COUNTING COLORINGS ON VARIETIES

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1.

1.1. **Introduction.** The goal of this note is to present a combinatorial mechanism for counting certain objects associated to a variety X defined over a finite field. The basic example, discussed in §2.2, is that of counting conjugacy classes in $\mathrm{GL}_n(\mathbb{F}_q)$, where $X = \mathbb{G}_m$ (the multiplicative group).

We give four different forms of the main formula (which is somewhat reminiscent of Polya's theory of counting). The principle that emerges is that in a given setup the counting generating functions for $X = \bullet$ (a point), $X = \mathbb{G}_a$ (the additive group) and $X = \mathbb{G}_m$ are related to one another in a simple way. Often one of the cases will be significantly easier to compute than the others yielding a closed formula for all three generating functions. For example, in §3 we describe how one can go from counting all matrices in $M_n(\mathbb{F}_q)$, corresponding to $X = \mathbb{G}_a$, to counting unipotent matrices in $M_n(\mathbb{F}_q)$, corresponding to $X = \bullet$.

None of the special cases considered here are really new; the point is, instead, to stress the main combinatorial principle. For more general applications (to quiver and character varieties) we refer the reader to [1] and [2].

1.2. **The Zeta Function of X.** Let $\mathbb{F}_q$ be a finite field with q elements. Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. For each $r \in \mathbb{N}$ let $\mathbb{F}_{q^r}$ be the unique subfield of $\overline{\mathbb{F}}_q$ of cardinality q^r. Let $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the Frobenius automorphism $x \mapsto x^q$. Then $\mathbb{F}_{q^r}$ is the fixed field of Frob_q^r.

Let X be an algebraic variety defined over $\mathbb{F}_q$. For each $r \in \mathbb{N}$ let $N_r(X) := \#X(\mathbb{F}_{q^r})$. The *zeta function* of X is defined as

$$Z(X,T) := \exp\left(\sum_{r\geq 1} N_r(X)\frac{T^r}{r}\right). \tag{1}$$

Let $\tilde{N}_d(X)$ be the number of Frobenius orbits in $X(\overline{\mathbb{F}}_q)$ of size d. Then

$$N_r(X) = \sum_{d|r} d\,\tilde{N}_d(X). \tag{2}$$

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We can write the zeta funcion as an Euler product

$$Z(X,T) = \prod_{d\geq 1}(1-T^d)^{-\tilde{N}_d(X)}. \tag{3}$$

1.3. **Colorings on X.** We consider the following general setup. Let $\mathcal{C}$ be a set, whose members we call *colors*, and

$$|\cdot| : \mathcal{C} \longrightarrow \mathbb{Z}_{\geq 0} \tag{4}$$

a function called *degree* such that

(1) there are finitely many colors of a given degree;
(2) there is a unique color $0 \in \mathcal{C}$ of degree 0.

A *coloring* on X is a map

$$\Lambda : X(\overline{\mathbb{F}}_q) \longrightarrow \mathcal{C}. \tag{5}$$

The *degree* of Λ is defined as

$$|\Lambda| := \sum_{x\in X(\overline{\mathbb{F}}_q)} |\Lambda(x)|. \tag{6}$$

We will only consider colorings of finite degree, so that $\Lambda(x) = 0$ for all but finitely many x. We let the Frobenius automorphism act on colorings via

$$\Lambda^{\mathrm{Frob}_q}(x) := \Lambda(\mathrm{Frob}_q(x)) \tag{7}$$

and say Λ is defined over $\mathbb{F}_{q^r}$ if Λ is fixed by Frob_q^r. In this case we will write: Λ is a coloring of $X/\mathbb{F}_{q^r}$.

Given a pair (d,λ), with $d \in \mathbb{N}$ and $\lambda \in \mathcal{C}$ a non-zero color, we define its *multiplicity* $m_{d,\lambda}$ in a coloring Λ of $X/\mathbb{F}_q$ to be the number of Frobenius orbits $\{x\}$ in $X(\overline{\mathbb{F}}_q)$ of degree d with $\Lambda(x) = \lambda$. Note that

$$|\Lambda| = \sum_{d\geq 1, \lambda\neq 0} m_{d,\lambda}\, d|\lambda|. \tag{8}$$

We call the combinatorial data $\{m_{d,\lambda}\}$ of multiplicities the *type* of Λ and denote it $\tau(\Lambda)$.

Example 1. Let $\mathcal{C} = \mathbb{Z}_{\geq 0}$ with degree function $|n| = n$. Then a coloring is an effective 0-cycle on X. The actions of Frobenius are compatible hence Λ is defined over $\mathbb{F}_{q^r}$ if and only if the corresponding 0-cycle is.

Example 2. If $X = \mathbb{G}_m$ and $\mathcal{C} = \mathcal{P}$ is the set of all partitions of non-negative integers with $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ if $\lambda = \lambda_1 \geq \lambda_2 \geq \ldots$ then colorings of degree n are in one-to-one correspondence with conjugacy classes in $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ by the Jordan decomposition theorem. Indeed, to each coloring Λ we may associate the direct sum of Jordan blocks with eigenvalue $x \in \overline{\mathbb{F}}_q^\star$ and size λ_i, where $\Lambda(x) = \lambda_1 \geq \lambda_2 \geq \ldots$. This correspondence preserves the action of Frobenius and therefore colorings defined over $\mathbb{F}_{q^r}$ are in bijection to conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_{q^r})$.

Similar statements hold for $X = \mathbb{G}_a$ with colorings corresponding to conjugacy classes in $M_n(\overline{\mathbb{F}}_q)$ instead.

We need one more ingredient. Let $R = \mathbb{Z}[[t_1, \dots, t_N]][t_1^{-1}, \dots, t_N^{-1}]$ be the ring of Laurent series with integer coefficients in the variables $t_1, \dots, t_N$. We let

$$W : \mathcal{C} \longrightarrow R \tag{9}$$

be a function called *weight* satisfying $W(0) = 1$.

We define the weight of a coloring Λ on $X/\mathbb{F}_q$ as

$$W(\Lambda) := \prod_{\{x\}} W(\Lambda(x))(t^{d(x)}), \tag{10}$$

where $\{x\}$ runs through the Frobenius orbits in $X(\overline{\mathbb{F}}_q)$, $d(x) = \#\{x\}$ is the degree of x (the size of its Frobenius orbit) and $t^d := t_1^d \cdots t_N^d$. Note that $W(\Lambda)$ only depends on the type $\tau(\Lambda)$:

$$W(\Lambda) = \prod_{d \geq 1, \lambda \neq 0} W(\lambda)(t^d)^{m_{d,\lambda}}. \tag{11}$$

We say that W is *homogeneous* if for each $\lambda \in \mathcal{C}$ we have that $W(\lambda) \in R$ is homogeneous of degree $|\lambda|$. In this case $W(\Lambda)$ is also homogeneous of degree $|\Lambda|$.

1.4. **Coloring Zeta Function of X.** Given the coloring data $C = (\mathcal{C}, |\cdot|, W(\cdot))$ we define the *coloring zeta function* of X as the formal power series in $R[[T]]$

$$Z_C(X, t, T) := \sum_{\Lambda} W(\Lambda) T^{|\Lambda|}, \tag{12}$$

where the sum runs over all colorings of $X/\mathbb{F}_q$. If $X = \bullet$ (a point) then the coloring zeta function simply reduces to

$$Z_C(\bullet, t, T) := \sum_{\lambda \in \mathcal{C}} W(\lambda) T^{|\lambda|}. \tag{13}$$

Example 3. In the *standard setup* $C = (\mathbb{Z}_{\geq 0}, |\cdot|, 1)$, $Z_C(X, T)$ is just the usual zeta function $Z(X, T)$. In particular, if $X = \bullet$ then

$$Z_C(\bullet, T) = \sum_{n \geq 0} T^n = (1 - T)^{-1}.$$

Example 4. In the *partition setup* $C = (\mathcal{P}, |\cdot|, 1)$ and if $X = \bullet$ then

$$Z_C(\bullet, T) = \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = \prod_{d \geq 1} (1 - T^d)^{-1}.$$

2.

2.1. **First Form.** This form of the main formula is similar to the Euler product (3) for the usual zeta function (to which it reduces to in the standard setup).

Theorem 1. *The following identity of generating functions holds*

$$Z_C(X, t, T) = \prod_{d \geq 1} Z_C(\bullet, t^d, T^d)^{\tilde{N}_d} \tag{14}$$

Proof. Write $Z_C(\bullet, t, T) = 1 + z(T)$. For $N \in \mathbb{N}$ we have

$$Z_C(\bullet, t, T)^N = 1 + \sum_{m \geq 1} N(N-1)\cdots(N-m+1)\frac{z(T)^m}{m!}$$

by the binomial theorem. On the other hand by the multinomial theorem

$$\frac{z(T)^m}{m!} = \sum_{m_\lambda} \prod_{\lambda \neq 0} \frac{W(\lambda)^{m_\lambda}}{m_\lambda!} T^{m_\lambda |\lambda|}$$

summed over all sequences of non-negative integers m_λ with $\sum_{\lambda \neq 0} m_\lambda = m$. Putting these two identities together we get that the coefficient of T^n on the right hand side of (14) equals

$$\sum_{m_{d,\lambda}} \prod_{d \geq 1, \lambda \neq 0} \tilde{N}_d(\tilde{N}_d - 1)\cdots(\tilde{N}_d - m_d + 1)\frac{W(\lambda)(t^d)^{m_{d,\lambda}}}{m_{d,\lambda}!}$$

summed over all $m_{d,\lambda}$ sequences of non-zero integers satisfying

$$n = \sum_{d \geq 1, \lambda \neq 0} m_{d,\lambda}\, d|\lambda|,$$

where $m_d := \sum_{\lambda \neq 0} m_{d,\lambda}$.

On the other hand to give a coloring of $X/\mathbb{F}_q$ with multiplicites $m_{d,\lambda}$ we need to pick $m_d = \sum_{\lambda \neq 0} m_{d,\lambda}$ Frobenius orbits of size d and color $m_{d,\lambda}$ of them with color $\lambda \neq 0$. There are $\binom{\tilde{N}_d}{m_d}$ ways of picking the orbits and $m_d!/\prod_{\lambda \neq 0} m_{d,\lambda}!$ ways to color them in this way and the weight of Λ is $W(\Lambda) = \prod_{d \geq 1, \lambda \neq 0} W(\lambda)(t^d)^{m_{d,\lambda}}$. It follows that the coefficients of T^n on both sides of (14) agree. □

2.2. **Second Form.**

Theorem 2. *The following identity of generating functions holds*

$$Z_C(X, t, T) = \prod_{m \in \mathbb{Z}^N, d \geq 1} Z(X, t^m T^d)^{v_{d,m}} \tag{15}$$

where the exponents $v_{d,m}$ are defined by the formal identity

$$Z_C(\bullet, t, T) = \prod_{m \in \mathbb{Z}^N, d \geq 1} (1 - t^m T^d)^{-v_{d,m}}. \tag{16}$$

Proof. Taking logarithms of both sides of (14) we get

$$\log Z_C(X, t, T) = \sum_{d \geq 1} \tilde{N}_d \log Z_C(\bullet, t^d, T^d).$$

By Möbius inversion of (2)

$$\tilde{N}_d = \frac{1}{d} \sum_{e|d} \mu(e) N_{d/e}.$$

Taking logarithm of both sides of (16) we get

$$\log Z_C(\bullet, t, T) = -\sum_{m\in\mathbb{Z}^N, d\geq 1} v_{d,m}\log(1-t^mT^d).$$

On the other hand,

$$T = -\sum_{r\geq 1}\mu(r)\log(1-t^r)$$

and hence

$$\begin{aligned}\log Z_C(X,t,T) &= -\sum_{r,s\geq 1}\frac{1}{rs}\mu(r)N_s\sum_{m\in\mathbb{Z}^N,d\geq 1} v_{d,m}\log(1-t^{msr}T^{dsr})\\ &= \sum_{s\geq 1}\frac{1}{s}N_s\sum_{m\in\mathbb{Z}^N,\,d\geq 1} v_{d,m}t^{sm}T^{sd}\\ &= \sum_{m\in\mathbb{Z}^N,\,d\geq 1} v_{d,m}\log Z(X,t^mT^d)\end{aligned}$$

proving our claim. □

Remark. It is easy to see by induction that the $v_{d,m}$ in (16) are integers uniquely determined by $Z_C(\bullet, t, T)$.

Example 5. In the *partition setup* $C = (\mathcal{P}, |\cdot|, 1)$ with $X = \mathbb{G}_m$ we have

$$Z_C(\mathbb{G}_m, T) = \sum_{n\geq 0} C_nT^n,$$

where C_n is the number of conjugacy classes in $\mathrm{GL}_n(\mathbb{F}_q)$ (see example 2); by (15) this equals

$$\prod_{n\geq 1}\left(\frac{1-T^n}{1-qT^n}\right) \tag{17}$$

as $Z(\mathbb{G}_m, T) = (1-T)/(1-qT)$. See [3], [5], [6].

Similarly, if we take $X = \mathbb{G}_a$ then we get the expression

$$\prod_{n\geq 1}(1-qT^n)^{-1} \tag{18}$$

for the generating funtion for the conjugacy classes in $M_n(\mathbb{F}_q)$ instead.

2.3. **Third Form.** This form of the expression for the coloring zeta function is a simple variant of the second form (15) but it is convenient to state it separately.

Let

$$Z_C(u,t,T) := \prod_{m\in\mathbb{Z}^N,d\geq 1}(1-u\,t^mT^d)^{-v_{d,m}} \tag{19}$$

where u is another formal variable and $v_{d,m}$ is as in (16).

Theorem 3. *Let*

$$Z(X,T) = \prod_i (1 - x_i T)^{-n_i}, \tag{20}$$

for some $x_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}$. Then with the above notation we have

$$Z_C(X,t,T) = \prod_i Z_C(x_i,t,T)^{n_i}. \tag{21}$$

Example 6. In the standard setup $Z_C(\bullet,T) = (1-T)^{-1}$ so that $Z_C(u,T) = (1-uT)^{-1}$ and (21) is simply a restatement of (20).

Remark. It is known by the work of Dwork that $Z(X,T)$ is a rational function of T of the form (20).

2.4. **Fourth Form.** Recall that $R = \mathbb{Z}[[t_1,\dots,t_N]][t_1^{-1},\dots,t_N^{-1}]$ is the ring of Laurent series in variables $t_1,\dots,t_N$ with integer coefficients. Given $Z \in 1+TR[[T]]$ we define, following Getzler [4]

$$\mathrm{Log}(Z) := \sum_{d\ge 1,\, m\in\mathbb{Z}^N} v_{d,m} t^m T^d \in R[[T]], \tag{22}$$

where

$$Z = \prod_{m\in\mathbb{Z}^N,\, d\ge 1} (1 - t^m T^d)^{-v_{d,m}},$$

as in (16).

In this section we assume that X is a polynomial count variety; i.e.

$$N_r(X) = N_X(q^r), \qquad r \in \mathbb{N},$$

for some fixed polynomial $N_X \in \mathbb{Z}[q]$. We also assume that one of the variables in R is q. To simplify the notation we relabel the variables as $q, t_1, \dots, t_N$ and the exponents as $i \in \mathbb{Z}$ for q and $m \in \mathbb{Z}^N$ for $t_1,\dots,t_N$. For example, with this relabeling (19) becomes

$$Z_C(u,t,T) = \prod_{i\in\mathbb{Z}, m\in\mathbb{Z}^N, d\ge 1} (1 - u\, q^i t^m T^d)^{-v_{d,i,m}}. \tag{23}$$

Theorem 4. *The following identity holds*

$$\mathrm{Log}(Z_C(X,t,T)) = N_X(q)\,\mathrm{Log}(Z_C(\bullet,t,T)). \tag{24}$$

Proof. The claim is a simple consequence of the third form (21) of our main formula. If $N_X(q) = \sum_j n_j q^j$ then

$$Z(X,T) = \prod_j (1 - q^j T)^{-n_j}.$$

Hence by (21)

$$\begin{aligned} Z_C(X,t,T) &= \prod_{j\in\mathbb{Z}} Z_C(q^j,t,T)^{n_j} \\ &= \prod_{i,j\in\mathbb{Z},m\in\mathbb{Z}^N,d\geq 1} (1-q^{i+j}t^mT^d)^{-n_jv_{d,i,m}} \\ &= \prod_{k\in\mathbb{Z},m\in\mathbb{Z}^N,d\geq 1} (1-q^kt^mT^d)^{-\sum_{i+j=k} n_jv_{d,i,m}}. \end{aligned}$$

Hence

$$\mathrm{Log}(Z_C(X,t,T)) = \sum_{k\in\mathbb{Z},m\in\mathbb{Z}^N,d\geq 1}\ \sum_{i+j=k} n_jv_{d,i,m}q^kt^mT^d$$

which equals the right hand side of (24) □

3.

3.1. **Unipotent matrices.** We consider the coloring data $\mathcal{C} = \mathcal{P}$ with the usual degree function $|\cdot|$ but with a non-trivial weight function. For all results and concepts related to partitions our reference will be [7], whose notation we will follow.

Let $X = \mathbb{G}_a$ and Λ a coloring of $X/\mathbb{F}_q$ corresponding to a conjugacy class c in $M_n(\mathbb{F}_q)$. The centralizer z_c of c in $G_n := \mathrm{GL}_n(\mathbb{F}_q)$ has order [7]

$$\prod_{\{x\}} a_{\Lambda(x)}(q^{d(x)}),$$

where, as before, $\{x\}$ runs through the Frobenius orbits, $d(x)$ is the size of $\{x\}$ and where for $\lambda \in \mathcal{P}$

$$a_\lambda(q) := q^{|\lambda|+2n(\lambda)}b_\lambda(q^{-1}), \tag{25}$$

with

$$n(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i \tag{26}$$

$$b_\lambda(q) := \prod_{i\geq 1}\phi_{m_i(\lambda)}(q) \tag{27}$$

$$\phi_m(q) := (1-q)(1-q^2)\cdots(1-q^m) \tag{28}$$

and, finally, $m_i(\lambda)$ is the multiplicity of i in λ.

It follows that if we define our weight function as

$$W(\lambda) := a_\lambda(q)^{-1} \in R$$

where $R = \mathbb{Z}[[q]][q^{-1}]$ then

$$W(\Lambda) = \frac{1}{|z_c|}. \tag{29}$$

Consequently, if we now take $X = \mathbb{G}_m$ then

$$\sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = 1$$

and therefore

$$Z_C(\mathbb{G}_m, q, T) = \sum_{n\geq 0} T^n = (1-T)^{-1}. \tag{30}$$

Applying (24) to this situation we find that

$$Z_C(\bullet, q, T) = \prod_{n\geq 0}(1-q^nT), \tag{31}$$

since $N_{\mathbb{G}_m} = q-1$ and $(q-1)^{-1} = -(1+q+q^2+\cdots)$; sincen $N_{\mathbb{G}_a} = q$ by (24) we also have the identity

$$Z_C(\mathbb{G}_a, q, T) = \prod_{n\geq 1}(1-q^nT). \tag{32}$$

On the other hand, for $X = \mathbb{G}_a$ we have

$$\sum_{|\lambda|=n} \frac{1}{a_\lambda(q)} = \sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = \frac{|M_n(\mathbb{F}_q)|}{|G_n|} = \frac{q^{\frac{1}{2}n(n+1)}}{(q^n-1)(q^{n-1}-1)\cdots(q-1)}$$

and we have therefore proved the following identity of Euler

$$\sum_{n\geq 0} \frac{q^{\frac{1}{2}n(n+1)}T^n}{(q^n-1)(q^{n-1}-1)\cdots(q-1)} = \prod_{n\geq 1}(1-q^nT). \tag{33}$$

If, instead, $X = \bullet$ we obtain

$$\sum_{\Lambda/\mathbb{F}_q, |\Lambda|=n} W(\Lambda) = \frac{u_n}{|G_n|},$$

where u_n is the number of unipotent matrices in G_n. Combining (31) with Euler's identity (33) with T replaced by T/q we find

$$\frac{u_n}{|G_n|} = \frac{q^{\frac{1}{2}n(n+1)-n}}{(q^n-1)(q^{n-1}-1)\cdots(q-1)}$$

we deduce the known result $u_n = q^{n^2-n}$ (see [8] for a general result on the number of unipotent elements in linear algebraic groups over finite fields).

3.2. **Commuting pairs of matrices.** We now consider a weight function arising from the centralizer algebra $\mathcal{Z}_A$ of a matrix $A \in M_n(\mathbb{F}_q)$. It is known that

$$\dim_{\mathbb{F}_q}(\mathcal{Z}_A) = \sum_{\{x\}} d(x)\langle\Lambda(x),\Lambda(x)\rangle$$

where for a partition λ we define $\langle\lambda,\lambda\rangle := |\lambda| + 2n(\lambda)$.

Since $|\mathcal{Z}_A|$ only depends on the conjugacy class $[A]$ of A, we can count ordered pairs of commuting matrices in $M_n(\mathbb{F}_q)$ as follows

$$\gamma_n := \#\{A, B \in M_n(\mathbb{F}_q) \ \mid \ AB = BA\} = \sum_{[A]} \#[A]\,|\mathcal{Z}_A|,$$

where $[A]$ runs through the conjugacy classes in $M_n(\mathbb{F}_q)$. Hence if we define

$$W(\lambda) := \frac{q^{\langle\lambda,\lambda\rangle}}{a_\lambda(q)}$$

then

$$W(\Lambda) = \frac{\#[A]}{|G_n|}|\mathcal{Z}_A|,$$

where $[A]$ corresponds to the coloring Λ on $\mathbb{G}_a/\mathbb{F}_q$, and therefore

$$\gamma_n = |G_n| \sum_{\Lambda} W(\Lambda).$$

Consequently,

$$Z_C(\mathbb{G}_a, q, T) = \sum_{n\geq 0} \frac{\gamma_n}{|G_n|}T^n. \tag{34}$$

On the other hand,

$$\begin{aligned} Z_C(\bullet, q, T) &= \sum_{\lambda} \frac{q^{\langle\lambda,\lambda\rangle}}{a_\lambda(q)} T^{|\lambda|} \\ &= \sum_{\lambda} \frac{T^{|\lambda|}}{b_\lambda(q^{-1})} \\ &= \prod_{i\geq 1} \sum_{m_i\geq 0} \frac{T^{im_i}}{\phi_{m_i}(q^{-1})} \\ &= \prod_{i,n\geq 1} (1 - q^n T^i) \end{aligned}$$

using Euler's identity (33). Applying (21) we recover (in an equivalent form) a result of Fine and Feit [3]

$$\sum_{n\geq 0} \frac{\gamma_n}{|G_n|}\, T^n = \prod_{i,n\geq 1} (1 - q^{n+1}T^i). \tag{35}$$

Similarly, we obtain

$$\sum_{n\geq 0} \frac{\gamma'_n}{|G_n|}\, T^n = Z_C(\mathbb{G}_m, q, T),$$

where

$$\gamma'_n := \#\{A \in \mathrm{GL}_n(\mathbb{F}_q), B \in M_n(\mathbb{F}_q) \ \mid\ AB = BA\}.$$

Again by (21) we find

$$Z_C(\mathbb{G}_m, q, T) = \prod_{i,n\geq 1} \left(\frac{1-q^{n+1}T^i}{1-q^nT^i}\right) = \prod_{i\geq 1}(1-qT^i)^{-1}.$$

We now recognize this generating series as (18) and conclude that $\gamma'_n/|G_n|$ is the number of conjugacy classes in $M_n(\mathbb{F}_q)$. This, in fact, can be proved directly by a simple application of Burnside's lemma to $\mathrm{GL}_n(\mathbb{F}_q)$ acting on $M_n(\mathbb{F}_q)$ by conjugation. By our main combinatorial principle, this means that we can run the argument backwards and prove (35) starting from (18).

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