Annals of Mathematics **177** (2013), 1147–1168
http://dx.doi.org/10.4007/annals.2013.177.3.8

Positivity for Kac polynomials and DT-invariants of quivers

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Abstract

We give a cohomological interpretation of both the Kac polynomial and the refined Donaldson-Thomas-invariants of quivers. This interpretation yields a proof of a conjecture of Kac from 1982 and gives a new perspective on recent work of Kontsevich–Soibelman. This is achieved by computing, via an arithmetic Fourier transform, the dimensions of the isotypical components of the cohomology of associated Nakajima quiver varieties under the action of a Weyl group. The generating function of the corresponding Poincaré polynomials is an extension of Hua's formula for Kac polynomials of quivers involving Hall–Littlewood symmetric functions. The resulting formulae contain a wide range of information on the geometry of the quiver varieties.

1. **The main results**

Let $\Gamma = (I, \Omega)$ be a quiver; that is, an oriented graph on a finite set $I = \{1, \ldots, r\}$ with Ω a finite multiset of oriented edges. In his study of the representation theory of quivers, Kac [14] introduced $A_{\mathbf{v}}(q)$, the number of isomorphism classes of absolutely indecomposable representations of Γ over the finite field $\mathbb{F}_q$ of dimension $\mathbf{v} = (v_1, \ldots, v_r)$ and showed they are polynomials in q. We call $A_{\mathbf{v}}(q)$ the *Kac polynomial* for Γ and $\mathbf{v}$. Following ideas of Kac [14], Hua [13] proved the following generating function identity:

(1.1)

$$\sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} A_{\mathbf{v}}(q)T^{\mathbf{v}} = (q-1)\mathrm{Log}\left(\sum_{\pi=(\pi^1,\ldots,\pi^r)\in\mathcal{P}^r} \frac{\prod_{i\to j\in\Omega} q^{\langle\pi^i,\pi^j\rangle}}{\prod_{i\in I} q^{\langle\pi^i,\pi^i\rangle}\prod_k\prod_{j=1}^{m_k(\pi^i)}(1-q^{-j})} T^{|\pi|}\right),$$

where $\mathcal{P}$ denotes the set of partitions of all positive integers, Log is the plethystic logarithm (see [11, §2.3.3]), $\langle , \rangle$ is the pairing on partitions defined by

$$\langle \lambda, \mu \rangle = \sum_{i,j} \min(i,j) m_i(\lambda) m_j(\mu)$$

with $m_j(\lambda)$ the multiplicity of the part j in the partition λ, $T^{\mathbf{v}} := T_1^{v_1} \cdots T_r^{v_r}$ for some variables T_i and finally $|\pi| := (|\pi_1| \cdots |\pi_r|)$.

Using such generating functions Kac [14] proved that in fact $A_{\mathbf{v}}(q)$ has integer coefficients and formulated two main conjectures. First, he conjectured that for quivers with no loops, the constant term $A_{\mathbf{v}}(0)$ equals the multiplicity of the root $\mathbf{v}$ in the corresponding Kac–Moody algebra. The proof of this conjecture was completed in [10]. We will give a general proof of Kac's second conjecture here.

Conjecture 1.1 ([14, Conj. 2]). *The Kac polynomial $A_{\mathbf{v}}(q)$ has nonnegative coefficients.*

This conjecture was settled for indivisible dimension vectors and any quiver by Crawley-Boevey and Van den Bergh [4] in 2004; they gave a cohomological interpretation of the Kac polynomial for indivisible dimension vectors in terms of the cohomology of an associated Nakajima quiver variety. More precisely [4, (1.1)], they showed that for $\mathbf{v}$ indivisible,

$$A_{\mathbf{v}}(q) = \sum \dim \left(H_c^{2i}(\mathcal{Q}_{\mathbf{v}}; \mathbb{C}) \right) q^{i-d_{\mathbf{v}}}, \tag{1.2}$$

where $\mathcal{Q}_{\mathbf{v}}$ is a certain smooth *generic* complex quiver variety of dimension $2d_{\mathbf{v}}$. Similarly, our proof of the general case will follow by interpreting the coefficients of $A_{\mathbf{v}}(q)$ as the dimensions of the sign isotypical component of cohomology groups of a smooth generic quiver variety $\mathcal{Q}_{\tilde{\mathbf{v}}}$ attached to an extended quiver (see (1.9)).

Recently, Mozgovoy [21] proved Conjecture 1.1 for any dimension vector for quivers with at least one loop at each vertex. His approach uses Efimov's proof [7] of a conjecture of Kontsevich–Soibelman [15] that implies positivity for certain refined DT-invariants of symmetric quivers with no potential.

The goal of Kontsevich–Soibelman's theory is to attach refined (or motivic, or quantum) Donaldson–Thomas invariants (or DT-invariants for short) to Calabi–Yau 3-folds X. The invariants should only depend on the derived category of coherent sheaves on X and some extra data; this raises the possibility of defining DT-invariants for certain Calabi–Yau 3-categories that share the formal properties of the geometric situation but are algebraically easier to study. The simplest of such examples are the Calabi–Yau 3-categories attached to quivers (symmetric or not) with no potential (cf. [9]).

Denote by $\overline{\Gamma} = (I, \overline{\Omega})$ the double quiver, that is $\overline{\Omega} = \Omega \coprod \Omega^{\mathrm{opp}}$, where Ω^{opp} is obtained by reversing all edges in Ω. The refined DT-invariants of $\overline{\Gamma}$ (a slight renormalization of those introduced by Kontsevich and Soibelman) are defined by the following combinatorial construction. For $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geq 0}$, let

$$\delta(\mathbf{v}) := \sum_{i=1}^{r} v_i, \qquad \gamma(\mathbf{v}) := \sum_{i=1}^{r} v_i^2 - \sum_{i \to j \in \Omega} v_i v_j.$$

Then

$$\sum_{\mathbf{v} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} \mathrm{DT}_{\mathbf{v}}(q)\, (-1)^{\delta(\mathbf{v})} T^{\mathbf{v}} := (q-1) \mathrm{Log} \sum_{\mathbf{v} \in \mathbb{Z}^r_{\geq 0}} \frac{(-1)^{\delta(\mathbf{v})} q^{-\frac{1}{2}(\gamma(\mathbf{v}) + \delta(\mathbf{v}))}}{\prod_{i=1}^{r} (1-q^{-1}) \cdots (1-q^{-v_i})} T^{\mathbf{v}}. \tag{1.3}$$

It was proved in [15] that $\mathrm{DT}_{\mathbf{v}}(q) \in \mathbb{Z}[q, q^{-1}]$. In fact, as a consequence of Efimov's proof [7] of [15, Conj. 1], $\mathrm{DT}_{\mathbf{v}}(q)$ actually has nonnegative coefficients. We will give an alternative proof of this in (1.10) by interpreting its coefficients as dimensions of cohomology groups of an associated quiver variety.

Remark 1.2. We should stress that we have restricted to double quivers for the benefit of exposition; our results extend easily to any symmetric quiver. We outline how to treat the general case in Section 3.2.

The technical starting point in this paper is a common generalization of (1.3) and Hua's formula (1.1). Namely, we consider

$$\mathbb{H}(\mathbf{x}_1, \ldots, \mathbf{x}_r; q) := (q-1) \cdot \mathrm{Log} \left(\sum_{\pi = (\pi^1, \ldots, \pi^r) \in \mathcal{P}^r} \frac{\prod_{i \to j \in \Omega} q^{\langle \pi^i, \pi^j \rangle}}{\prod_{i \in I} q^{\langle \pi^i, \pi^i \rangle} \prod_k \prod_{j=1}^{m_k(\pi^i)} (1-q^{-j})} \prod_{i=1}^{r} \tilde{H}_{\pi^i}(\mathbf{x}_i; q) \right), \tag{1.4}$$

where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots)$ is a set of infinitely many independent variables and $\tilde{H}_{\pi^i}(\mathbf{x}_i; q)$ denote the (transformed) Hall–Littlewood polynomial (see [11, §2.3.2]), which is a symmetric function in $\mathbf{x}_i$ and polynomial in q. From (1.4) we can extract many rational functions in q just by pairing against other symmetric functions. For a multi-partition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^r$, we let $s_{\boldsymbol{\mu}} := s_{\mu^1}(\mathbf{x}_1) \cdots s_{\mu^r}(\mathbf{x}_r)$, where $s_{\mu^i}(\mathbf{x}_i)$ is the Schur symmetric function attached to the partition μ^i. Define

$$\mathbb{H}^s_{\boldsymbol{\mu}}(q) := \langle \mathbb{H}(\mathbf{x}_1, \ldots, \mathbf{x}_r; q), s_{\boldsymbol{\mu}} \rangle, \tag{1.5}$$

where $\langle , \rangle$ is the natural extension to r variables of the Hall pairing on symmetric functions, defined by declaring the basis $s_{\boldsymbol{\mu}}$ orthonormal (see [11, (2.3.1)]). Note that *a priori* the $\mathbb{H}^s_{\boldsymbol{\mu}}$ are rational functions in q; we will prove that they are actually polynomials with nonnegative integer coefficients. More precisely, we will show (Theorem 1.4(i)) that the coefficients of $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ are the dimensions of certain cohomology groups.

Our first formal observation is the following

PROPOSITION 1.3. *For any* $\mathbf{v} = (v_1, \ldots, v_r) \in \mathbb{Z}^r_{\geq 0}$, *we have*

(i)

$$A_{\mathbf{v}}(q) = \mathbb{H}^s_{\mathbf{v}^1}(q), \tag{1.6}$$

where $\mathbf{v}^1 := ((v_1), \ldots, (v_r)) \in \mathcal{P}^r$ *and*

(ii)

$$\mathrm{DT}_{\mathbf{v}}(q) = \mathbb{H}^s_{1^{\mathbf{v}}}(q), \tag{1.7}$$

where $1^{\mathbf{v}} := ((1^{v_1}), \ldots, (1^{v_r})) \in \mathcal{P}^r$.

We will prove Part (i) as a consequence of Theorem 2.11 since $h_{\mathbf{v}^1} = s_{\mathbf{v}^1}$, and part (ii) is a special case of Proposition 3.5.

Fix a nonzero multi-partition $\boldsymbol{\mu} \in \mathcal{P}^r$, and let $\mathbf{v} = |\boldsymbol{\mu}| := (|\mu^1|, \ldots, |\mu^r|)$. Associated to the pair $(\Gamma, \mathbf{v})$ we construct a new quiver by attaching a leg of length $v_i - 1$ at the vertex i of Γ, where $v_i := |\mu^i|$. We denote it by $\tilde{\Gamma}_{\mathbf{v}} = (\tilde{I}_{\mathbf{v}}, \tilde{\Omega}_{\mathbf{v}})$. We extend the dimension vector $\mathbf{v} : I \to \mathbb{Z}_{\geq 0}$ to $\tilde{\mathbf{v}} : \tilde{I}_{\mathbf{v}} \to \mathbb{Z}_{\geq 0}$ by placing decreasing dimensions $v_i - 1, v_i - 2, \ldots, 1$ at the extra leg starting with v_i at the original vertex i. We also consider the subgroup $W_{\mathbf{v}} < W$ of the Weyl group of the quiver generated by the reflections at the extra vertices $\tilde{I} \setminus I$. We may identify $W_{\mathbf{v}}$ with $S_{v_1} \times \cdots \times S_{v_r}$, the Weyl group of the group $\mathrm{GL}_{\mathbf{v}} := \mathrm{GL}_{v_1} \times \cdots \times \mathrm{GL}_{v_r}$.

Because by construction $\tilde{\mathbf{v}}$ is indivisible, we can define the corresponding smooth *generic* complex quiver variety $\mathcal{Q}_{\tilde{\mathbf{v}}}$. Note that $\tilde{\mathbf{v}}$ is left invariant by $W_{\mathbf{v}}$ and thus $W_{\mathbf{v}}$ acts on $H^*_c(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C})$ by work of Nakajima [22], [23], Lusztig [18], Maffei [20] and Crawley-Boevey–Holland [5]. We denote by $\chi^{\boldsymbol{\mu}} = \chi^{\mu^1} \cdots \chi^{\mu^r} : W_{\mathbf{v}} \to \mathbb{C}^\times$ the exterior product of the irreducible characters χ^{μ_i} of the symmetric groups S_{v_i} in the notation of [19, §I.7]. In particular, $\chi^{(v_i)}$ is the trivial character and $\chi^{(1^{v_i})}$ is the sign character $\varepsilon_i : S_{v_i} \to \{\pm 1\} < \mathbb{C}^\times$. If $\boldsymbol{\mu}' := ((\mu^1)', \ldots, (\mu^r)')$, where $(\mu^i)'$ is the dual partition of μ^i, then $\chi^{\boldsymbol{\mu}'} = \varepsilon \chi^{\boldsymbol{\mu}}$ with $\varepsilon := \varepsilon_1 \cdots \varepsilon_r$ the sign character of $W_{\mathbf{v}}$.

We may decompose the representation of $W_{\mathbf{v}}$ on $H^*_c(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C})$ into its isotypical components

$$H^*_c(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C}) \cong \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_{\mathbf{v}}} H^*_c(\mathcal{Q}_{\tilde{\mathbf{v}}}; \mathbb{C})_{\chi^{\boldsymbol{\mu}}},$$

where $\mathcal{P}_{\mathbf{v}}$ denotes the set of multi-partitions $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r)$ of size $\mathbf{v} = (v_1, \ldots, v_r)$.

More generally, for a multi-partition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r) \in \mathcal{P}_{\mathbf{v}}$, with $\mu^i = (\mu^i_1, \mu^i_2, \ldots, \mu^i_{l_i})$ and l_i the length of μ^i, denote by $\Gamma_{\boldsymbol{\mu}}$ the quiver obtained from $(\Gamma, \boldsymbol{\mu})$ by adding at each vertex $i \in I$ a leg with $l_i - 1$ edges. We denote by $\mathbf{v}_{\boldsymbol{\mu}}$

the dimension vector of $\Gamma_{\boldsymbol{\mu}}$ with coordinates $v_i, v_i-\mu^i_1, v_i-\mu^i_1-\mu^i_2, \dots, \mu^i_{l_i}$ at the i-th leg. Define

$$d_{\boldsymbol{\mu}} := 1 - \tfrac{1}{2}{}^t\mathbf{v}_{\boldsymbol{\mu}}\mathbf{C}_{\boldsymbol{\mu}}\mathbf{v}_{\boldsymbol{\mu}}, \tag{1.8}$$

with $\mathbf{C}_{\boldsymbol{\mu}}$ the Cartan matrix of $\Gamma_{\boldsymbol{\mu}}$. Notice that if $\boldsymbol{\mu} = (1^{\mathbf{v}})$, then $\tilde{\Gamma}_{\mathbf{v}} = \Gamma_{1^{\mathbf{v}}}$, $\mathbf{v}_{\boldsymbol{\mu}} = \tilde{\mathbf{v}}$; we will write $d_{\tilde{\mathbf{v}}}$ instead of $d_{1^{\mathbf{v}}}$. The quiver variety $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is nonempty if and only if $\tilde{\mathbf{v}}$ is a root of $\Gamma_{\tilde{\mathbf{v}}}$, in which case it has dimension $2d_{\tilde{\mathbf{v}}}$ [2, Th. 1.2].

Our main geometric result is the following

Theorem 1.4. (i) *We have*

$$\mathbb{H}^s_{\boldsymbol{\mu}}(q) = \sum_i \left\langle \varrho^{2i}, \varepsilon\chi^{\boldsymbol{\mu}} \right\rangle_{W_{\mathbf{v}}} q^{i-d_{\tilde{\mathbf{v}}}},$$

where $\langle \varrho^{2i}, \varepsilon\chi^{\boldsymbol{\mu}} \rangle_{W_{\mathbf{v}}}$ *is the multiplicity of* $\varepsilon\chi^{\boldsymbol{\mu}}$ *in the representation* ϱ^{2i} *of* $W_{\mathbf{v}}$ *in* $H^{2i}_c(\mathcal{Q}_{\tilde{\mathbf{v}}};\mathbb{C})$.

(ii) $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ *is nonzero if and only if* $\mathbf{v}_{\boldsymbol{\mu}}$ *is a root of* $\Gamma_{\boldsymbol{\mu}}$, *in which case it is a monic polynomial of degree* $d_{\boldsymbol{\mu}}$. *Moreover,* $\mathbb{H}^s_{\boldsymbol{\mu}}(q) = 1$ *if and only if* $\mathbf{v}_{\boldsymbol{\mu}}$ *is a real root.*

In combination with Proposition 1.3, Theorem 1.4 implies the following

Corollary 1.5. (i) *We have*

$$A_{\mathbf{v}}(q) = \sum_i \dim\left(H^{2i}_c(\mathcal{Q}_{\tilde{\mathbf{v}}};\mathbb{C})_{\varepsilon}\right) q^{i-d_{\tilde{\mathbf{v}}}} \tag{1.9}$$

and

$$\mathrm{DT}_{\mathbf{v}}(q) = \sum_i \dim\left(H^{2i}_c(\mathcal{Q}_{\tilde{\mathbf{v}}};\mathbb{C})^{W_{\mathbf{v}}}\right) q^{i-d_{\tilde{\mathbf{v}}}}. \tag{1.10}$$

(ii) *In particular,* $A_{\mathbf{v}}(q)$ *and* $\mathrm{DT}_{\mathbf{v}}(q)$ *have nonnegative integer coefficients.*

(iii) *Conjecture* 1.1 *holds for any quiver and dimension vector* $\mathbf{v}$.

(iv) *The polynomial* $\mathrm{DT}_{\mathbf{v}}(q)$ *is nonzero if and only if* $\tilde{\mathbf{v}}$ *is a root of* $\tilde{\Gamma}_{\mathbf{v}}$, *in which case it is monic of degree* $d_{\tilde{\mathbf{v}}}$. *Moreover,* $\mathrm{DT}_{\mathbf{v}}(q) = 1$ *if and only if* $\tilde{\mathbf{v}}$ *is a real root.*

Acknowledgments. We would like to thank Sergey Mozgovoy for explaining his work and for useful comments. We thank William Crawley-Boevey, Bernard Keller, Maxim Kontsevich, Andrea Maffei, Hiraku Nakajima, Philippe Satgé, Yan Soibelman and Balázs Szendrői for useful comments. The first author thanks the Royal Society for funding his research 2005–2012 in the form of a Royal Society University Research Fellowship as well as the Mathematical Institute and Wadham College in Oxford for a very productive environment. The second author is supported by Agence Nationale de la Recherche grant ANR-09-JCJC-0102-01. The third author is supported by the NSF grant DMS-1101484 and a Research Scholarship from the Clay Mathematical Institute.

2. **Proof of Theorem 1.4**

2.1. *The quiver varieties* $\mathcal{Q}_{\tilde{\mathbf{v}}}$. Let Γ, I, Ω be as in the introduction. Let $\mathbb{K}$ be any algebraically closed field. For a dimension vector $\mathbf{v} = (v_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$, put

$$\mathbb{K}^{\mathbf{v}} := \bigoplus_{i\in I} \mathbb{K}^{v_i}, \quad \mathrm{GL}_{\mathbf{v}} := \prod_{i\in I} \mathrm{GL}_{v_i}(\mathbb{K}), \quad \mathfrak{gl}_{\mathbf{v}} := \bigoplus_{i\in I} \mathfrak{gl}_{v_i}(\mathbb{K}).$$

By a graded subspace of $V \subseteq \mathbb{K}^{\mathbf{v}}$ we will mean a subspace of the form

$$V = \bigoplus_{i\in I} V_i, \qquad V_i \subseteq \mathbb{K}^{v_i}.$$

The group $\mathrm{GL}_{\mathbf{v}}$ acts on $\mathfrak{gl}_{\mathbf{v}}$ by conjugation. For an element $X = (X_i)_{i\in I} \in \mathfrak{gl}_{\mathbf{v}}$, we put $\mathrm{Tr}(X) := \sum_{i\in I} \mathrm{Tr}(X_i)$. We denote by $T_{\mathbf{v}}$ the maximal torus of $\mathrm{GL}_{\mathbf{v}}$ whose elements are of the form $(g_i)_{i\in I}$ with g_i a diagonal matrix for each $i \in I$. The Weyl group $W_{\mathbf{v}} := N_{\mathrm{GL}_{\mathbf{v}}}(T_{\mathbf{v}})/T_{\mathbf{v}}$ of $\mathrm{GL}_{\mathbf{v}}$ with respect to $T_{\mathbf{v}}$ is isomorphic to $\prod_{i\in I} S_{v_i}$, where S_v denotes the symmetric group in v letters. Recall that a semisimple element $X \in \mathfrak{gl}_{\mathbf{v}}$ is *regular* if $C_{\mathrm{GL}_{\mathbf{v}}}(X)$ is a maximal torus of $\mathrm{GL}_{\mathbf{v}}$; i.e., the eigenvalues of the coordinates of X are all with multiplicity 1.

We say that an adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_{\mathbf{v}}$ is *generic* if $\mathrm{Tr}(X) = 0$ and if for any graded subspace $V \subseteq \mathbb{K}^{\mathbf{v}}$ stable by some $X \in \mathcal{O}$ such that

$$\mathrm{Tr}\left(X|_V\right) = 0,$$

then either $V = 0$ or $V = \mathbb{K}^{\mathbf{v}}$. We fix such a generic regular semisimple adjoint orbit $\mathcal{O} \subset \mathfrak{gl}_{\mathbf{v}}$. (We can prove as in [11, §2.2] that such a choice is always possible.)

Let $\overline{\Gamma}$ be the *double quiver* of Γ; namely, $\overline{\Gamma}$ has the same vertices as Γ, but for each arrow $\gamma \in \Omega$ going from i to j, we add a new arrow γ^* going from j to i. We denote by $\overline{\Omega} = \{\gamma, \gamma^* \mid \gamma \in \Omega\}$ the set of arrows of $\overline{\Gamma}$. Consider the space

$$\mathrm{Rep}_{\mathbb{K}}\left(\overline{\Gamma}, \mathbf{v}\right) := \bigoplus_{i\to j\in\overline{\Omega}} \mathrm{Mat}_{v_j,v_i}(\mathbb{K})$$

of representations of $\overline{\Gamma}$ with dimension $\mathbf{v}$. Recall that $\mathrm{GL}_{\mathbf{v}}$ acts on $\mathrm{Rep}_{\mathbb{K}}\left(\overline{\Gamma}, \mathbf{v}\right)$ as

$$(g\cdot\varphi)_{i\to j} = g_j \varphi_{i\to j} g_i^{-1}$$

for any $g = (g_i)_{i\in I} \in \mathrm{GL}_{\mathbf{v}}$, $\varphi = (\varphi_\gamma)_{\gamma\in\overline{\Omega}} \in \mathrm{Rep}_{\mathbb{K}}\left(\overline{\Gamma}, \mathbf{v}\right)$ and any arrow $i \to j \in \overline{\Omega}$.

Let $\tilde{\Gamma}_{\mathbf{v}}$ on vertex set $\tilde{I}_{\mathbf{v}}$ be the quiver obtained from $(\Gamma, \mathbf{v})$ by adding at each vertex $i \in I$ a leg of length $v_i - 1$ with the edges all oriented toward the vertex i. Define $\tilde{\mathbf{v}} \in \mathbb{Z}^{\tilde{I}_{\mathbf{v}}}_{\geq 0}$ as the dimension vector with coordinate v_i at $i \in I \subset \tilde{I}_{\mathbf{v}}$ and with coordinates $(v_i - 1, v_i - 2, \ldots, 1)$ on the leg attached to the vertex $i \in I$.

Let $\mathcal{Q}_{\tilde{\mathbf{v}}}$ be the quiver variety over $\mathbb{K}$ attached to the quiver $\tilde{\Gamma}$ and parameter set defined from the eigenvalues of $\mathcal{O}$. (See [11] and the reference therein.)

Concretely, define the moment map $\mu_{\mathbf{v}} : \mathrm{Rep}_{\mathbb{K}}\left(\overline{\Gamma}, \mathbf{v}\right) \to \mathfrak{gl}_{\mathbf{v}}^0$, $(x_\gamma)_{\gamma\in\overline{\Omega}} \mapsto \sum_{\gamma\in\Omega}[x_\gamma, x_{\gamma^*}]$, where $\mathfrak{gl}_{\mathbf{v}}^0 := \{X \in \mathfrak{gl}_{\mathbf{v}} \mid \mathrm{Tr}\,(X) = 0\}$. Then $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is the affine GIT quotient

$$\mu_{\mathbf{v}}^{-1}(\mathcal{O})\,//\mathrm{GL}_{\mathbf{v}} := \mathrm{Spec}\,\left(\mathbb{K}[\mu_{\mathbf{v}}^{-1}(\mathcal{O})]^{\mathrm{GL}_{\mathbf{v}}}\right). \tag{2.1}$$

Note that the one-dimensional torus $\mathbb{G}_m$ embeds naturally in $\mathrm{GL}_{\mathbf{v}}$ as $t \mapsto (t \cdot I_{v_i})_{i\in I}$ where I_v is the identity matrix of GL_v. The action of $\mathrm{GL}_{\mathbf{v}}$ on $\mu_{\mathbf{v}}^{-1}(\mathcal{O})$ factorizes through an action of $G_{\mathbf{v}} := \mathrm{GL}_{\mathbf{v}}/\mathbb{G}_m$.

We have the following theorem whose proof is similar to that of [11, Th. 2.2.4].

THEOREM 2.1. (i) *The variety $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is nonsingular and the quotient map $\mu_{\mathbf{v}}^{-1}(\mathcal{O}) \to \mathcal{Q}_{\tilde{\mathbf{v}}}$ is a principal $G_{\mathbf{v}}$-bundle in the étale topology.*

(ii) *The odd degree cohomology of $\mathcal{Q}_{\tilde{\mathbf{v}}}$ vanishes.*

2.2. *Weyl group action.*

2.2.1. *Weyl group action.* Let $\mathbb{K}$ denote an arbitrary algebraically closed field as before. Let ℓ be a prime different from the characteristic $\mathrm{char}\,\mathbb{K}$ of $\mathbb{K}$, and for an algebraic variety over $\mathbb{K}$, denote by $H_c^i(X;\overline{\mathbb{Q}}_\ell)$ the compactly supported ℓ-adic cohomology.

Denote by $\mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$ the generic regular semisimple elements of the Lie algebra $\mathfrak{t}_{\mathbf{v}}$ of $T_{\mathbf{v}}$. For $\sigma \in \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$, define

$$\begin{aligned}\mathcal{M}_\sigma := \Big\{&(\varphi, X, gT_{\mathbf{v}}) \in \mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v})\\ &\times \mathfrak{gl}_{\mathbf{v}} \times (\mathrm{GL}_{\mathbf{v}}/T_{\mathbf{v}}) \mid g^{-1}Xg = \sigma,\, \mu_{\mathbf{v}}(\varphi) = X\Big\}//\mathrm{GL}_{\mathbf{v}},\end{aligned}$$

where $\mathrm{GL}_{\mathbf{v}}$ acts by

$$g \cdot (\varphi, X, hT_{\mathbf{v}}) = (g \cdot \varphi, gXg^{-1}, ghT_{\mathbf{v}}).$$

The following lemma is immediate.

LEMMA 2.2. *The projection $(\varphi, X, gT_{\mathbf{v}}) \to \varphi$ induces an isomorphism from $\mathcal{M}_\sigma$ onto the quiver variety associated to the adjoint orbit of σ as in* (2.1).

For $w \in W_{\mathbf{v}}$, denote by $w : \mathcal{M}_\sigma \to \mathcal{M}_{\dot{w}\sigma\dot{w}^{-1}}$ the isomorphism $(\varphi, X, gT_{\mathbf{v}}) \mapsto (\varphi, X, g\dot{w}^{-1}T_{\mathbf{v}})$ and by w^* the induced isomorphism on cohomology. The first aim of this section is to prove the following theorem.

THEOREM 2.3. *There exists a prime p_0 such that if* $\mathrm{char}\,\mathbb{K} \geq p_0$ *or if* $\mathbb{K} = \mathbb{C}$, *then for any $\sigma, \tau \in \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$ there exists a canonical isomorphism $i_{\sigma,\tau}$: $H_c^i(\mathcal{M}_\sigma;\overline{\mathbb{Q}}_\ell) \to H_c^i(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell)$ that commutes with w^*. Moreover for all $\sigma, \tau, \zeta \in \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$, we have $i_{\sigma,\tau} \circ i_{\zeta,\sigma} = i_{\zeta,\tau}$.*

Before writing the proof let us explain the rough strategy. Put

$$\mathcal{M} := \Big\{(\varphi, X, gT_{\mathbf{v}}, \sigma) \in \mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v}) \times \mathfrak{gl}_{\mathbf{v}} \times (\mathrm{GL}_{\mathbf{v}}/T_{\mathbf{v}}) \times \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}} \,\Big|\, g^{-1}Xg = \sigma, \mu_{\mathbf{v}}(\varphi) = X\Big\} //\mathrm{GL}_{\mathbf{v}},$$

where $\mathrm{GL}_{\mathbf{v}}$ acts on the first three coordinates as before and trivially on the last one. Denote by $\pi : \mathcal{M} \to \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$ the projection to the last coordinate. Note that the stalk at σ of the sheaf $R^i\pi_!\overline{\mathbb{Q}}_\ell$ is $H_c^i(\pi^{-1}(\sigma); \overline{\mathbb{Q}}_\ell) = H_c^i(\mathcal{M}_\sigma; \overline{\mathbb{Q}}_\ell)$. Since π commutes with Weyl group actions, to prove Theorem 2.3 we need to prove that the sheaf $R^i\pi_!\overline{\mathbb{Q}}_\ell$ is constant. Unfortunately we do not know any algebraic proof of this last statement so we do not have a proof that works independently from $\mathrm{char}\,\mathbb{K}$. We follow the same strategy as in [4, Proof of Prop. 2.3.1] proving first the statement with $\mathbb{K} = \mathbb{C}$ using the hyperkähler structure on quiver varieties and then proving the positive characteristic case by reducing modulo p.

We prefer to work with étale $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves instead of ℓ-adic sheaves. We will show that the sheaves $R^i\pi_!\mathbb{Z}/\ell^n\mathbb{Z}$ are constant for all $n \geq 1$ assuming that $\mathrm{char}\,\mathbb{K}$ is either sufficiently large or equal to 0. This will prove Theorem 2.3 for the étale cohomology $H_c^i(\mathcal{M}_\sigma; \mathbb{Z}/\ell^n\mathbb{Z})$ with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$. We then pass to the direct limit to get the statement for ℓ-adic cohomology.

Proof of Theorem 2.3. (i) Assume that $\mathbb{K} = \mathbb{C}$. From the equivalence of categories between constructable étale sheaves on a complex variety and constructable sheaves on the underlying topological space (see [1, §6] for more details), we are reduced to prove that if $\mathbb{Z}/\ell^n\mathbb{Z}$ is the constant sheaf (for the analytic topology) on the complex variety $\mathcal{M}$, then the analytic sheaf $R^i\pi_!\mathbb{Z}/\ell^n\mathbb{Z}$ is constant on $\mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$. But this is exactly what is proved in [20, Lemma 48] where the hyperkähler structure on quiver varieties is used as an essential ingredient.

We note that in [20] it is assumed that the quiver does not have loops. The proof of [20, Lemma 48], however, remains valid in the general case. The only nontrivial ingredient is the proof of the surjectivity [20, CON3 §5] of the hyperkähler moment map over the regular locus in the case for quivers with possible loops. A more recent result [3, Th. 2] implies that as long as the dimension vector is a root, the complex moment map is surjective for any quiver. By hyperkähler rotation we get that the corresponding hyperkähler moment map is also surjective.

(ii) The $\mathbb{C}$-schemes $\mathcal{M}, \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$ and the projection $\pi : \mathcal{M} \to \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$ to the last coordinate can actually be defined over $\mathbb{Z}$ (see [4, App. B]). We will denote these by $\mathcal{M}/_{\mathbb{Z}}$, $\mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}/_{\mathbb{Z}}$, $\pi/_{\mathbb{Z}} : \mathcal{M}/_{\mathbb{Z}} \to \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}/_{\mathbb{Z}}$ and denote by $\mathcal{F} = \mathcal{F}_{\mathbb{Z}}$ the sheaf $R^i(\pi/_{\mathbb{Z}})_!\mathbb{Z}/\ell^n\mathbb{Z}$. Recall that if $f : X \to \mathrm{Spec}\,\mathbb{Z}$ denotes the structure map of a $\mathbb{Z}$-scheme, then by Deligne [6, Th. 1.9], for any constructable $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf $\mathcal{E}$ on X, there exists an open dense subset U of $\mathrm{Spec}\,\mathbb{Z}$ such that for any base change $S \to U \subset \operatorname{Spec}\mathbb{Z}$, we have $(f_*\mathcal{E})_S \simeq (f_S)_*(\mathcal{E}_S)$. Denote by $t/_{\mathbb{Z}}$ the structure map of $\mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}/_{\mathbb{Z}}$. We are thus reduced to prove that the canonical map

$$\eta : (t/_{\mathbb{Z}})^*(t/_{\mathbb{Z}})_*\mathcal{F} \to \mathcal{F}$$

given by adjointness is an isomorphism over an open subset U of $\operatorname{Spec}\mathbb{Z}$. Indeed, this will prove that for any prime p such that $p\mathbb{Z} \in U$, the map $\eta/_{\overline{\mathbb{F}}_p}$ obtained from η by base change is an isomorphism, which is equivalent to saying that the sheaf $\mathcal{F}_{\overline{\mathbb{F}}_q} \simeq R^i(\pi/_{\overline{\mathbb{F}}_q})_!\mathbb{Z}/\ell^n\mathbb{Z}$ is constant. By (i) we know that the sheaf $\mathcal{F}_{\mathbb{C}}$ is constant; i.e., the isomorphism $\eta_{\mathbb{C}} : (t/_{\mathbb{C}})^*(t/_{\mathbb{C}})_*\mathcal{F}_{\mathbb{C}} \to \mathcal{F}_{\mathbb{C}}$ of étale sheaves obtained from η by base change is an isomorphism. Hence if $\mathcal{K}$ and $\mathcal{C}$ denote respectively the kernel and co-kernel of η, then $\mathcal{K}_{\mathbb{C}} = 0$ and $\mathcal{C}_{\mathbb{C}} = 0$. Since by Deligne [6, Th. 1.9] the sheaf $(t/_{\mathbb{Z}})^*(t/_{\mathbb{Z}})_*\mathcal{F}$ is constructable over an open subset V of $\operatorname{Spec}\mathbb{Z}$, the sheaves $\mathcal{K}$ and $\mathcal{C}$ are also constructable over V and so the support of $\mathcal{K}_V$ and $\mathcal{C}_V$ are constructable sets. Since $\mathcal{K}_{\mathbb{C}} = 0$ and $\mathcal{C}_{\mathbb{C}} = 0$, the supports do not contain the generic point and so there exists an open subset U of V such that $\mathcal{K}_U = \mathcal{C}_U = 0$. □

Assume that $\operatorname{char}\mathbb{K}$ is as in Theorem 2.3. For $w \in W_{\mathbf{v}}$ and $\tau \in \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$, define

$$\rho^i(w) : H_c^i(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell) \to H_c^i(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell)$$

as the composition $i_{w\tau w^{-1},\tau} \circ (w^{-1})^*$. The following proposition is a straightforward consequence of Theorem 2.3.

Proposition 2.4. *The map*

$$\begin{array}{rccc} \rho^i = \rho^i_{\mathbb{K}} : & W_{\mathbf{v}} & \to & \mathrm{GL}\left(H_c^i(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell)\right) \\ & w & \mapsto & \rho^i(w) \end{array}$$

is a representation of $W_{\mathbf{v}}$ *that does not depend on the choice of* $\tau \in \mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}}$.

The following proposition follows formally from the base change techniques used in the proof of Theorem 2.3.

Proposition 2.5. *Assume that* $\operatorname{char}\mathbb{K} \gg 0$. *Then the* $\overline{\mathbb{Q}}_\ell$*-representations* $\rho^i_{\mathbb{K}}$ *and* $\rho^i_{\mathbb{C}}$ *are isomorphic.*

Note that the representation $\rho^i_{\mathbb{C}}$ of $W_{\mathbf{v}}$ on $H_c^i(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell)$ is defined so that it agrees via the comparison theorem with the action ϱ^i of $W_{\mathbf{v}}$ on the compactly supported cohomology $H_c^i(\mathcal{M}_\tau;\mathbb{C})$ as defined from [20, Lemma 48].

2.2.2. *Introducing Frobenius.* Here $\mathbb{K}$ is an algebraic closure of a finite field $\mathbb{F}_q$. We use the same letter F to denote the Frobenius endomorphism on $\operatorname{Rep}_{\mathbb{K}}(\Gamma,\mathbf{v})$ and $\mathfrak{gl}_{\mathbf{v}}$ that raises entries of matrices to their q-th power. There is a well-defined map from the set of F-stable regular semisimple orbits onto the set of conjugacy classes of the Weyl group $W_{\mathbf{v}}$ of $\mathrm{GL}_{\mathbf{v}}$. This map is given by Frobenius action on eigenvalues of F-stable orbits.

Given $w \in W_{\mathbf{v}}$, choose a generic regular semisimple adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_{\mathbf{v}}(\mathbb{F}_q)$ mapping to w and denote by $\mathcal{Q}^w_{\tilde{\mathbf{v}}}$ the corresponding quiver variety associated to $\mathcal{O}$ defined by (2.1). If $w = 1$, then we will denote these simply by $\mathcal{Q}_{\tilde{\mathbf{v}}}$ instead of $\mathcal{Q}^1_{\tilde{\mathbf{v}}}$; note that in this case the orbit $\mathcal{O}$ has all its eigenvalues in $\mathbb{F}_q$. Since $\mathcal{O}$ is F-stable, the quiver variety $\mathcal{Q}^w_{\tilde{\mathbf{v}}}$ inherits an action of the Frobenius endomorphism, which we also denote by F.

By Lemma 2.2 and Proposition 2.4, we have a well-defined representation ρ^i of $W_{\mathbf{v}}$ in $H^i_c(\mathcal{Q}_{\tilde{\mathbf{v}}}, \overline{\mathbb{Q}}_\ell)$ assuming that the characteristic of $\mathbb{K}$ is large enough.

The aim of this section is to prove the following theorem.

THEOREM 2.6. *Assume that* $\operatorname{char}\mathbb{K} \gg 0$. *We have*

$$\#\mathcal{Q}^w_{\tilde{\mathbf{v}}}(\mathbb{F}_q) = \sum_i \operatorname{Tr}\left(\rho^{2i}(w), H^{2i}_c(\mathcal{Q}_{\tilde{\mathbf{v}}}; \overline{\mathbb{Q}}_\ell)\right) q^i.$$

We keep the notation introduced in Section 2.2.1. The Frobenius morphism $(\varphi, X, gT_{\mathbf{v}}) \mapsto (F(\varphi), F(X), F(g)T_{\mathbf{v}})$ defines a bijective morphism $F : \mathcal{M}_\sigma \to \mathcal{M}_{F(\sigma)}$ with induced morphism $F^* : H^i_c(\mathcal{M}_\tau; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathcal{M}_{F^{-1}(\tau)}; \overline{\mathbb{Q}}_\ell)$. Since the map $\pi : \mathcal{M} \to \mathfrak{t}^{\mathrm{gen}}_{\mathbf{v}}$ commutes with F, the canonical maps $i_{\sigma,\tau}$, with $\sigma, \tau \in \mathfrak{t}^{\mathrm{gen}}_{\mathbf{v}}$, commute with F^*.

For $w \in W_{\mathbf{v}}$, consider the w-twisted Frobenius endomorphism wF on $\mathfrak{gl}_{\mathbf{v}}$ defined as $X \mapsto \dot{w}F(X)\dot{w}^{-1}$, where $\dot{w}$ is a representative of w in $N_{\mathrm{GL}_{\mathbf{v}}}(T_{\mathbf{v}})$. Let $\sigma \in (\mathfrak{t}^{\mathrm{gen}}_{\mathbf{v}})^{wF}$. Since $wF(\sigma) = \sigma$, we get a Frobenius endomorphism

$$\begin{array}{rccc} wF: & \mathcal{M}_\sigma & \to & \mathcal{M}_\sigma \\ & (\varphi, X, gT_{\mathbf{v}}) & \mapsto & (F(\varphi), F(X), F(g)\dot{w}^{-1}T_{\mathbf{v}}). \end{array}$$

Let $\tau \in (\mathfrak{t}^{\mathrm{gen}}_{\mathbf{v}})^F$. By Theorem 2.3, the following diagram commutes:

$$\begin{tikzcd} H^i_c(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell) \arrow[r, "\rho^i(w)"] & H^i_c(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell) \arrow[r, "F^*"] & H^i_c(\mathcal{M}_\tau;\overline{\mathbb{Q}}_\ell) \\ H^i_c(\mathcal{M}_\sigma;\overline{\mathbb{Q}}_\ell) \arrow[u, "i_{\sigma,\tau}"] \arrow[r, "w^*"] & H^i_c(\mathcal{M}_{F(\sigma)};\overline{\mathbb{Q}}_\ell) \arrow[u, "i_{F(\sigma),\tau}"] \arrow[r, "F^*"] & H^i_c(\mathcal{M}_\sigma;\overline{\mathbb{Q}}_\ell). \arrow[u, "i_{\sigma,\tau}"'] \end{tikzcd} \tag{2.2}$$

Note that the arrow labeled by w^* is well defined as $F(\sigma) = \dot{w}^{-1}\sigma\dot{w}$.

Using the above diagram, the fact that $\mathcal{M}_\tau$ has vanishing odd cohomology and applying the Grothendieck trace formula to $wF : \mathcal{M}_\sigma \to \mathcal{M}_\sigma$, we find that

$$\#\mathcal{M}_\sigma(\mathbb{K})^{wF} = \sum_i \operatorname{Tr}\left(F^* \circ \rho^{2i}(w), H^{2i}_c(\mathcal{M}_\tau; \overline{\mathbb{Q}}_\ell)\right). \tag{2.3}$$

It is well known [4] that the cohomology of any generic quiver variety defined over $\mathbb{F}_q$ (i.e. with parameters in $\mathbb{F}_q^I$) is *pure*, in the sense that the eigenvalues of the Frobenius F^* on the compactly supported i-th cohomology have absolute value $q^{i/2}$. (It is proved in [11], [10] that over $\mathbb{C}$ the cohomology has pure mixed Hodge structure.) It is also well known that these quiver varieties are *polynomial count*, i.e., there exists a polynomial $P(T) \in \mathbb{Q}[T]$ such that for any finite field extension $\mathbb{F}_{q^n}$, the evaluation of P at q^n counts the number of points of the variety over $\mathbb{F}_{q^n}$. (See, for instance, [4], [10], [11].) Since $\tau \in (\mathfrak{t}_{\mathbf{v}}^{\mathrm{gen}})^F$, the variety $\mathcal{M}_\tau$ is thus pure and polynomial count. Hence by [4, App. A], the automorphism F^* on $H_c^{2i}(\mathcal{M}_\tau; \overline{\mathbb{Q}}_\ell)$ has a unique eigenvalue q^i. Hence using that the two automorphisms $\rho^i(w)$ and F^* commute for all i and $w \in W_{\mathbf{v}}$, we deduce from formula (2.3) that

$$\#\mathcal{M}_\sigma(\mathbb{K})^{wF} = \sum_i \mathrm{Tr}\left(\rho^{2i}(w), H_c^{2i}(\mathcal{M}_\tau; \overline{\mathbb{Q}}_\ell)\right) q^i. \tag{2.4}$$

Hence Theorem 2.6 follows from (2.4) and the following lemma.

Lemma 2.7. *Let $w \in W_{\mathbf{v}}$, and let σ be a representative of the orbit $\mathcal{O}^w$ in $(\mathfrak{t}_{\mathbf{v}})^{wF}$. Then*

$$\#\mathcal{M}_\sigma(\mathbb{K})^{wF} = \#\mathcal{Q}_{\tilde{\mathbf{v}}}^w(\mathbb{K})^F.$$

Proof. The proof follows from the fact that the isomorphism $\mathcal{M}_\sigma \to \mathcal{Q}_{\tilde{\mathbf{v}}}^w$, $(\varphi, X, gT_{\mathbf{v}}) \mapsto \varphi$ of Lemma 2.2 commutes with wF and F. □

2.3. *Counting points of $\mathcal{Q}_{\tilde{\mathbf{v}}}^w$ over finite fields.* In this section we will evaluate $\#\mathcal{Q}_{\tilde{\mathbf{v}}}^w(\mathbb{F}_q)$. As in Section 1 we label the vertices of Γ by $\{1, \ldots, r\}$ and we denote by $\mathcal{P}_{\mathbf{v}}$ the set of all multi-partitions $(\mu^1, \ldots, \mu^r)$ of size $(v_1, \ldots, v_r)$. The conjugacy class of an element $w = (w_1, \ldots, w_r) \in W_{\mathbf{v}}$ determines a multi-partition $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r) \in \mathcal{P}_{\mathbf{v}}$, where λ^i is the cycle type of $w_i \in S_{v_i}$. We will call $\boldsymbol{\lambda}$ the *cycle type* of w.

Let $p_{\boldsymbol{\lambda}} := p_{\lambda^1}(\mathbf{x}_1) \cdots p_{\lambda^r}(\mathbf{x}_r)$, where for a partition λ, $p_\lambda(\mathbf{x}_i)$ is the corresponding power symmetric function in the variables of $\mathbf{x}_i = \{x_{i,1}, x_{i,2}, \ldots\}$ (see [19, Chap. I, §2]). Recall that ε denotes the sign character of $W_{\mathbf{v}}$. Denote by $\tilde{\mathbf{C}}_{\mathbf{v}}$ the Cartan matrix of the quiver $\tilde{\Gamma}_{\mathbf{v}}$. Then

$$d_{\tilde{\mathbf{v}}} := 1 - \tfrac{1}{2}{}^t\tilde{\mathbf{v}}\tilde{\mathbf{C}}_{\mathbf{v}}\tilde{\mathbf{v}}$$

equals $\frac{1}{2}\dim \mathcal{Q}_{\tilde{\mathbf{v}}}$ if $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is nonempty.

The aim of this section is to prove the following theorem.

Theorem 2.8. *Let $w \in W_{\mathbf{v}}$ have cycle type $\boldsymbol{\lambda} \in \mathcal{P}_{\mathbf{v}}$. Then*

$$\#\mathcal{Q}_{\tilde{\mathbf{v}}}^w(\mathbb{F}_q) = q^{d_{\tilde{\mathbf{v}}}} \varepsilon(w) \left\langle \mathbb{H}(\mathbf{x}_1, \ldots, \mathbf{x}_r; q), p_{\boldsymbol{\lambda}} \right\rangle. \tag{2.5}$$

Fix a nontrivial additive character $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$, and for $X, Y \in \mathfrak{gl}_{\mathbf{v}}$, put $\langle X, Y \rangle := \mathrm{Tr}(XY)$. Denote by $C(\mathfrak{gl}_{\mathbf{v}})$ the $\mathbb{C}$-vector space of all functions $\mathfrak{gl}_{\mathbf{v}}^F \to \mathbb{C}$, and define the Fourier transform $\mathcal{F} : C(\mathfrak{gl}_{\mathbf{v}}) \to C(\mathfrak{gl}_{\mathbf{v}})$ by

$$\mathcal{F}(f)(X) := \sum_{Y \in \mathfrak{gl}_{\mathbf{v}}^F} \Psi\left(\langle X, Y \rangle\right) f(Y),$$

with $f \in C(\mathfrak{gl}_{\mathbf{v}})$ and $X \in \mathfrak{gl}_{\mathbf{v}}^F$. Basic properties of $\mathcal{F}$ can be found, for instance, in [16]. For an F-stable adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_{\mathbf{v}}$, we denote by $1_{\mathcal{O}} \in C(\mathfrak{gl}_{\mathbf{v}})$ the characteristic function of the adjoint orbit $\mathcal{O}^F$ of $\mathfrak{gl}_{\mathbf{v}}^F$; i.e., $1_{\mathcal{O}}(X) = 1$ if $X \in \mathcal{O}^F$ and $1_{\mathcal{O}}(X) = 0$ if $X \notin \mathcal{O}^F$.

PROPOSITION 2.9. *For any F-stable generic adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_{\mathbf{v}}^0$, we have*

(2.6)

$$\begin{aligned}\#\left(\mu_{\mathbf{v}}^{-1}(\mathcal{O})//\mathrm{GL}_{\mathbf{v}}\right)^F &= \frac{q-1}{|\mathrm{GL}_{\mathbf{v}}^F|}\ \#\mu_{\mathbf{v}}^{-1}(\mathcal{O})^F \\ &= \frac{(q-1)\,|\mathrm{Rep}_{\mathbb{F}_q}(\Gamma,\mathbf{v})|}{|\mathrm{GL}_{\mathbf{v}}^F|\cdot|\mathfrak{gl}_{\mathbf{v}}^F|}\sum_{X\in\mathfrak{gl}_{\mathbf{v}}^F}\#\left\{\varphi\in\mathrm{Rep}_{\mathbb{F}_q}(\Gamma,\mathbf{v})\,\Big|\,[X,\varphi]=0\right\}\mathcal{F}(1_{\mathcal{O}})(X),\end{aligned}$$

where $[X,\varphi]=0$ means that for each arrow $\gamma = i \to j$ in Ω, we have $X_j\varphi_\gamma = \varphi_\gamma X_i$.

Proof. The first equality comes from the fact that $G_{\mathbf{v}} = \mathrm{GL}_{\mathbf{v}}/\mathbb{G}_m$ is connected and acts freely on $\mu_{\mathbf{v}}^{-1}(\mathcal{O})$ (see Theorem 2.1(i)). For the second, write

$$\#\,\mu_{\mathbf{v}}^{-1}(\mathcal{O})^F = \sum_{z\in\mathcal{O}^F}\#\,\mu_{\mathbf{v}}^{-1}(z)^F$$

and use [10, Prop. 2]. □

In order to compute the right-hand side of formula (2.6), we need to introduce some notation. Denote by $\mathfrak{O}$ the set of all F-orbits of $\mathbb{K}$. The adjoint orbits of $\mathfrak{gl}_n^F$ are parametrized by the maps $h : \mathfrak{O} \to \mathcal{P}$ such that $\sum_{\gamma\in\mathfrak{O}}|\gamma|\cdot|h(\gamma)| = n$. Denote by $0 \in \mathcal{P}$ the unique partition of 0. The *type* of an adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_n^F$ corresponding to $h : \mathfrak{O} \to \mathcal{P}$ is defined as the map $\omega_{\mathcal{O}}$ that assigns to a positive integer d and a nonzero partition λ the number of Frobenius orbits $\gamma \in \mathfrak{O}$ of degree d such that $h(\gamma) = \lambda$.

It is sometimes also convenient (see [11]) to write a type as follows. Choose a total ordering $\geq$ on partitions that we extend to a total ordering on the set $\mathbb{Z}_{>0}\times(\mathcal{P}\backslash\{0\})$ as $(d,\lambda) \geq (d',\lambda')$ if $d \geq d'$ and $\lambda \geq \lambda'$. Then we may write the type $\omega_{\mathcal{O}}$ as a the strictly decreasing sequence $(d_1,\lambda^1)^{n_1}(d_2,\lambda^2)^{n_2}\cdots(d_s,\lambda^s)^{n_s}$ with $n_i = \omega_{\mathcal{O}}(d_i,\lambda^i)$. The set of all nonincreasing sequences $(d_1,\lambda^1)(d_2,\lambda^2)\cdots(d_s,\lambda^s)$ of size n (i.e., $\sum_{i=1}^s d_i|\lambda^i| = n$) denoted by $\mathbb{T}_n$ parametrizes the types of the adjoint obits of $\mathfrak{gl}_n^F$.

It is easy to extend this to adjoint orbits of $\mathfrak{gl}_{\mathbf{v}}^F$. They are parametrized by the set of all maps $h = (h_1,\dots,h_r) : \mathfrak{O} \to \mathcal{P}^r$ such that for each $i = 1,\dots,r$, we have $\sum_{\gamma\in\mathfrak{O}}|\gamma|\cdot|h_i(\gamma)| = v_i$. A *type* of an adjoint orbit $\mathcal{O}$ of $\mathfrak{gl}_{\mathbf{v}}^F$ corresponding to $h : \mathfrak{O} \to \mathcal{P}^r$ is now a map $\boldsymbol{\omega}_{\mathcal{O}}$ that assigns to a positive integer d and a nonzero multi-partition $\boldsymbol{\lambda} = (\lambda^1,\dots,\lambda^r)$ the number of Frobenius orbits $\gamma \in \mathfrak{O}$ of degree d such that $h(\gamma) = \boldsymbol{\lambda}$.

As above, after choosing a total ordering on the set $\mathbb{Z}_{>0}\times(\mathcal{P}^r\backslash\{0\})$ we may write $\boldsymbol{\omega}_{\mathcal{O}}$ as a (strictly) decreasing sequence $(d_1,\boldsymbol{\lambda}_1)^{n_1}(d_2,\boldsymbol{\lambda}_2)^{n_2}\cdots(d_s,\boldsymbol{\lambda}_s)^{n_s}$ with $n_i=\boldsymbol{\omega}_{\mathcal{O}}(d_i,\boldsymbol{\lambda}_i)$. We denote by $\mathbb{T}_{\mathbf{v}}$ the set of all nonincreasing sequences $(d_1,\boldsymbol{\lambda}_1)\ldots(d_s,\boldsymbol{\lambda}_s)$ of size $\mathbf{v}$ so that $\mathbb{T}_{\mathbf{v}}$ parametrizes the types of the adjoint orbits of $\mathfrak{gl}_{\mathbf{v}}^F$. We may also write a type $\boldsymbol{\omega}\in\mathbb{T}_{\mathbf{v}}$ as $(\omega_1,\ldots,\omega_r)$, where $\omega_i=(d_1,\lambda^{i,1})(d_2,\lambda^{i,2})\cdots$, with $\lambda^{i,j}$ the i-th coordinate of $\boldsymbol{\lambda}_j$, is a type in $\mathbb{T}_{v_i}$.

Given any family $\{A_{\boldsymbol{\mu}}(\mathbf{x}_1,\ldots,\mathbf{x}_r;q)\}_{\boldsymbol{\mu}\in\mathcal{P}^r}$ of functions separately symmetric in each set $\mathbf{x}_1,\ldots,\mathbf{x}_k$ of infinitely many variables with $A_0=1$, we extend its definition to types $\boldsymbol{\omega}=(d_1,\boldsymbol{\lambda}_1)\cdots(d_s,\boldsymbol{\lambda}_s)\in\mathbb{T}_{\mathbf{v}}$ as

$$A_{\boldsymbol{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_r;q):=\prod_{i=1}^{s}A_{\boldsymbol{\lambda}_i}(\mathbf{x}_1^{d_i},\ldots,\mathbf{x}_r^{d_i};q^{d_i}),$$

where $\mathbf{x}^d$ stands for all the variables $x_1,x_2,\ldots$ in $\mathbf{x}$ replaced by $x_1^d,x_2^d,\ldots$.

For $\pi=(\pi^1,\ldots,\pi^r)\in\mathcal{P}^r$, put

$$\mathcal{A}_\pi(q)=\prod_{i\to j\in\Omega}q^{\langle\pi^i,\pi^j\rangle},$$

$$\mathcal{Z}_\pi(q):=\prod_{i\in I}q^{\langle\pi^i,\pi^i\rangle}\prod_k\prod_{j=1}^{m_k(\pi^i)}(1-q^{-j}),\qquad \mathcal{H}_\pi(q):=\frac{\mathcal{A}_\pi(q)}{\mathcal{Z}_\pi(q)},$$

where we use the same notation as in Section 1. Then by [13, Th. 3.4], for any element X in an adjoint orbit of $\mathfrak{gl}_{\mathbf{v}}^F$ of type $\boldsymbol{\omega}\in\mathbb{T}_{\mathbf{v}}$, we have

$$\#\left\{\varphi\in\mathrm{Rep}_{\mathbb{F}_q}(\Gamma,\mathbf{v})\,\middle|\,[X,\varphi]=0\right\}=\mathcal{A}_{\boldsymbol{\omega}}(q),\qquad |C_{\mathrm{GL}_{\mathbf{v}}^F}(X)|=\mathcal{Z}_{\boldsymbol{\omega}}(q).$$

Hence,

$$\frac{1}{|\mathrm{GL}_{\mathbf{v}}^F|}\sum_{X\in\mathfrak{gl}_{\mathbf{v}}^F}\#\left\{\varphi\in\mathrm{Rep}_{\mathbb{F}_q}(\Gamma,\mathbf{v})\,\middle|\,[X,\varphi]=0\right\}\mathcal{F}(1_{\mathcal{O}})(X)=\sum_{\boldsymbol{\omega}\in\mathbb{T}_{\mathbf{v}}}\mathcal{H}_{\boldsymbol{\omega}}(q)\sum_{\mathcal{O}'}\mathcal{F}(1_{\mathcal{O}})(\mathcal{O}'),\tag{2.7}$$

where the last sum is over the adjoint orbits $\mathcal{O}'$ of $\mathfrak{gl}_{\mathbf{v}}^F$ of type $\boldsymbol{\omega}$ and $\mathcal{F}(1_{\mathcal{O}})(\mathcal{O}')$ denotes the common value $\mathcal{F}(1_{\mathcal{O}})(X)$ for $X\in\mathcal{O}'$.

For a type $\boldsymbol{\omega}=(d_1,\boldsymbol{\lambda}_1)\cdots(d_s,\boldsymbol{\lambda}_s)\in\mathbb{T}_{\mathbf{v}}$, put

$$C_{\boldsymbol{\omega}}^o:=\begin{cases}\frac{\mu(d)}{d}(-1)^{s-1}\frac{(s-1)!}{\prod_{\boldsymbol{\lambda}}m_{d,\boldsymbol{\lambda}}(\boldsymbol{\omega})!} & \text{if } d_1=d_2=\cdots=d_s=d,\\ 0 & \text{otherwise,}\end{cases}$$

where $m_{d,\boldsymbol{\lambda}}(\boldsymbol{\omega})$ denotes the multiplicity of the pair $(d,\boldsymbol{\lambda})$ in $\boldsymbol{\omega}$ and where μ denotes the ordinary Möbius function.

Recall that we defined a map (see the beginning of Section 2.2) from regular semisimple adjoint orbits of $\mathfrak{gl}_{\mathbf{v}}(\mathbb{F}_q)$ to $W_{\mathbf{v}}$.

Proposition 2.10. *Let $w = (w_1, \ldots, w_r) \in W_{\mathbf{v}}$ have cycle type $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r) \in \mathcal{P}_{\mathbf{v}}$. Let $\mathcal{O}$ be a generic regular semisimple adjoint orbit of $\mathfrak{gl}_{\mathbf{v}}(\mathbb{F}_q)$ mapping to w, and let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ be any type in $\mathbb{T}_{\mathbf{v}}$. Then*

$$\sum_{\mathcal{O}'} \mathcal{F}(1_{\mathcal{O}})(\mathcal{O}') = \varepsilon(w)\, q^{1+\delta_{\mathbf{v}}/2}\, C^o_{\boldsymbol{\omega}} \prod_{i=1}^r \left\langle \tilde{H}_{\omega_i}(\mathbf{x}_i; q), p_{\lambda^i}(\mathbf{x}_i) \right\rangle,$$

where the sum is over the adjoint orbits $\mathcal{O}'$ of $\mathfrak{gl}_{\mathbf{v}}^F$ of type $\boldsymbol{\omega}$, $\mathcal{F}(1_{\mathcal{O}})(\mathcal{O}')$ denotes the common value $\mathcal{F}(1_{\mathcal{O}})(X)$ for $X \in \mathcal{O}'$ and $\delta_{\mathbf{v}} = \dim \mathrm{GL}_{\mathbf{v}} - \dim T_{\mathbf{v}} = \sum_i v_i^2 - \sum_i v_i$.

Proof. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $d \in \mathbb{Z}_{>0}$, define $d \cdot \lambda := (d\lambda_1, \ldots, d\lambda_m)$, and for a type $\tau = (d_1, \tau^1) \cdots (d_s, \tau^s) \in \mathbb{T}_n$, put $[\tau] := \cup_i d_i \cdot \tau^i$, a partition of n. We will say that two types $\nu = (d_1, \nu^1) \cdots (d_s, \nu^s)$ and $\omega = (e_1, \omega^1) \cdots (e_l, \omega^l)$ are *compatible*, which we denote $\nu \sim \omega$, if $s = l$ and for each $i = 1, \ldots, s$ we have $d_i = e_i$ and $|\nu^i| = |\omega^i|$. For two partitions of same size λ, μ let $Q^\lambda_\mu(q)$ be the Green polynomial as defined, for instance, in [19, Chap. III §7]. For two compatible types ν and ω, we put $Q^\omega_\nu(q) := \prod_i Q^{\omega^i}_{\nu^i}(q^{d_i})$ and $Q^\omega_\nu(q) := 0$ otherwise. Let z_λ be the order of the centralizer of an element of cycle type λ in $S_{|\lambda|}$. For a type $\nu = (d_1, \nu_1), \ldots, (d_r, \nu_r)$, set $z_\nu = \prod_i z_{\nu^i}$.

Notice that for λ a partition $(d_1, d_2, \ldots, d_s)$ of n, we have $p_\lambda(\mathbf{x}) = s_\tau(\mathbf{x})$ where $\tau \in \mathbb{T}_n$ is the type $(d_1, 1)(d_2, 1) \cdots (d_s, 1)$. Hence by [11, Lemma 2.3.5], for any $\omega \in \mathbb{T}_n$ and any partition λ of n,

$$\left\langle \tilde{H}_\omega(\mathbf{x}; q), p_\lambda(\mathbf{x}) \right\rangle = z_\lambda \sum_{\{\nu \in \mathbb{T}_n \,\mid\, [\nu] = \lambda\}} \frac{Q^\omega_\nu(q)}{z_\nu}.$$

We are therefore reduced to prove that

$$\sum_{\mathcal{O}'} \mathcal{F}(1_{\mathcal{O}})(\mathcal{O}') = \varepsilon(w)\, q^{1+\delta_{\mathbf{v}}/2}\, C^o_{\boldsymbol{\omega}} \prod_{i=1}^r z_{\lambda^i} \sum_{\{\nu \in \mathbb{T}_n \,\mid\, [\nu] = \lambda^i\}} \frac{Q^{\omega_i}_\nu(q)}{z_\nu}. \tag{2.8}$$

The proof of this formula is similar to that of [11, Th. 4.3.1(2)], although the context is different and will require some new calculations. Embed $\mathrm{GL}_{\mathbf{v}}$ in GL_N with $N = \sum_{i=1}^r v_i$. Write $\boldsymbol{\omega} = (d_1, \boldsymbol{\mu}_1) \cdots (d_s, \boldsymbol{\mu}_s)$, and define $\omega = (d_1, \cup\boldsymbol{\mu}_1) \cdots (d_s, \cup\boldsymbol{\mu}_s) \in \mathbb{T}_N$ where for a multi-partition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^r$, we put $\cup\boldsymbol{\mu} = \cup_i \mu^i$. If $\mathcal{O}'$ is an F-stable adjoint orbit of $\mathfrak{gl}_{\mathbf{v}}$ of type $\boldsymbol{\omega}$, then the unique GL_N-adjoint orbit of $\mathfrak{gl}_N$ that contains $\mathcal{O}'$ is of type ω. Consider a representative of an adjoint orbit of $\mathfrak{gl}_{\mathbf{v}}^F \subset \mathfrak{gl}_N^F$ of type $\boldsymbol{\omega}$ with Jordan form $\sigma + u$ where σ is semisimple and u is nilpotent. Put $L := C_{\mathrm{GL}_N}(\sigma)$, and denote by $\mathfrak{l}$ the Lie algebra of L and by $z_{\mathfrak{l}}$ the center of $\mathfrak{l}$. Note that $\mathfrak{l}$ is not contained in $\mathfrak{gl}_{\mathbf{v}}$ unless each for each i, the multi-partition $\boldsymbol{\mu}_i$ has a unique nonzero coordinate, in which case $L = M := C_{\mathrm{GL}_{\mathbf{v}}}(\sigma)$. However we always have $z_{\mathfrak{l}} \subset \mathfrak{gl}_{\mathbf{v}}$. Put $(z_{\mathfrak{l}})_{\mathrm{reg}} := \{y \in z_{\mathfrak{l}} \mid C_{\mathrm{GL}_N}(y) = L\}$. The map that sends $z \in (z_{\mathfrak{l}})^F_{\mathrm{reg}}$ to the

$\mathrm{GL}_{\mathbf{v}}$-orbits of $z+u$ surjects onto the set of adjoint orbits $\mathcal{O}'$ of $\mathfrak{gl}_{\mathbf{v}}^F$ of type $\boldsymbol{\omega}$. The fibers of this map can be identified with

$$\{g \in \mathrm{GL}_{\mathbf{v}}^F \mid gLg^{-1} = L, gC_ug^{-1} = C_u\}/M,$$

where C_u is the M-orbit of u. Hence we may in turn identify these fibers with

$$W(\boldsymbol{\omega}) := \prod_{d,\boldsymbol{\lambda}} (\mathbb{Z}/d\mathbb{Z})^{m_{d,\boldsymbol{\lambda}}(\boldsymbol{\omega})} \times S_{m_{d,\boldsymbol{\lambda}}(\boldsymbol{\omega})}.$$

It follows that

$$\sum_{\mathcal{O}'} \mathcal{F}(1_{\mathcal{O}})(\mathcal{O}') = \frac{1}{|W(\boldsymbol{\omega})|} \sum_{z \in (z_{\mathfrak{l}})^F_{\mathrm{reg}}} \prod_{i=1}^r \mathcal{F}^{\mathfrak{gl}_{v_i}}\left(1_{\mathcal{O}_i}\right)(z_i+u_i), \tag{2.9}$$

where $\mathcal{F}^{\mathfrak{gl}_{v_i}}$ denotes the Fourier transform on $\mathfrak{gl}_{v_i}^F$, $\mathcal{O}_i$ is the i-th coordinate of $\mathcal{O}$ (a $\mathrm{GL}_{v_i}^F$-orbit of $\mathfrak{gl}_{v_i}^F$) and z_i, u_i are the i-th coordinates of z, u respectively.

It is known [16, Th. 7.3.3][11, formulas (2.5.4), (2.5.5)] that

$$\begin{aligned}\mathcal{F}^{\mathfrak{gl}_{v_i}}\left(1_{\mathcal{O}_i}\right)(z_i+u_i &= \varepsilon(w_i) q^{\frac{1}{2}(v_i^2-v_i)} |M_i^F|^{-1}) \\ &\times \sum_{\{h \in \mathrm{GL}_{v_i}^F \mid h^{-1}z_ih \in \mathfrak{t}_{\lambda^i}\}} Q^{M_i}_{hT_{\lambda^i}h^{-1}}(u_i+1) \Psi\left(\left\langle X_i, h^{-1}z_ih\right\rangle\right),\end{aligned}$$

where X_i is a fixed element in $\mathcal{O}_i^F$, T_{λ^i} is the unique F-stable maximal torus of GL_{v_i} whose Lie algebra $\mathfrak{t}_{\lambda^i}$ contains X_i, $M_i = C_{\mathrm{GL}_{\mathbf{v}}}(z_i)$ and where $Q^{M_i}_{hT_{\lambda^i}h^{-1}}$ is the Green function defined by Deligne and Lusztig. (The values of these functions are products of usual Green polynomials.)

It follows that

$$\begin{aligned}\sum_{\mathcal{O}'} \mathcal{F}(1_{\mathcal{O}})(\mathcal{O}') &= \frac{1}{|W(\boldsymbol{\omega})|} \varepsilon(w) q^{\frac{1}{2}(\sum_i v_i^2 - \sum_i v_i)} \\ &\quad \times \sum_{h=(h_1,\dots,h_r)} \Phi_h(u) \sum_{z \in (z_{\mathfrak{l}})^F_{\mathrm{reg}}} \prod_{i=1}^r \Psi\left(\left\langle X_i, h_i^{-1} z_i h_i\right\rangle\right) \\ &= \frac{1}{|W(\boldsymbol{\omega})|} \varepsilon(w) q^{\frac{1}{2}(\sum_i v_i^2 - \sum_i v_i)} \sum_h \Phi_h(u) \sum_{z \in (z_{\mathfrak{l}})^F_{\mathrm{reg}}} \Psi\left(\left\langle X, h^{-1}zh\right\rangle\right),\end{aligned}$$

where $h = (h_1, \dots, h_r)$ runs over the set

$$\{h \in \mathrm{GL}_{\mathbf{v}}^F \mid hT_{\boldsymbol{\lambda}}h^{-1} \subset M\} = \{h \in \mathrm{GL}_{\mathbf{v}}^F \mid z \in hT_{\boldsymbol{\lambda}}h^{-1}\},$$

with $T_{\boldsymbol{\lambda}} := \prod_{i=1}^r T_{\lambda^i}$, $X := (X_i)_{i=1,\dots,r} \in \mathfrak{t}_{\boldsymbol{\lambda}} \cap \mathcal{O}$ and where to simplify, we set

$$\Phi_h(u) := \prod_{i=1}^r |M_i^F|^{-1} Q^{M_i}_{h_iT_{\lambda^i}h_i^{-1}}(u_i+1).$$

To finish the proof it suffices to check the following two formulas:

$$\sum_{z\in(z_{\mathfrak{l}})^F_{\mathrm{reg}}} \Psi\left(\left\langle X, h^{-1}zh\right\rangle\right) = \begin{cases} (-1)^{s-1}\mu(d)q(s-1)! & \text{if } d_i = d \text{ for all } \ i=1,\ldots,s,\\ 0 & \text{otherwise,}\end{cases}$$

$$\sum_h \Phi_h(u) = \prod_{i=1}^r z_{\lambda^i} \sum_{\{\nu\in\mathbb{T}_n \mid [\nu]=\lambda^i\}} \frac{Q_\nu^{\omega_i}(q)}{z_\nu},$$

where recall that $\boldsymbol{\omega} = (d_1, \boldsymbol{\mu}^1), (d_2, \boldsymbol{\mu}^2), \ldots$.

The proof of the second formula is contained in the proof of [11, Th. 4.3.1(2)]. For the first formula, by [17, Prop. 6.8.3], it is enough to prove that the linear character $\Theta : \mathfrak{t}_{\boldsymbol{\lambda}}^F \to \mathbb{C}^\times$, $z \mapsto \Psi(\langle X, z\rangle)$ is a *generic* character; i.e., the restriction of Θ to $z_{\mathfrak{gl}_N}^F$ is trivial and for any F-stable Levi subgroup L (of some parabolic subgroup) of GL_N that contains $T_{\boldsymbol{\lambda}}$, the restriction of Θ to $z_{\mathfrak{l}}^F \subset \mathfrak{t}_\lambda^F$ is nontrivial unless $L = \mathrm{GL}_N$.

But Θ is generic because the adjoint orbit $\mathcal{O}$ is generic. Indeed, since $\mathrm{Tr}(\mathcal{O}) = 0$, we have $\Theta|_{z^F_{\mathfrak{gl}_N}} = 1$. Now assume that $L \supset T_{\boldsymbol{\lambda}}$ satisfies $\Theta|_{z_{\mathfrak{l}}^F} = 1$. There is a decomposition $\mathbb{K}^N = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, with $W_j \neq 0$, such that $\mathfrak{l}$ is GL_N-conjugate to $\bigoplus_i \mathfrak{gl}(W_i)$. An element $z \in z_{\mathfrak{l}}$ is of the form $(\xi_1\iota_1, \ldots, \xi_s\iota_s)$, where $\xi_1, \ldots, \xi_s \in \mathbb{K}$ and where ι_j denotes the identity endomorphism of W_j. Denote by X^j the $\mathfrak{gl}(W_j)$ coordinate of X. Since $\Theta|_{z_{\mathfrak{l}}^F} = 1$, we must have $\langle X, z\rangle = \sum_{j=1}^s \xi_j \,\mathrm{Tr}(X^j) = 0$ for all $z = (\xi_1\iota_1, \ldots, \xi_s\iota_s)$, and so $\mathrm{Tr}(X^j) = 0$ for all $j = 1, \ldots, s$. Now $\mathfrak{gl}_{\mathbf{v}}$ and $\mathfrak{l}$ are two Levi sub-algebras of $\mathfrak{gl}_N$ that contain $\mathfrak{t}_{\boldsymbol{\lambda}}$, hence $\mathfrak{gl}_{\mathbf{v}} \cap \mathfrak{l} \simeq \bigoplus_{i,j} \mathfrak{gl}(U_{i,j})$ where $W_j = \bigoplus_i U_{i,j}$. For each $j = 1, \ldots s$, the space $\bigoplus_i U_{i,j}$ is also a graded subspace of $\mathbb{K}^N = \mathbb{K}^{\mathbf{v}}$ on which X acts by X^j, and so by the genericity assumption we must have $W_j = \mathbb{K}^N$ i.e. $L = \mathrm{GL}_N$. □

Proof of Theorem 2.8. By definition (1.8),

$$d_{\tilde{\mathbf{v}}} = \sum_{i\to j\in\Omega} v_i v_j - \sum_i v_i^2 + \delta_{\mathbf{v}} + 2 = \dim \mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v}) - \dim \mathfrak{gl}_{\mathbf{v}} + 1 + \delta_{\mathbf{v}}/2.$$

Hence applying formula (2.7) and Proposition 2.10, we find ([11, Formula (2.3.9)]) that

$$\begin{aligned} \#\, \mathcal{Q}_{\tilde{\mathbf{v}}}^w(\mathbb{F}_q) &= \varepsilon(w)(q-1)q^{d_{\tilde{\mathbf{v}}}} \left\langle \sum_{\boldsymbol{\omega}\in\mathbb{T}_{\mathbf{v}}} C^o_{\boldsymbol{\omega}} \mathcal{H}_{\boldsymbol{\omega}}(q) \prod_{i=1}^r \tilde{H}_{\omega_i}(\mathbf{x}_i; q), p_{\boldsymbol{\lambda}} \right\rangle \\ &= \varepsilon(w)(q-1)q^{d_{\tilde{\mathbf{v}}}} \left\langle \mathrm{Log}\left(\sum_{\pi=(\pi_1,\ldots\pi_r)\in\mathcal{P}^r} \mathcal{H}_\pi(q) \prod_{i=1}^r \tilde{H}_{\pi_i}(\mathbf{x}_i; q)\right), p_{\boldsymbol{\lambda}} \right\rangle. \ \square \end{aligned}$$

2.4. *Kac polynomials.* We start with a general fact about extracting Kac polynomials of the quivers $\Gamma_{\boldsymbol{\mu}}$ from the generating function $\mathbb{H}(\mathbf{x}_1,\ldots,x_r;q)$ defined in (1.4). This is similar to [12, Th. 3.2.7], and hence we omit the proof. For any multi-partition $\boldsymbol{\mu}\in\mathcal{P}^r$, denote by $A_{\boldsymbol{\mu}}(q)$ the Kac polynomial associated with $(\Gamma_{\boldsymbol{\mu}},\mathbf{v}_{\boldsymbol{\mu}})$, where $\mathbf{v}_{\boldsymbol{\mu}}$ is defined as in Section 1 with $\mathbf{v}=|\boldsymbol{\mu}|$. For a partition λ, denote by h_λ the complete symmetric function as in [19].

THEOREM 2.11. *For any $\boldsymbol{\mu}\in\mathcal{P}_{\boldsymbol{\mu}}$, we have $\langle\mathbb{H}(\mathbf{x}_1,\ldots,\mathbf{x}_r;q),h_{\boldsymbol{\mu}}\rangle=A_{\boldsymbol{\mu}}(q)$.*

Proof of Theorem 1.4. We start by proving (i). Let us denote here by $\mathcal{Q}_{\tilde{\mathbf{v}}}/_{\mathbb{C}}$ and $\mathcal{Q}_{\tilde{\mathbf{v}}}/_{\overline{\mathbb{F}}_q}$ the associated quiver varieties over the indicated field. Assume also that the characteristic of $\mathbb{F}_q$ is large enough so that the results of Section 2.2 apply. Combining Theorem 2.6 and Theorem 2.8, we find that

$$\langle\mathbb{H}(\mathbf{x}_1,\ldots,\mathbf{x}_r;q),p_{\boldsymbol{\lambda}}\rangle=\varepsilon(w)\sum_i\operatorname{Tr}\left(\rho^{2i}(w),H_c^{2i}\left(\mathcal{Q}_{\tilde{\mathbf{v}}}/_{\overline{\mathbb{F}}_q};\overline{\mathbb{Q}}_\ell\right)\right)q^{i-d_{\tilde{\mathbf{v}}}}. \tag{2.10}$$

The Schur functions $s_{\boldsymbol{\mu}}$, with $\boldsymbol{\mu}\in\mathcal{P}_{\mathbf{v}}$, decompose into power symmetric functions as $s_{\boldsymbol{\mu}}=\sum_{\boldsymbol{\lambda}\in\mathcal{P}_{\mathbf{v}}}\chi^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}p_{\boldsymbol{\lambda}}$, where $\chi^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}=\chi^{\boldsymbol{\mu}}(w)$ is the value of the irreducible character $\chi^{\boldsymbol{\mu}}$ of $W_{\mathbf{v}}$ at an element $w\in W_{\mathbf{v}}$ of cycle type $\boldsymbol{\lambda}$. We then deduce from formula (2.10) that

$$\langle\mathbb{H}(\mathbf{x}_1,\ldots,\mathbf{x}_r;t),s_{\boldsymbol{\mu}}\rangle=t^{-d_{\tilde{\mathbf{v}}}}\sum_i\left\langle\chi^{\boldsymbol{\mu}'},\rho^{2i}\right\rangle_{W_{\mathbf{v}}}t^i.$$

From Proposition 2.5 and the comment below that proposition, the above formula remains true if we replace ρ^i by ϱ^i, and Theorem 1.4(i) follows. Note that $\mathbb{H}^s_{\boldsymbol{\mu}}(t)$ is a polynomial since $H^i_c(\mathcal{Q}_{\tilde{\mathbf{v}}};\mathbb{C})=0$ unless $d_{\tilde{\mathbf{v}}}\leq i\leq 2d_{\tilde{\mathbf{v}}}$ as the variety $\mathcal{Q}_{\tilde{\mathbf{v}}}$ is affine.

We now proceed with the proof of (ii). Consider the partial ordering $\trianglelefteq$ on partitions defined as $\lambda\trianglelefteq\mu$ if for all i, we have $\sum_i\lambda_i\leq\sum_i\mu_i$. Extend this ordering on multi-partitions by declaring that $\boldsymbol{\alpha}\trianglelefteq\boldsymbol{\beta}$ if and only if for all i, we have $\alpha^i\trianglelefteq\beta^i$. A simple calculation shows that if $\boldsymbol{\alpha}\trianglelefteq\boldsymbol{\beta}$ and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$ for any two multi-partitions in $\mathcal{P}^r$, then $d_{\boldsymbol{\beta}}<d_{\boldsymbol{\alpha}}$.

Using the relations between Schur functions and complete symmetric functions [19, p. 101] together with Theorem 2.11, we find that

$$A_{\boldsymbol{\lambda}}(q)=\sum_{\boldsymbol{\mu}\trianglerighteq\boldsymbol{\lambda}}K'_{\boldsymbol{\lambda}\boldsymbol{\mu}}\mathbb{H}^s_{\boldsymbol{\mu}}(q),\qquad\qquad \mathbb{H}^s_{\boldsymbol{\mu}}(q)=\sum_{\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}}K^*_{\boldsymbol{\mu}\boldsymbol{\lambda}}A_{\boldsymbol{\lambda}}(q), \tag{2.11}$$

where $K=(K_{\boldsymbol{\lambda}\boldsymbol{\mu}})$ is the matrix of Kostka numbers and where K' and K^* are respectively the transpose and the transpose inverse of K. Recall [14, §1.15] that, if nonzero, $A_{\boldsymbol{\mu}}(q)$ is monic of degree $d_{\boldsymbol{\mu}}$, and $A_{\boldsymbol{\mu}}(q)$ is nonzero if and only if $\mathbf{v}_{\boldsymbol{\mu}}$ is a root of $\Gamma_{\boldsymbol{\mu}}$, with $A_{\boldsymbol{\mu}}(q)=1$ if and only if $\mathbf{v}_{\boldsymbol{\mu}}$ is real [14, §1.10]. Since $K^*_{\boldsymbol{\mu}\boldsymbol{\mu}}=1$ and since the polynomials $A_{\boldsymbol{\lambda}}(q)$, with $\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}$, $\boldsymbol{\lambda}\neq\boldsymbol{\mu}$, are of degree strictly less than $d_{\boldsymbol{\mu}}$, we deduce from the second equality (2.11) that if $\mathbf{v}_{\boldsymbol{\mu}}$ is a root, then $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ is monic of degree $d_{\boldsymbol{\mu}}$. Conversely if $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ is nonzero, then by the first formula (2.11), the polynomial $A_{\boldsymbol{\mu}}(q)$ must be nonzero (i.e., $\mathbf{v}_{\boldsymbol{\mu}}$ is a root) as the Kostka numbers are nonnegative, $K_{\boldsymbol{\mu}\boldsymbol{\mu}} = 1$ and $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ has nonnegative coefficients. This completes the proof. $\square$

3. DT-invariants for symmetric quivers

3.1. *Preliminaries.* Denote by Λ the ring of symmetric functions in the variables $\mathbf{x} = \{x_1, x_2, \dots\}$ with coefficients in $\mathbb{Q}(q)$ and Λ_n those functions in Λ homogeneous of degree n. We define the u-specialization of symmetric functions as the ring homomorphism $\Lambda \to \mathbb{Q}[u]$ that on power sums behaves as $p_r(\mathbf{x}) \mapsto 1 - u^r$. In plethystic notation this is denoted by $f \mapsto f[1-u]$.

Note that for any $f \in \Lambda_n$, the u-specialization $f[1-u]$ is a polynomial in u of degree at most n. We will need to consider the effect of taking top degree coefficients in u after u-specialization. Define the top degree of $f \in \Lambda_n$ as

$$[f] := u^n f\left[1 - u^{-1}\right]\Big|_{u=0}.$$

It is a crucial fact for what follows that the operations of u-specialization and of taking its top degree coefficient commute with the Log map. More precisely, we have the following facts whose proofs we only sketch.

PROPOSITION 3.1. *Let $\Omega(\mathbf{x};T) = \sum_{n\geq 0} A_n(\mathbf{x})\ T^n \in \Lambda[[T]]$ be a power series with $A_n(\mathbf{x}) \in \Lambda_n$, and let $V_n(\mathbf{x}) \in \Lambda$ be defined by*

$$\sum_{n\geq 1} V_n(\mathbf{x})\ T^n := \operatorname{Log}\ \Omega(\mathbf{x};T).$$

Then we have

$$\sum_{n\geq 1} V_n[1-u]\ T^n = \operatorname{Log} \sum_{n\geq 0} A_n[1-u]\ T^n, \quad \sum_{n\geq 1} [V_n]\ T^n = \operatorname{Log} \sum_{n\geq 0} [A_n]\ T^n. \tag{3.1}$$

Proof. The key point is the identity $p_{dr}[1-u] = 1 - u^{rd} = p_r[1-u^d]$. $\square$

PROPOSITION 3.2. *For any partition $\lambda \in \mathcal{P}$, we have*

(i) $\tilde{H}_\lambda(q)[1-u] = (u)_l$, *where $l := l(\lambda)$ is the length of λ and* $(u)_l := \prod_{i=1}^{l}(1 - q^{i-1}u)$.

(ii) $[\tilde{H}_\lambda]$ *is zero unless $\lambda = (1^n)$ when it equals $(-1)^n q^{\binom{n}{2}}$.*

Proof. The specialization (i) follows from the corresponding result for Macdonald polynomials; see [8, Cor. 2.1]. The second claim is an immediate consequence of (i). $\square$

LEMMA 3.3. *For the Schur function s_λ, we have that $s_\lambda[1-u]$ is zero unless $\lambda = (r, 1^{n-r})$ with $1 \leq r \leq n$ is a hook, in which case it equals $(-u)^{n-r}(1-u)$. In particular, for $f \in \Lambda_n$, we have*

$$[f] = (-1)^n \langle f, s_{(1^n)} \rangle. \tag{3.2}$$

Proof. The identity follows from the known u-specialization of the Schur functions [8, (2.15)]. $\square$

3.2. DT-*invariants.* In this section we prove a somewhat more general case of Proposition 1.3(ii). We work with a *symmetric quiver* (a quiver with as many arrows going from the vertex i to j as arrows going from j to i) instead of the double of a quiver (see Remark 1.2). The only difference is that the double of a quiver has an even number of loops at every vertex whereas a symmetric quiver may not. We deal with this by attaching an arbitrary number of legs to each vertex instead of just one. In general, the parity of the number of legs required at a vertex i is the opposite of that of the number of loops at i.

Concretely, attach $k_i \geq 1$ infinite legs to each vertex $i \in I$ of Γ. The orientation of the arrows ultimately does not matter, but say all the arrows on the new legs point towards the vertex. Consider the following generalization of (1.4):

$$\mathbb{H}(\mathbf{x};q) := (q-1)\mathrm{Log}\left(\sum_{\pi\in\mathcal{P}^r} \mathcal{H}_\pi(q)\tilde{H}_\pi(\mathbf{x};q)\right), \text{ with} \tag{3.3}$$

$$\tilde{H}_\pi(\mathbf{x};q) := \prod_{i=1}^{r}\prod_{j=1}^{k_i} \tilde{H}_{\pi^i}(\mathbf{x}^{i,j};q),$$

and $\mathbf{x}^{i,j} = (x_1^{i,j}, x_2^{i,j}, \ldots)$ for $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$ are independent sets of infinitely many variables.

Given a multi-partition $\boldsymbol{\mu} = (\mu^{i,j})$ where $i = 1, \ldots, r$ and $j = 1, \ldots, k_i$, define

$$s_{\boldsymbol{\mu}}(\mathbf{x}) := \prod_{i=1}^{r}\prod_{j=1}^{k_i} s_{\mu^{i,j}}(\mathbf{x}^{i,j}), \qquad \mathbb{H}^s_{\boldsymbol{\mu}}(q) := \langle \mathbb{H}(\mathbf{x};q), s_{\boldsymbol{\mu}}(\mathbf{x})\rangle. \tag{3.4}$$

Note that $\mathbb{H}^s_{\boldsymbol{\mu}}(q)$ is zero unless $|\mu^{i,1}| = |\mu^{i,2}| = \cdots = |\mu^{i,k_i}|$ for each $i = 1, \cdots, r$. For $\mathbf{v} \in \mathbb{Z}^r_{\geq 0}\backslash\{0\}$, denote by $1^{\mathbf{v}}$ the multi-partition $(\mu^{i,j})$ where for every $j = 1, \ldots, k_i$, either $\mu^{i,j} = (1^{v_i})$ if $v_i > 0$ or $\mu^{i,j} = 0$ otherwise.

Proposition 3.4. *We have*

$$(q-1)\mathrm{Log}\left(\sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}} \frac{q^{-\frac{1}{2}(\gamma(\mathbf{v})+\delta(\mathbf{v}))}}{(q^{-1})_{\mathbf{v}}}\,(-1)^{\delta(\mathbf{v})}T^{\mathbf{v}}\right) = \sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}\backslash\{0\}} \mathbb{H}^s_{1^{\mathbf{v}}}(q)(-1)^{\delta(\mathbf{v})}T^{\mathbf{v}}, \tag{3.5}$$

where

$$\gamma(\mathbf{v}) := \sum_{i=1}^{r}(2-k_i)v_i^2 - 2\sum_{i\to j\in\Omega} v_iv_j, \qquad \delta(\mathbf{v}) := \sum_{i=1}^{r} k_iv_i,$$

and $(q)_{\mathbf{v}} := (q)_{v_1}\cdots(q)_{v_r}$ *with* $(q)_s := (1-q)\cdots(1-q^s)$.

Proof. By (3.4), we have $\mathbb{H}(\mathbf{x};q) = \sum_{\boldsymbol{\mu}} \mathbb{H}^s_{\boldsymbol{\mu}}(q)s_{\boldsymbol{\mu}}(\mathbf{x})$. Apply Proposition 3.1(ii) to all the variables $\mathbf{x}^{i,j}$ in (3.3) to get

$$\sum_{\boldsymbol{\mu}} \mathbb{H}^s_{\boldsymbol{\mu}}(q)[s_{\boldsymbol{\mu}}(\mathbf{x})]T^{|\boldsymbol{\mu}|} = (q-1)\,\mathrm{Log}\left(\sum_{\pi\in\mathcal{P}^r} \mathcal{H}_\pi(q)[\tilde{H}_\pi(\mathbf{x};q)]T^{|\pi|}\right),$$

where $\boldsymbol{\mu}$ runs through the nonzero multi-partitions $(\mu^{i,j})$ with $|\mu^{i,j}| = v_i$ for some $v_i \in \mathbb{Z}_{\geq 0}$ independent of j, $T^{|\boldsymbol{\mu}|} := \prod_i T_i^{v_i}$ and $T^{|\pi|} = \prod_i T_i^{|\pi_i|}$. A calculation using Proposition 3.2(ii) shows that the right-hand side equals

$$(q-1)\,\mathrm{Log}\left(\sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}} \frac{q^{-\frac{1}{2}(\gamma(\mathbf{v})+\delta(\mathbf{v}))}}{(q^{-1})_{\mathbf{v}}}\,(-1)^{\delta(\mathbf{v})}T^{\mathbf{v}}\right).$$

Finally, (3.2) shows that the left-hand side equals $\sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}\backslash\{0\}} \mathbb{H}^s_{1^{\mathbf{v}}}(q)(-1)^{\delta(\mathbf{v})}T^{\mathbf{v}}$ and our claim is proved. □

Now let $\Gamma' = (I, \Omega')$ be any symmetric quiver with r vertices. The Donaldson–Thomas invariants for a symmetric quiver Γ', as defined by Kontsevich and Soibelman, are given as follows in an equivalent formulation. Let $c_{\mathbf{v},k}$ be the coefficients in the generating function identity

$$\mathrm{Log}\sum_{\mathbf{v}} \frac{(-q^{\frac{1}{2}})^{\gamma'(\mathbf{v})}}{(q)_{\mathbf{v}}}\,T^{\mathbf{v}} = (1-q)^{-1}\sum_{\mathbf{v}}\sum_k (-1)^k c_{\mathbf{v},k}\,q^{k/2}\,T^{\mathbf{v}}, \tag{3.6}$$

where $\gamma'(\mathbf{v}) := \sum_{i=1}^r v_i^2 - \sum_{i\to j\in\Omega'} v_i v_j$. We know ([7, p.15]) that $\sum_k c_{\mathbf{v},k}\,q^{k/2}$ is a Laurent polynomial in $q^{\frac{1}{2}}$ and if $c_{\mathbf{v},k}$ is nonzero, then $k \equiv \gamma'(\mathbf{v}) \bmod 2$ ([7, Th. 4.1]). Since Γ' is symmetric, we also have $\gamma'(\mathbf{v}) \equiv \delta'(\mathbf{v}) \bmod 2$, where δ' is a fixed linear form

$$\delta'(v) := \sum_{i=1}^r k'_i v_i, \qquad k'_i \in \mathbb{Z}_{>0}, \qquad k'_i \equiv a'_{i,i} - 1 \bmod 2,$$

with $a'_{i,i}$ the number loops at the vertex i of Γ'. Changing $q \mapsto q^{-1}$ and then $T_i \mapsto q^{-k'_i/2}T_i$, we extend the definition of $\mathrm{DT}_{\mathbf{v}}$ given in 1.3 to Γ' by setting

$$(q-1)\,\mathrm{Log}\sum_{\mathbf{v}} \frac{q^{-\frac{1}{2}(\gamma'(\mathbf{v})+\delta'(\mathbf{v}))}}{(q^{-1})_{\mathbf{v}}}\,(-1)^{\delta'(\mathbf{v})}T^{\mathbf{v}} =: \sum_{\mathbf{v}\in\mathbb{Z}^r_{\geq 0}\backslash\{0\}} \mathrm{DT}_{\mathbf{v}}(q)\,(-1)^{\delta'(\mathbf{v})}T^{\mathbf{v}}. \tag{3.7}$$

Up to powers of q, the definition of $\mathrm{DT}_{\mathbf{v}}(q)$ is independent of the choice of linear form δ', and we do not include it in the notation. We would like to match (3.7) with (3.5) by making appropriate choices for Γ and k_i. Denote by $a_{i,j}$ (resp. $a'_{i,j}$) the number of arrows of Γ (resp. Γ') going from i to j. To match γ with γ' requires that

$$a'_{i,j} + a'_{j,i} = 2(a_{i,j}+a_{j,i}), \quad i \neq j, \qquad k_i - 2 + 2a_{i,i} = -1 + a'_{i,i}. \tag{3.8}$$

This we can always do (typically in more than one way) because Γ' is symmetric. We have then

Proposition 3.5. *With the above notation, let Γ be a quiver and k_i be integers satisfying* (3.8). *Then for all $\mathbf{v} \in \mathbb{Z}^r_{\geq 0}\backslash\{0\}$,*

$$\mathrm{DT}_{\mathbf{v}}(q) = q^{\delta(v)-\delta'(v)}\mathbb{H}^s_{1^{\mathbf{v}}}(q). \tag{3.9}$$

A special case of Proposition 3.5 is when $\Gamma' = \overline{\Gamma}$ for some quiver Γ. In this case we may take $k_i = k_i' = 1$ for all i and (3.9) is Proposition 1.3(ii).

References

[1] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces, I* (Luminy, 1981), *Astérisque* **100**, Soc. Math. France, Paris, 1982, pp. 5–171. MR 0751966. Zbl 0536.14011.

[2] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, *Compositio Math.* **126** (2001), 257–293. MR 1834739. Zbl 1037.16007. http://dx.doi.org/10.1023/A:1017558904030.

[3] W. Crawley-Boevey, Quiver algebras, weighted projective lines, and the Deligne-Simpson problem, in *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, pp. 117–129. MR 2275591. Zbl 1125.16007.

[4] W. Crawley-Boevey and M. Van den Bergh, Absolutely indecomposable representations and Kac-Moody Lie algebras, *Invent. Math.* **155** (2004), 537–559, with an appendix by Hiraku Nakajima. MR 2038196. Zbl 1065.16009. http://dx.doi.org/10.1007/s00222-003-0329-0.

[5] W. Crawley-Boevey and M. P. Holland, Noncommutative deformations of Kleinian singularities, *Duke Math. J.* **92** (1998), 605–635. MR 1620538. Zbl 0974.16007. http://dx.doi.org/10.1215/S0012-7094-98-09218-3.

[6] P. Deligne, *Cohomologie étale*, *Lecture Notes in Mathematics* **569**, Springer-Verlag, New York, 1977, Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. MR 0463174. Zbl 0349.14008.

[7] A. I. Efimov, Cohomological Hall algebra of a symmetric quiver, *Compos. Math.* **148** (2012), 1133–1146. MR 2956038. Zbl 06111062. http://dx.doi.org/10.1112/S0010437X12000152.

[8] A. M. Garsia and M. Haiman, A remarkable q,t-Catalan sequence and q-Lagrange inversion, *J. Algebraic Combin.* **5** (1996), 191–244. MR 1394305. Zbl 0853.05008. http://dx.doi.org/10.1023/A:1022476211638.

[9] V. Ginzburg, Calabi-Yau algebras, 2006. arXiv math/0612139.

[10] T. Hausel, Kac's conjecture from Nakajima quiver varieties, *Invent. Math.* **181** (2010), 21–37. MR 2651380. Zbl 1198.16016. http://dx.doi.org/10.1007/s00222-010-0241-3.

[11] T. Hausel, E. Letellier, and F. Rodriguez-Villegas, Arithmetic harmonic analysis on character and quiver varieties, *Duke Math. J.* **160** (2011), 323–400. MR 2852119. Zbl 1246.14063. http://dx.doi.org/10.1215/00127094-1444258.

[12] T. Hausel, E. Letellier, and F. Rodriguez-Villegas, Arithmetic harmonic analysis on character and quiver varieties II, *Adv. Math.* **234** (2013), 85–128. MR 3003926. http://dx.doi.org/10.1016/j.aim.2012.10.009.

[13] J. Hua, Counting representations of quivers over finite fields, *J. Algebra* **226** (2000), 1011–1033. MR 1752774. Zbl 0972.16006. http://dx.doi.org/10.1006/jabr.1999.8220.

[14] V. G. Kac, Root systems, representations of quivers and invariant theory, in *Invariant Theory* (Montecatini, 1982), *Lecture Notes in Math.* **996**, Springer-Verlag, New York, 1983, pp. 74–108. MR 0718127. Zbl 0534.14004. http://dx.doi.org/10.1007/BFb0063236.

[15] M. Kontsevich and Y. Soibelman, Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants, *Commun. Number Theory Phys.* **5** (2011), 231–352. MR 2851153. Zbl 1248.14060.

[16] E. Letellier, *Fourier Transforms of Invariant Functions on Finite Reductive Lie Algebras*, *Lecture Notes in Math.* **1859**, Springer-Verlag, New York, 2005. MR 2114404. Zbl 1076.43001. http://dx.doi.org/10.1007/b104209.

[17] E. Letellier, Quiver varieties and the character ring of general linear groups over finite fields, to appear in J. European Math. Soc. arXiv 1103.2759.

[18] G. Lusztig, Quiver varieties and Weyl group actions, *Ann. Inst. Fourier* (*Grenoble*) **50** (2000), 461–489. MR 1775358. Zbl 0958.20036. http://dx.doi.org/10.5802/aif.1762.

[19] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., *Oxford Math. Monogr.*, The Clarendon Press Oxford University Press, New York, 1995. MR 1354144. Zbl 0824.05059.

[20] A. Maffei, A remark on quiver varieties and Weyl groups, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1** (2002), 649–686. MR 1990675. Zbl 1143.14309. Available at http://www.numdam.org/item?id=ASNSP_2002_5_1_3_649_0.

[21] S. Mozgovoy, Motivic Donaldson-Thomas invariants and Kac conjecture, 2010. arXiv 1103.2100.

[22] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, *Duke Math. J.* **76** (1994), 365–416. MR 1302318. Zbl 0826.17026. http://dx.doi.org/10.1215/S0012-7094-94-07613-8.

[23] H. Nakajima, Reflection functors for quiver varieties and Weyl group actions, *Math. Ann.* **327** (2003), 671–721. MR 2023313. Zbl 1060.16017. http://dx.doi.org/10.1007/s00208-003-0467-0.

(Received: April 20, 2012)
(Revised: November 29, 2012)

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