(\delta)) > 0$. Lemma 1.1 in [Y4] is still valid when $\mathrm{Im}\,\sigma(\delta) < 0$ if we replace $\mathrm{sign}(\alpha)$ by $\mathrm{sign}(\mathrm{Im}(\sigma(\delta\alpha)))$. When v is split, we verify the case $v \in S_{\chi\bar{\eta}} \cap S_{\mathrm{sp}}$ and leave other cases to the reader. By [Y, (2.30)], one has

$$\phi_{\bar{\eta}_v} = \mathrm{meas}\left(\pi_v^{k_v}\mathbb{O}_v\right)^{-(1/2)}\rho\Big(\mathrm{char}\left(1+\pi_v^{k_v}\mathbb{O}_v\right)\Big),$$

where ρ is an isometry defined by [Y, (2.28)] ($|x_0^3\alpha|^{1/3}$ should be $|x_0^3\alpha|^{1/2}$ in the formula). Let $\omega_{\chi,v}$ be the “natural” realization of the Weil representation of G_v on $S(F_v)$ given by [Y, Corollary 2.10], where the subscript v is omitted. Since ρ is an isometry, one has for $g \in \mathbb{O}_v^*$,

$$\omega_{\alpha,\chi,v}(g)\phi_{\bar{\eta}_v}(x) = \mathrm{meas}\left(\pi_v^{k_v}\mathbb{O}_v\right)^{-(1/2)}\rho\Big(\omega_{\chi,v}(g)\,\mathrm{char}\left(1+\pi_v^{k_v}\mathbb{O}_v\right)\Big)(x).$$

Applying [Y, Corollary 2.10] and [Y, (2.28)], one has then

$$\begin{aligned}\omega_{\alpha,\chi,v}(g)\phi_{\bar{\eta}_v}(x) &= \mathrm{meas}\left(\pi_v^{k_v}\mathbb{O}_v\right)^{-(1/2)}\chi_v(g)\rho\Big(\mathrm{char}\left(g^{-1}+\pi_v^{k_v}\mathbb{O}_v\right)\Big)(x)\\ &= q_v^{-(k_v/2)+(n_v/2)-(n(\psi_v)/4)}\chi_v(g)\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)\psi_v\left(x_v^3\alpha x g^{-1}\right)\\ &\quad\cdot\mathrm{char}\left(\pi_v^{n_v-k_v}\mathbb{O}_v\right)(x).\end{aligned}$$

Here we have used the fact $\mathrm{meas}(\mathbb{O}_v) = q_v^{n(\psi_v)/2}$ and the assumption $n((x_v^3\alpha/4)\psi_v) \le 0$, specified in (0.8) So,

$$I_{v,C}(x) = q_v^{-(k_v/2)+(n_v/2)-(n(\psi_v)/4)}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)I\left(x_v^3\alpha x\psi_v,(\chi\eta)_v^{-1}\right).$$

2. The main formula. According to Rohrlich [Roh4, p. 518], a canonical Hecke character of E is a Hecke character χ_{can} satisfying the three following conditions for some CM type Φ of E:

(i) $\chi_{\mathrm{can}}(\bar{\mathfrak{A}}) = \overline{\chi_{\mathrm{can}}(\mathfrak{A})}$ for ideals $\mathfrak{A}$ prime to the conductor of χ_{can}.

(ii) $\chi_{\mathrm{can}}(\alpha\mathbb{O}_E) = \pm\alpha^{\Phi}$ for $\alpha \in E^*$ prime to the conductor of χ_{can}.

(iii) The conductor of χ_{can} is divisible only by primes of E that are ramified over F.

When E is generic and every prime of F dividing 2 is split in E, which we assume throughout this paper, canonical characters of E exist (see [Roh4]). They differ from each other by an ideal class character of E trivial on the ideal class group of F. The following lemma and its proof are due to Rohrlich. We thank him for kindly allowing us to publish it here.

PROPOSITION 2.0 (Rohrlich). *Every Hecke character of E satisfying (i) and (ii) has the form $\chi_{\mathrm{can}}\tilde{\mu}$ for some canonical Hecke character of E and some quadratic Hecke character μ of F. Here $\tilde{\mu} = \mu \circ N_{E/F}$.*

Proof. Let χ_{can} be a canonical Hecke character of E of CM type Φ. Let χ be a Hecke character of E satisfying (i) and (ii); then the conductor of χ is divisible by all primes of E that are ramified over F (see [Roh4, Proposition 1]). Put $\xi = \chi/\chi_{\mathrm{can}}$ and view ξ as a character of the idèle class group $[E^*] = E^*\backslash E^*_{\mathbb{A}}$. Since E is generic, the natural map $[F^*] \longrightarrow [E^*]$ is injective. By [Roh4, Proposition 1], ξ is trivial on $[F^*]$. Condition (ii) implies that ξ is quadratic on $E^*\backslash E^*U_E$, where U_E is as in the proof of Proposition 1.3. So the identity $\xi(\bar{a}) = \overline{\xi(a)}$, which follows from (i), becomes $\xi(\bar{a}/a) = 1$ for $a \in E^*\backslash E^*U_E$. Thus there is a quadratic character μ of $F^*\backslash F^*N_{E/F}(U_E)F^{*2}_{\mathbb{A}}$ such that $\xi(a) = \mu(N_{E/F}(a))$ for every $a \in E^*U_E F^*_{\mathbb{A}}$. Extend μ to a character μ of $[F^*]$. Then $\tilde{\mu} = \mu \circ N_{E/F}$ is a character of $[E^*]$ such that $\tilde{\mu} = \xi$ on $E^*U_E F^*_{\mathbb{A}}$. So there is a character φ of $E^*U_E F^*_{\mathbb{A}}\backslash E^*_{\mathbb{A}} \cong \mathrm{CL}^-(E)$ such that $\xi = \varphi\tilde{\mu}$. After replacing χ_{can} by $\chi_{\mathrm{can}}\varphi$ (which is another canonical Hecke character), we may assume that φ is 1. Then $\xi = \tilde{\mu}$ as idèle class characters of E. This equation implies that $\xi(\bar{a}) = \xi(a)$ for every $a \in [E^*]$. On the other hand, $\xi|_{[F^*]} = 1$ implies $\xi(\bar{a})\xi(a) = 1$. Therefore ξ is quadratic. It follows that μ is quadratic when restricted to $N_{E/F}([E^*])$. But according to class field theory, the group $N_{E/F}([E^*])$ has index 2 in $[F^*]$, and a representative for the nontrivial coset is the idèle c that is -1 at one infinite place and 1 everywhere else. Since $c^2 = 1$, we have $\mu(c)^2 = \mu(c^2) = 1$, and we conclude that μ is quadratic on all of $[F^*]$, as desired.

By the proposition, the characters in the introduction are just Hecke characters of E satisfying (i) and (ii). Throughout this paper, we denote the characters by $\chi_{\mathrm{can}}\tilde{\mu}$ instead of the simplified notation χ we used in the introduction. We do this for two reasons. First, it clearly indicates the different roles that χ_{can} and μ play in the formula. Second, χ is reserved for the unitary character $\chi_{\mathrm{can}}|\,|_{\mathbb{A}}^{1/2}$ in the rest of this paper. Notice that $\chi|^*_{F_{\mathbb{A}}} = \epsilon_{E/F}$ (refer to [Roh4, Proposition 1]). From now on, we fix μ and χ_{can}, and we let $\mathfrak{f}$ be the conductor of μ. Since $\tilde{\mu}|_{F^*_{\mathbb{A}}} = 1$, there is a unique character η of $[E^1]$ such that $\tilde{\mu} = \tilde{\eta}$. Notice that the superscript $\tilde{}$ has different meanings over μ and η. We write $\eta_k = \eta\chi^k|_{E^1_{\mathbb{A}}}$, so $\tilde{\eta_k} = \tilde{\eta}\chi^{2k}$.

LEMMA 2.1. (1) *Let θ be a Hecke character of E such that its restriction on $F^*_{\mathbb{A}}$ is the quadratic Hecke character ϵ associated to E/F. Then the local root numbers satisfy*

$$\prod_{w|v} \epsilon\left(\frac{1}{2}, \theta_w, \frac{1}{2}\psi_{E_w}\right) = \begin{cases} \theta_w(-1) & \text{if } v \in S_{\mathrm{sp}}, \\ i^{|2m_\sigma+1|} & \text{if } v = \sigma \in \Phi, \\ \epsilon\left(\frac{1}{2}, \epsilon_v, \psi_v\right)\epsilon_v(2)(-1)^{n(\theta_w)}\theta_w(\delta) & \text{if } v \in S_{\mathrm{in}}, \\ \epsilon\left(\frac{1}{2}, \epsilon_v, \psi_v\right) & \text{if } v \in S_{\mathrm{ra}}. \end{cases}$$

Here the product takes over places w of E above a place v of F, and m_σ is given by $\theta_\sigma(z) = (|z|/z)^{2m_\sigma+1}$.

(2) *Assume that every prime factor of* $2\mathfrak{f}$ *splits in* E. *Then the global root number of* $(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}$ *is* $(-1)^{kt}\mu_\infty(-1)$.

Proof. (1) When $v = w\bar{w}$ is split, we identify $E_w \cong E_{\bar{w}}$ with F_v and $\psi_w \cong \psi_{\bar{w}}$ with ψ_v. Under the identification, $\theta_{\bar{w}} = \bar{\theta}_w$, and so the product is equal to

$$\epsilon\left(\frac{1}{2}, \theta_w, \frac{1}{2}\psi_w\right)\epsilon\left(\frac{1}{2}, \bar{\theta}_w, \frac{1}{2}\psi_w\right) = \theta_w(-1).$$

The infinity case is well known, the inert case follows from the proof of [Roh5, Proposition 3], and the ramified case is from [Roh4, Proposition 8]. This proves (1). Notice that the additive character used here is $1/2\psi_{E_v}$, while Rohrlich uses ψ_{E_v}. As for (2), first note that the root number of $(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}$ is that of $\chi\tilde{\eta}_k$. Now apply (1) to $\theta = \chi\tilde{\eta}_k$. Since every prime factor of $2\mathfrak{f}$ is split in E and χ is unramified outside $\delta\mathcal{O}_E$, one has $\epsilon_v(2)(-1)^{n(\theta_w)}\theta_w(\delta) = 1$ for all $v \in S_{\mathrm{in}}$. Since the root number of ϵ is 1 and $\epsilon_v = 1$ for split v, one has

$$\prod_{v\in S_{\mathrm{in}}\cup S_{\mathrm{ra}}} \epsilon\left(\frac{1}{2}, \epsilon_v, \psi_v\right) = \prod_{v=\sigma\in\Phi} \epsilon\left(\frac{1}{2}, \epsilon_v, \psi_v\right)^{-1} = i^{-t}.$$

Also, $m_\sigma = k$ for all $\sigma \in \Phi$ in this case. Finally,

$$\prod_{v\in S_{\mathrm{sp}}} \mu_v(-1) = \prod_{v|\infty} \mu_v(-1) = \mu_\infty(-1)$$

since $\mu_v(-1) = 1$ for every nonsplit and finite place v. Taking the product over v in (1) for $\theta = \chi\tilde{\eta}_k$, one proves (2).

From now on, we assume that every prime of F dividing $2\mathfrak{f}$ is split in E/F and $(-1)^{kt}\mu_\infty(-1) = 1$ so that the root number of $(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}$ is 1. Then there is $\alpha \in F^*$ such that $(\chi = \chi_{\mathrm{can}}|\,|^{1/2}, \eta_k, \alpha)$ satisfies (0.5) and (0.8).

Since E is a generic CM number field, $n(\chi_v) = 1$ for $v \in S_{\mathrm{ra}}$ (see [Roh4, Proposition 3]). The next two lemmas follow from Lemma 2.1 and [Y4, Proposition 1.2, Corollary 1.4] easily.

LEMMA 2.2. *Assume that* E/F *is ramified at* v. *Then* $\mathrm{char}(\pi_v^{[n_v/2]}\mathcal{O}_v)$ *is an eigenfunction of* G_v *with eigencharacter* $\bar{\eta}_{k,v} = \eta_v$. *Note that* $n_v = (1/2)n((\delta\alpha/4)\psi_{E_v})$ *is an integer.*

LEMMA 2.3. *Assume that* E/F *is inert at* v. *Then* $n_v = (1/2)n((\delta\alpha/4)\psi_{E_v})$ *is an integer, and* $\mathrm{char}(\pi_v^{n_v}\mathcal{O}_v)$ *is an eigenfunction of* G_v *with eigencharacter* $\bar{\eta}_{k,v} = 1$.

For a totally real number field F of degree t, we identify $F\otimes\mathbb{R}$ with $\mathbb{R}^t$ via the real embeddings of F, and we extend the identification to $F\otimes\mathbb{C} \cong \mathbb{C}^t$. Given a function f on $\mathbb{C}$, we define a function on $\mathbb{C}^t$, still denoted by f, via

$$f(z) = \prod_{i=1}^{t} f(z_i).$$

In particular, $f(x)=\prod f(\sigma(x))$ for $x\in F$, where the product runs over all real embeddings of F. Let $\mathcal{H}^t=\{z=(z_1,\dots,z_t)\in\mathbb{C}^t:\operatorname{Im}(z_i)>0\}$ be the Hilbert upper plane. For a CM number field with maximal totally real number field F and a CM type Φ, we embed E into $\mathbb{C}^t$ via Φ. A CM point in $\mathcal{H}^t$ is just a point in the image of this embedding for some E and Φ.

For each finite place v, let $\mathfrak{p}_v$ be the prime ideal of F corresponding to v. Let

$$\mathfrak{l}=\prod_{v\in S_{\mathrm{ra}}}\mathfrak{p}_v^{-[n_v/2]}\prod_{v\in S_{\mathrm{in}}\cup S_{\mathrm{sp}}}\mathfrak{p}_v^{-n_v+k_v}. \tag{2.1}$$

We refer to the introduction for the definitions of n_v and k_v. We fix once and for all a square root $\sqrt{\Delta}$ of $\Delta \bmod 4\prod_{v\in S_{\mathrm{sp}}}\mathfrak{p}_v^{-n_v+2k_v}$. For each ideal class $C\in\mathrm{CL}^-(E)$, we choose a primitive ideal $\mathfrak{A}\in C^{-1}$ relatively prime to $2\delta\alpha\mathfrak{f}$ (recall that $\mathfrak{f}$ is the conductor of μ). Let $\mathfrak{a}=N_{E/F}\mathfrak{A}$ and choose $b\in\mathcal{O}_F$ satisfying

$$\begin{aligned} &b^2\equiv\Delta \bmod \mathfrak{a}^2,\\ &b\equiv\sqrt{\Delta}\bmod 2\mathfrak{p}_v^{-n_v+2k_v}\qquad \text{for } v\in S_{\mathrm{sp}},\\ &b\equiv 0\bmod \mathfrak{p}_v^{n_v-2[n_v/2]}\qquad \text{for } v\in S_{\mathrm{ra}}.\end{aligned} \tag{2.2}$$

Finally, let

$$\theta_{\mu,k,\mathfrak{A}}(z)=\operatorname{Im}(2z)^{-(k/2)}\sum_{x\in(\mathfrak{l}\mathfrak{a})^{-1}}\gamma_\mu(x)H_{k,F}\Big(x\sqrt{\operatorname{Im}(2z)}\Big)e\big(x^2z\big), \tag{2.3}$$

where

$$\gamma_\mu(x)=\prod_{v\in S_\mu}q_v^{-(k_v/2)}G\big(x_v^3\alpha x\psi_v,\mu_v\big)\operatorname{char}\big(\pi_v^{n_v-k_v}\mathcal{O}_v^*\big)(x)$$

only depends on μ. Here $G(\psi,\chi)$ is the usual Gauss sum. Then $\theta_{\mu,k,\mathfrak{A}}$ is a non-holomorphic Hilbert modular form on F of weight $(2k+1/2)=((2k+1/2),\dots,(2k+1/2))$ (see [G, Theorem 5.3, p. 154]).

THEOREM 2.4. *Let F be a totally real number field of degree t over $\mathbb{Q}$. Let μ be a quadratic Hecke character of F of conductor $\mathfrak{f}$. Let E be a generic CM quadratic extension of F such that every prime of F dividing $2\mathfrak{f}$ is split in E/F. Let $k\geq 0$ be an integer with $(-1)^{kt}\mu_\infty(-1)=1$, and let s be the number of prime ideals of F ramified in E. Then, for (δ,α) satisfying (0.3), (0.5), and (0.8), the central L-value*

$$L(k+1,(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1})=\kappa\left|\sum_{C\in\mathrm{CL}^-(E)}\frac{(N\mathfrak{a})^k}{(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}(\bar{\mathfrak{A}})}\theta_{\mu,k,\mathfrak{A}}(\tau)\right|^2. \tag{2.4}$$

Here $N=N_{F/\mathbb{Q}}$, $\tau=-(\Delta\alpha(b+\delta)/2)\in E$ is a CM point, and

$$\kappa=\frac{2^{1+(1/2)t-s}\pi^t\left|N\big(\Delta^3\alpha^2\big)\right|^{(2k+1/4)}}{|N\Delta|^{1/2}N(\mathfrak{l})}\left(\frac{(4\pi)^k}{k!}\right)^t.$$

Proof. First, at an infinite place $v=\sigma\in\Phi$, one has $k_\sigma=k\ge 0$ and $\operatorname{Im}(\sigma(\delta\alpha))>0$, by (0.5) and Lemma 2.1. So $\operatorname{Im}\sigma(\tau)>0$, and the right-hand side of (2.4) is meaningful. Second, note that $S_{\mathrm{sp}}\cap S_{\chi\tilde\eta_k}=S_\mu$. For $v\in S_\mu$, one has, by Proposition 1.7 and Lemma 1.2,

$$I_{v,C}(x)=q_v^{-(3k_v/2)+(n_v/2)-(n(\psi_v)/4)}\big(1-q_v^{-1}\big)^{-1}\cdot\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)G\big(x_v^3\alpha x\psi_v,\mu_v\big)\operatorname{char}\big(\pi_v^{n_v-k_v}\mathbb{O}_v^*\big)(x).$$

Here we have used the fact that χ is unramified at v.

For $v\in S_{\mathrm{sp}}-S_\mu$ but $v\ne v_i$, the assumption $n\big((x_v^3\alpha/4)\psi_v\big)\le 0$ implies (by Lemma 1.1)

$$I_{v,C}(x)=q_v^{(n_v/2)-\big(n(\psi_v)/4\big)}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)\operatorname{char}(\pi_v^{n_v}\mathbb{O}_v)(x).$$

So one has, by Lemma 2.2, Lemma 2.3, and Proposition 1.7,

$$I_{v,C}(x)=\begin{cases} q_v^{(n_v/2)-(n(\psi_v)/4)}\operatorname{char}\big(\pi_v^{n_v}\mathbb{O}_v\big)(x) & \text{if } v\in S_{\mathrm{in}},\\ q_v^{(1/2)[n_v/2]-(n(\psi_v)/4)}\operatorname{char}\big(\pi_v^{[n_v/2]}\mathbb{O}_v\big)(x) & \text{if } v\in S_{\mathrm{ra}},\\ \big|2\sigma\big(\delta^3\alpha\big)\big|^{1/4}\left(\frac{(4\pi)^k}{k!}\right)^{1/2}\phi^k_{|\sigma(\delta)|,|\sigma(\alpha)|}(\sigma(x)) & \text{if } v=\sigma\in\Phi,\\ \frac{1}{\chi_{\mathrm{can}}\tilde\eta_k(\bar{\mathfrak{P}}_i^{e_i})}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)\operatorname{char}\big(\pi_v^{-e_i}\mathbb{O}_v\big)(x) & \text{if } v=v_i,\\ q_v^{-(3k_v/2)+(n_v/2)-(n(\psi_v)/4)}\big(1-q_v^{-1}\big)^{-1}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right) & \\ \quad\cdot G\big(x_v^3\alpha x\psi_v,\mu_v\big)\operatorname{char}\big(\pi_v^{n_v-k_v}\mathbb{O}_v^*\big)(x) & \text{if } v\in S_\mu,\\ q_v^{(n_v/2)-(n(\psi_v)/4)}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)\operatorname{char}\big(\pi_v^{n_v}\mathbb{O}_v\big)(x) & \text{otherwise.}\end{cases}$$

Putting everything together, one obtains

$$I_C(\eta_k)=\left(\frac{(4\pi)^k}{k!}\right)^{t/2}\frac{D_F^{1/4}c^{1/2}2^{t/4}\big|N\big(\Delta^3\alpha^2\big)\big|^{1/8}(N\mathfrak{a})^k}{\big(\chi_{\mathrm{can}}\tilde\mu\big)^{2k+1}(\bar{\mathfrak{A}})(N\mathfrak{l})^{1/2}}J_k(\mathfrak{a}).$$

Here

$$J_k(\mathfrak{a})=\sum_{x\in(\mathfrak{l}\mathfrak{a})^{-1}}\gamma_\mu(x)\prod_{v\in S_{\mathrm{sp}}}\psi_v\left(\frac{1}{2}x_v^3\alpha x^2\right)\prod_{\sigma\in\Phi}\phi^k_{|\sigma(\delta)|,|\sigma(\alpha|)}(x).$$

For $v=w\bar w\in S_{\mathrm{sp}}$, we identify $E_v=E_w\oplus E_{\bar w}=F_v\oplus F_v$ and write $\delta=(x_v,-x_v)$. We choose and fix w so that $x_v=\sqrt{\Delta}\bmod 2\pi_v^{-n_v+2k_v}$. So

$$b\equiv x_v\bmod 2\pi_v^{-n_v+2k_v}\qquad\text{for } v\in S_{\mathrm{sp}}.$$

We may assume that

$$b \equiv x_v \bmod \mathfrak{a}^2 \quad \text{for } v = v_i.$$

So

$$\psi_v\left(\frac{1}{2}\Delta b \alpha x^2\right) = \begin{cases} \psi_v\left(\frac{1}{2}x_v^3 \alpha x^2\right) & \text{if } v \in S_{\mathrm{sp}}, \\ 1 & \text{if } v \in S_{\mathrm{in}} \cup S_{\mathrm{ra}}. \end{cases}$$

Also note that $\operatorname{Im}(\sigma(\tau)) = -\sigma(\delta^3\alpha) = |\sigma(\delta^3\alpha)| = |\Delta^3\alpha^2|^{1/2}$. Applying (0.14) and (0.15), one gets, after some computation,

$$J_k(\mathfrak{a}) = \left|\Delta^3\alpha^2\right|^{k/4}\theta_{\mu,k,\mathfrak{A}}(\tau).$$

Now applying Corollary 1.4 and using the fact $D_E = D_F^2|N\Delta|$, one proves (2.4).

Now we consider the special case where F has ideal class number 1. We write

$$\mathfrak{l} = l\mathcal{O}_F, \qquad \mathfrak{f} = f\mathcal{O}_F, \qquad \mathfrak{a} = a\mathcal{O}_F.$$

Choose a lattice decomposition (it always exists since F has ideal class number 1 and every prime factor of 2 splits in E)

$$\mathfrak{A}^2 = \left[a^2, \frac{b+\delta}{2}\right] \tag{2.5}$$

such that b satisfies (2.2). Recall that, in general, associated to a Hecke character μ of a number field F of class number 1 with conductor $\mathfrak{f}$, there is a character μ' of the group $(\mathcal{O}_F/\mathfrak{f})'$ given by

$$\mu'(x) = \mu(x\mathcal{O}_F)\mu_\infty(x). \tag{2.6}$$

Here $(\mathcal{O}_F/\mathfrak{f})'$ is the multiplicative group of elements in $\mathcal{O}_F$ relative prime to $\mathfrak{f}$ modulo those who are $1 \bmod \mathfrak{f}$ and positive with respect to every real embedding of F. In terms of local characters, one has

$$\mu'(x) = \prod_{v \in S_\mu} \mu_v(x).$$

Define, for the fixed quadratic Hecke character μ of F and an integer $k \geq 0$,

$$\theta_{\mu,k}(z) = \left(\operatorname{Im}(2z)\right)^{-(k/2)} \sum_{x \in \mathcal{O}_F, (x,f)=1} \mu'(x) H_k\left(x\sqrt{\operatorname{Im}(2z)}\right) e\left(x^2 z\right). \tag{2.7}$$

Then $\theta_{\mu,k}$ is a nonholomorphic Hilbert modular form of weight $(2k+1)/2$ which *only* depends on (F, μ, k) and not on the CM number field E. Notice that $\theta_{\mu,k}$ is holomorphic when $k = 0$ or 1. We also remark that

$$\theta_{\mu,k}\left(c^2 z\right) = \mu'(c)\operatorname{sign}(Nc)^k \theta_{\mu,k}(z), \tag{2.8}$$

for every unit $c \in \mathcal{O}_F^*$, where N is the norm map from F to $\mathbb{Q}$. In particular, $\theta_{\mu,k}(z) = 0$ unless $\mu'(-1)(-1)^{kt} = \mu_\infty(-1)(-1)^{kt} = 1$; that is, the global root number of $(\chi_{\mathrm{can}}\tilde{\eta})^{2k+1}$ is 1. A simple calculation gives

$$\theta_{\mu,k,\mathfrak{A}}(z) = \gamma' N(la)^{-k}\mu'(a)\theta_{\mu,k}\left(\frac{z}{(la)^2}\right). \tag{2.9}$$

Here $\gamma' \in \mathbb{C}^1$ is some constant independent of the ideal class $C \in \mathrm{CL}(E)$. We also mention that $\mu_\infty(-1) = \mu'(-1)$. Plugging (2.9) into (2.4), we have the following theorem.

THEOREM 2.5. *Let F be a totally real number field of ideal class number* 1 *and degree t over $\mathbb{Q}$. Let μ be a quadratic Hecke character of F of conductor $f\mathcal{O}_F$ such that $(2, f) = 1$. Let E be a generic CM quadratic extension of F such that every prime of F dividing $2f$ is split in E/F. Let $k \geq 0$ be an integer with $(-1)^{kt}\mu_\infty(-1) = 1$, and let s be the number of prime ideals of F ramified in E. Then there exist δ and α satisfying (0.3), (0.5), and (0.8). For such a pair (δ, α), let $\mathfrak{l} = l\mathcal{O}_F$ be defined by (2.1). For an ideal class $C \in \mathrm{CL}(E)$, we choose a primitive ideal $\mathfrak{A} \in C^{-1}$ relatively prime to $2\alpha\delta f\mathcal{O}_E$, and we write*

$$\mathfrak{A}^2 = \left[a^2, \frac{b+\delta}{2}\right], \quad a, b \in \mathcal{O}_F,$$

such that b satisfies (2.2). Let

$$\tau_{\mathfrak{A}} = -\frac{\Delta\alpha(b+\delta)}{2(la)^2}$$

and

$$\theta_{\mu,k}(\mathfrak{A}) = \mu'(a)\,\mathrm{sign}(Na)^k\theta_{\mu,k}(\tau_{\mathfrak{A}}).$$

Then $\theta_{\mu,k}(\mathfrak{A})$ is independent of the choice of a and b (subject to condition (2.2)) and the central L-value

$$L\big(k+1, (\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}\big) = \kappa\left|\sum_{C\in\mathrm{CL}(E)}\frac{\theta_{\mu,k}(\mathfrak{A})}{(\chi_{\mathrm{can}}\tilde{\mu})^{2k+1}(\bar{\mathfrak{A}})}\right|^2. \tag{2.10}$$

Here

$$\kappa = \frac{2^{1+(1/2)t-s}\pi^t\left|N\left(\Delta^3\alpha^2\right)\right|^{(2k+1)/4}}{|N(\Delta)|^{1/2}|N(l)|^{2k+1}}\left(\frac{(4\pi)^k}{k!}\right)^t.$$

Proof. Since every prime of F above 2 splits in E, and F has class number 1, one can choose δ satisfying (0.3). For the same reason, every primitive ideal $\mathfrak{A}$ of E relative prime to $2\delta\mathcal{O}_E$ has a lattice decomposition

$$\mathfrak{A} = \left[a, \frac{b+\delta}{2}\right],$$

where $a\mathcal{O}_F = \mathfrak{A}\bar{\mathfrak{A}}$ and

$$b^2 \equiv \Delta \bmod 4a.$$

Changing b in the coset $b + 2a\mathcal{O}_F$, one can assume that b satisfies (2.2) if $\mathfrak{A}$ is relatively prime to $2f\delta\mathcal{O}_E$. Since the root number of $(\chi_{\text{can}}\tilde{\mu})^{2k+1}$ is 1, one can find $\alpha \in F^*$, unique up to a factor from $N_{E/F}E^*$ satisfying (0.5). Replacing α by its multiple from $N_{E/F}E^*$ if necessary, we can assume that α also satisfies (0.8). This proves the existence of δ and α satisfying (0.3), (0.5), and (0.8). Now the theorem follows from Theorem 2.4 and the computation above.

Although Theorem 2.5 is a special case of Theorem 2.4, let us point out that its hypotheses guarantee the remarkable fact that the theta function involved only depends on F and μ and not on E. Neither the condition that F has ideal class number 1 nor the condition that every prime factor of $\mathfrak{f}$ splits in E (see [Y4]) can be dropped to have the independence.

For a CM type Φ of E, we write $k\Phi$ for $\sum_{\sigma\in\Phi} k\sigma$. According to Rohrlich [Roh, p. 700], there is a group homomorphism

$$h_{k\Phi} : \mathrm{CL}(E) \longrightarrow \mathrm{CL}\left(E^{k\Phi}\right). \tag{2.11}$$

Here $E^{k\Phi}$ is the number field generated by $z^{k\Phi} = \prod_{\sigma\in\Phi} \sigma(z)^k$, $z \in E^*$.

THEOREM 2.6. *Let the notation and assumptions be as in Theorem 2.4. Assume that* $\mathrm{CL}_{\mathrm{ra}}(E)\,\mathrm{CL}(F) \supset \ker h_{(2k+1)\Phi}$. *Then the following statements are equivalent.*

(a) *The central L-value* $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$.

(b) *The global theta lifting* $\theta_{\alpha,\chi}(\eta_k) = 0$.

(c) *For every ideal class* $C \in \mathrm{CL}(E)$, $I_C(\eta_k) = 0$.

(d) *For every ideal class* $C \in \mathrm{CL}(E)$, $\tau_{\mathfrak{A}}$ *is a root of the theta function* $\theta_{\mu,k,\mathfrak{A}}$.

Proof. Claims (a) and (b) are equivalent by a theorem of Rogawski ([Ro, Theorem 1.1]; see also [Y, Theorem 0.4]). Claims (c) and (d) are equivalent by the proof of Theorem 2.4. So it suffices to prove that (a) and (c) are equivalent. This follows from a trick of Rohrlich. Let $\tilde{\xi}$ be an ideal class character of E trivial on $\mathrm{CL}_{\mathrm{ra}}(E)\,\mathrm{CL}(F)$. Then it is trivial on $\ker h_{(2k+1)\Phi}$ by assumption. By a theorem of Rohrlich [Roh3, Theorem 1], one has $\chi_{\text{can}}^{2k+1}\tilde{\xi}\tilde{\mu} = ((\chi_{\text{can}}\tilde{\mu})^{2k+1})^{\sigma}$ for some $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/E^{(2k+1)\Phi})$. Applying [S2, Theorem 1], one has that $L(k+1, (\chi_{\text{can}}\tilde{\mu})^{2k+1}) = 0$ if and only if $L(1/2, \chi\tilde{\xi}\tilde{\eta}_k) = 0$ for every ideal class character ξ of E trivial on $\mathrm{CL}_{\mathrm{ra}}(E)\,\mathrm{CL}(F)$ (notice the shift of center). By Corollaries 1.4, 1.5, and 1.6, this is equivalent to $I_C(\eta_k) = 0$ for every ideal class $C \in \mathrm{CL}(E)/\mathrm{CL}_{\mathrm{ra}}(E)\,\mathrm{CL}(F)$.

3. $F = \mathbb{Q}$. In this section, let $d \equiv 1 \bmod 4$ be a squarefree integer and let μ be the quadratic Hecke character of $\mathbb{Q}$ associated to the quadratic field $\mathbb{Q}(\sqrt{d})$. Then $\mu' = (\frac{d}{\cdot})$. Let $D \equiv 7 \bmod 8$ be a positive squarefree integer, and let $E = E_D = \mathbb{Q}(\sqrt{-D})$. We view E as a subfield of $\mathbb{C}$ by fixing a choice $\delta = \sqrt{-D} = i\sqrt{D} \in i\mathbb{R}_{>0}$. We write

$\chi_{D,d}$ for $\chi_{\text{can}}\tilde{\mu}$ in this section. Here we have chosen and fixed the CM type Φ to be the fixed complex embedding of E, and we ignore it from now on. The other choice of Φ does not give anything new. Let χ, η, and η_k be as in Section 2.

LEMMA 3.1. *Assume that every prime factor of d is split in E_D, and $(-1)^k = \operatorname{sign}(d)$. Then, for a decomposition $D = D_1 D_2$ with $D_i > 0$, there are exactly $h_D/2^{s-1}$ canonical characters of E_D such that*

$$\epsilon\left(\frac{1}{2}, (\chi\tilde{\eta}_k)_v, \frac{1}{2}\psi_{E_v}\right)\big(\chi\tilde{\eta}_k\big)_v(\delta) = \epsilon_v(D_1) = \epsilon_v(D_2) \tag{3.1}$$

for every place v of $\mathbb{Q}$. Here s is the number of positive prime factors of D, and $\epsilon = \epsilon_D = (\frac{-D}{})$. Conversely, for every canonical character χ_{can}, there is a (unique up to order) decomposition $D = D_1 D_2$ satisfying (3.1) and $D_i > 0$.

Proof. Since 2 is split in E/F, $\epsilon_2(2) = 1$; and then (3.1) is true for every place $v \nmid D$ by Lemma 2.1. Fix a canonical Hecke character χ_{can} of E, so other canonical Hecke characters of E are just $\chi_{\text{can}}\xi$, where ξ runs over all ideal class characters of E. Let

$$i_v = \epsilon\left(\frac{1}{2}, \big(\chi\tilde{\eta}_k\big)_v, \frac{1}{2}\psi_{E_v}\right)\big(\chi\tilde{\eta}_k\big)_v(\delta) = \pm 1.$$

Then $(i_v)_{v|D} \in S$, where

$$S = \Big\{(d_v)_{v|D} : d_v = \pm 1, \ \prod d_v = 1\Big\}$$

is a subgroup of $(\mathbb{Z}^*)^s$ of order 2^{s-1}. First, we define a map

$$F = (F_v) : \mathrm{CL}(E)^\wedge \longrightarrow S, \qquad F_v(\xi) = \frac{\epsilon\big(1/2, \big(\chi\xi\tilde{\eta}_k\big)_v, (1/2)\psi_{E_v}\big)\big(\chi\xi\tilde{\eta}_k\big)_v(\delta)}{i_v}.$$

Here $\mathrm{CL}(E)^\wedge$ is the character group of $\mathrm{CL}(E)$. By Lemma 2.1, $F_v(\xi) = \xi_v(\delta) = \xi(\mathfrak{P}_v)$, where $\mathfrak{P}_v$ is the prime ideal of E_D above v. So F is a group homomorphism. Moreover, $F(\xi) = 1$ if and only if ξ is trivial on $\mathrm{CL}_{\text{ra}} = \mathrm{CL}_2$. So F induces a group injection $F : (\mathrm{CL}(E)/\mathrm{CL}_2)^\wedge \longrightarrow S$. By genus theory, $h_2 = |\mathrm{CL}_2| = 2^{s-1}$, so the induced map is an isomorphism. On the other hand, we have a map from the set of decompositions $D = D_1 D_2$ with $D_i > 0$ to S via

$$\big(D = D_1 D_2\big) \mapsto \big(\epsilon_v(D_1)\big)_{v|D} = \big(\epsilon_v(D_2)\big)_{v|D}.$$

One finds by counting that this map is also a bijection. This proves the lemma.

THEOREM 3.2. *Let $D \equiv 7 \bmod 8$ be a positive squarefree integer with s prime factors. Let $d \equiv 1 \bmod 4$ be a squarefree integer such that every prime factor of d is split in E_D. Assume $(-1)^k = \operatorname{sign}(d)$. Let $D = D_1 D_2$ with $D_i > 0$ and let $\chi_{\text{can}} = \chi_{\text{can}, D_1, D_2}$ be a canonical character satisfying (3.1). Fix a square root r*

of $-D \bmod 16d^2$*. For every ideal class* $C \in \mathrm{CL}(E)/\mathrm{CL}_2$*, choose a primitive ideal* $\mathfrak{A} \in C^{-1}$ *relatively prime to* $2d$*, and write*

$$\mathfrak{A}^2 = \left[a^2, \frac{b+\sqrt{-D}}{2} \right] \tag{3.2}$$

with $a > 0$ *and*

$$b \equiv r \bmod 8d^2 \quad \textit{and} \quad b \equiv 0 \bmod D_1. \tag{3.3}$$

Then

$$L\big(k+1, (\chi_{D,d})^{2k+1}\big) = \kappa \left| \sum_{C \in \mathrm{CL}(E)/\mathrm{CL}_2} \frac{1}{\chi_{D,d}^{2k+1}(\bar{\mathfrak{A}})} \theta_{d,k}(\tau_{\mathfrak{A},D_1}) \right|^2. \tag{3.4}$$

Here

$$\kappa = \frac{2^{s-(3/2)} \pi^{k+1}}{k! |d|^{2k+1} \sqrt{D}} \left(\frac{D_2}{D_1} \right)^{(2k+1/4)}$$

and

$$\tau_{\mathfrak{A},D_1} = \frac{b+\sqrt{-D}}{8D_1 d^2 a^2},$$

and

$$\theta_{d,k}(z) = \big(\mathrm{Im}(2z)\big)^{-(k/2)} \sum_{(x,d)=1} \left(\frac{d}{x} \right) H_k\Big(x\sqrt{\mathrm{Im}(2z)}\Big) e\big(x^2 z\big)$$

is a (nonholomorphic) modular form for $\Gamma_0(4d^2)$ *of weight* $(2k+1)/2$*, where* $e(z) = e^{2\pi i z}$ *as usual.*

Proof. This is just a special case of Theorem 2.5 where we can choose α explicitly. Set $\alpha = (4/D_2)$; then (χ, η_k, α) satisfies (0.3), (0.5), and (0.8). Moreover,

$$n_v = \begin{cases} 0 & \text{if } v \nmid 2D_1, \\ 1 & \text{if } v | D_1, \\ 2 & \text{if } v = 2, \end{cases}$$

and

$$k_v = \begin{cases} 0 & \text{if } v \nmid d\infty, \\ 1 & \text{if } v | d, \\ k & \text{if } v = \infty. \end{cases}$$

Also $l = 4D_1$, $f = d$. Now applying Theorem 2.5 and Corollary 1.5, one gets (3.4).

Remark 3.3. If we take $\alpha = (4/D_1)$, we get a variation of formula (3.4) with the positions of D_1 and D_2 switched. These two formulae can be derived from each other by the Poisson summation formula. In general, one can take any $\alpha \in D_1 N_{E/\mathbb{Q}} E^*$ satisfying (0.8) and apply Theorem 2.5 to get a variation of formula (3.4). This gives various relations among theta functions.

THEOREM 3.4. *Let (D, d, k) be as in Theorem 3.2.*

(1) *If the central L-value $L(k+1, (\chi_{D,d})^{2k+1}) = 0$ and $(2k+1, h_D) = 1$, then for any positive factor D_1 of D, and for any primitive ideal $\mathfrak{A}$ relative prime to $2Dd$, the CM point $\tau_{\mathfrak{A},D_1}$ defined in Theorem* 3.2 *is a root of the modular form $\theta_{d,k}$.*

(2) *Conversely, for a fixed positive divisor D_1 of D, if, for every ideal class $C \in \mathrm{CL}(E)$, there is a primitive ideal $\mathfrak{A}$ relative prime to $2Dd$ such that the CM point $\tau_{\mathfrak{A},D_1}$ is a root of the modular form $\theta_{d,k}$, then the central value $L(k+1, (\chi_{D,d})^{2k+1}) = 0$ for every Hecke character $\chi_{D,d}$ of E_D.*

Proof. The proof follows from Theorem 2.6, Theorem 3.2, and the following fact: When $L(k+1, (\chi_{D,d})^{2k+1}) = 0$ for one canonical Hecke character of E_D, it is zero for every canonical Hecke character of E_D, since they are Galois conjugate to each other under the condition $(2k+1, h_D) = 1$ (see [Roh3, Theorem 1]).

COROLLARY 3.5. *Let $p \equiv 7 \bmod 8$ be a prime, and let $d \equiv 1 \bmod 4$ be a squarefree integer. Assume that every prime factor of d is split in E_p. Let $k \geq 0$ be an integer with $(-1)^k = \mathrm{sign}(d)$, and let $(2k+1, h_p) = 1$. Then the central L-value $L(k+1, \chi_{p,d}^{2k+1}) = 0$ if and only if all the Heegner points of $X_0(4d^2)$ with endomorphism ring $\mathcal{O}$ are roots of the theta function $\theta_{d,k}$. Here $\mathcal{O}$ is the ring of integers of E_p.*

Proof. It is clear that $\tau_{\mathfrak{A},1}$ is a Heegner point of $X_0(4d^2)$ with endomorphism ring $\mathcal{O} = \mathcal{O}_{E_p}$. By Theorem 3.4, it now suffices to prove that the $\tau_{\mathfrak{A},1}$ account for all such Heegner points when $[\mathfrak{A}]^2$ and $r \bmod 8d^2$ vary. Here $[\mathfrak{A}]$ denotes the ideal class of E containing $\mathfrak{A}$, and r is a square root of $-p \bmod 16d^2$ (see Theorem 3.2). This can be shown by counting. Indeed, since p is a prime, h_p is odd, and so $\mathrm{CL}(E)^2 = \mathrm{CL}(E)$. Furthermore,

$$\#\left\{r \bmod 8d^2 : r \text{ is a square root of } -p \bmod 16d^2\right\} = 2^t,$$

where t is the number of prime factors of $4d^2$. So the total number of points $\tau_{\mathfrak{A},1}$ is $2^t h_p$. On the other hand, recall that the Heegner points of $X_0(4d^2)$ can be parameterized by $(\mathcal{O}, \mathfrak{n}, [\mathfrak{A}])$, where $\mathfrak{n}$ is an integral ideal of $\mathcal{O}$ such that $\mathcal{O}/\mathfrak{n} \cong \mathbb{Z}/4d^2$, and $[\mathfrak{A}]$ are ideal classes of E (see [Gr2, p. 89]. So there are also $2^t h_p$ such Heegner points.

Proof of Theorem 0.2. Let $c = e^{-(\pi\sqrt{p}/4d^2)} < 1$. By Theorem 3.4, it suffices to prove that $\tau = (b + \sqrt{-p})/8d^2$ is not a root of $\theta_{d,k}$. We assume $d \neq 1$, so that the theta function $\theta_{d,k}$ has no constant term. The case $d = 1$ is similar (and a little bit easier) and is left to the reader.

Let us first assume $k=0$. Recall that $H_0(x)=1$. For $c<.8$, one has

$$\frac{1}{2}|\theta_{d,0}(\tau)| \geq c-\sum_{n=2}^{\infty} c^{n^2} \geq c-c^4-c^9\sum_{n=0}^{\infty} c^{7n} > 0.$$

So $\theta_{d,0}(\tau)\neq 0$ if $-(\pi\sqrt{p}/4d^2)\leq \log 4/5$; that is, $p\geq .081d^4$.

Secondly when $k=1$, $H_1(x)=x$. In this case, one has

$$\begin{aligned}\frac{1}{2}|\theta_{d,1}(\tau)| &\geq c-\sum_{n=2}^{\infty} nc^{n^2}\\ &> c-2c^4-c^3\sum_{n=3}^{\infty} n\left(c^3\right)^{n-1}\\ &= c\left[1-2c^3-\frac{3c^8-2c^9}{\left(1-c^3\right)^2}\right]\\ &\geq 0 \quad \text{ if } 0<c\leq 0.7.\end{aligned}$$

So when $c\leq 0.7$, that is, $p\geq .206d^4$, one has $\theta_{d,1}(\tau)\neq 0$.

Finally, let $k\geq 0$ be an arbitrary integer with $(-1)^k=\operatorname{sign}(d)$. Since H_k is a polynomial of degree k, there is a constant $C_1=C_1(k)$ such that

$$\left|H_k(x)\right| \leq C_1|x|^k \quad \text{for all } |x|\geq 1.$$

Since H_k has finitely many roots, there are positive constants C_2 and $C_3>1$ such that

$$\left|H_k(x)\right|\geq C_2 \quad \text{for all } x\geq\sqrt{C_3}.$$

So when $2\operatorname{Im}(\tau)=(\sqrt{p}/4d^2)\geq C_3$, one has ($\theta_{d,k}$ is even)

$$\begin{aligned}\frac{\left(2\operatorname{Im}(\tau)\right)^{k/2}}{2}|\theta_{d,k}(\tau)| &\geq C_2c-C_1\sum_{n=2}^{\infty}\left(\frac{\sqrt{p}}{4d^2}\right)^k n^k c^{n^2}\\ &= cf(c),\end{aligned}$$

where

$$f(c)=C_2-\pi^{-k}C_1(-\log c)^k c^3\sum_{n=2}^{\infty} n^k c^{n^2-4}.$$

Notice that $f(c)$ is continuous and independent of d or p in the interval $(0,1)$, and $\lim_{c\mapsto 0+} f(c)=C_2>0$. So there is a constant $0<C_4<e^{-C_3}<1$ such that $f(c)>0$ for all $0<c\leq C_4$. Take $M(k)=(4/\pi)\log(1/C_4)$; then for $p>Md^4$, one has $\theta_{d,k}(\tau)\neq 0$, and so $L(k+1,\chi_{p,d}^{2k+1})\neq 0$.

Remark 3.6. Theorem 0.2 remains true when one replaces p by any positive squarefree integer $D\equiv 7 \bmod 8$. The proof is the same.

References

[G] P. Garrett, *Holomorphic Hilbert Modular Forms*, Wadsworth & Brooks/Cole Math. Ser., Wadsworth & Brooks/Cole Adv. Books Software, Pacific Grove, Calif., 1990.

[Gr] B. Gross, *Arithmetic on Elliptic Curves with Complex Multiplication*, Lecture Notes in Math. **776**, Springer-Verlag, Berlin, 1980.

[Gr2] ———, "Heegner points on $X_0(N)$" in *Modular Forms,* ed. R. Rankin, Ellis Horwood Ser. Math. Appl. Statist. Oper. Res. Comput. Math., Horwood, Chichester, 1984, 87–105.

[HKS] M. Harris, S. Kudla, and W. Sweet, *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. **9** (1996), 941–1004.

[K] S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), 361–401.

[MRoh] H. Montgomery and D. Rohrlich, *On the L-functions of canonical Hecke characters of imaginary quadratic fields, II*, Duke Math. J. **49** (1982), 937–942.

[RR] R. Ranga Rao, *On some explicit formulas in the theory of Weil representations*, Pacific J. Math. **157** (1993), 335–371.

[RV] F. Rodriguez Villegas, *On the square root of special values of certain L-series*, Invent. Math **106** (1991), 549–573.

[RV2] ———, *Square root formulas for central values of Hecke L-series, II,* Duke Math. J. **72** (1993), 431–440.

[RVZ] F. Rodriguez Villegas and D. Zagier, "Square roots of central values of Hecke L-series" in *Advances in Number Theory (Kingston, Ontario, 1991)*, Oxford Sci. Publ., Oxford Univ. Press, New York, 1993, 81–99.

[RVZ2] ———, "Which primes are sums of two cubes" in *Number Theory (Halifax, Nova Scotia, 1994)*, CMS Conf. Proc. **15**, Amer. Math. Soc., Providence, 1995, 295–306.

[Ro] J. Rogawski, "The multiplicity formula for A-packets" in *The Zeta Functions of Picard Modular Surfaces*, ed. R. P. Langlands and D. Ramakrishnan, Univ. Montréal, Montréal, 1992, 395–419.

[Roh] D. Rohrlich, *Galois conjugacy of unramified twists of Hecke characters*, Duke Math J. **47** (1980), 695–543.

[Roh2] ———, *The nonvanishing of certain Hecke L-functions at the center of the critical strip*, Duke Math J. **47** (1980), 223–232.

[Roh3] ———, *On the L-functions of canonical Hecke characters of imaginary quadratic fields*, Duke Math J. **47** (1980), 547–557.

[Roh4] ———, *Root numbers of Hecke L-functions of CM fields*, Amer. J. Math. **104** (1982), 517–543.

[Roh5] ———, *Galois theory, elliptic curves, and root numbers*, Compositio Math. **100** (1996), 311–349.

[Ru] K. Rubin, *Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer*, Invent. Math. **64** (1981), 455–470.

[S] G. Shimura, *On the factors of the Jacobian variety of a modular function field*, J. Math. Soc. Japan **25** (1973), 523–544.

[S2] ———, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), 783–804.

[S3] ———, *On the periods of modular forms*, Math. Ann. **229** (1977), 211–221.

[W] L. Washington, *Introduction to Cyclotomic Fields*, Grad. Texts in Math. **83**, Springer-Verlag, New York, 1982.

[We] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

[Y] T. H. Yang, *Theta liftings and Hecke L-functions*, J. Reine Angew. Math. **485**

(1997), 25–53.

[Y2] ———, *Common zeros and central Hecke L-values of CM number fields of degree 4,* Proc. Amer. Math. Soc. **126** (1998), 999–1004.

[Y3] ———, *Eigenfunction of the Weil representations of unitary groups of one variable,* Trans. Amer. Math. Soc. **350** (1998), 2393–2407.

[Y4] ———, *Nonvanishing of central value of Hecke characters and the rank of their associated elliptic curves,* to appear in Compositio Math.

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