COMPUTATION OF CENTRAL VALUE OF QUADRATIC TWISTS OF MODULAR L–FUNCTIONS

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1. Introduction

Let $f(z) \in S_2(p)$ be a newform of weight two, prime level p. If $f(z) = \sum_{m=1}^{\infty} a_m q^m$, where $q = e^{2\pi i z}$, and D is a fundamental discriminant, we define the twisted L-function

$$L(f, D, s) = \sum_{m=1}^{\infty} a_m m^{-s} \left(\frac{D}{m}\right).$$

It will be convenient to also allow $D = 1$ as a fundamental discriminant, in which case we write simply $L(f, s)$ for $L(f, 1, s)$.

In this paper we consider the question of computing the twisted central values $\{L(f, D, 1) \,:\, |D| \leq x\}$ for some x.

It is well known that the fact that f is an eigenform for the Fricke involution yields a rapidly convergent series for $L(f, D, 1)$. Computing $L(f, D, 1)$ by means of this series, which we call the *standard method*, takes time very roughly proportional to $|D|$ and therefore time very roughly proportional to x^2 to compute $L(f, D, 1)$ for $|D| \leq x$. We will see that this can be improved to $x^{3/2}$ by using an explicit version of Waldspurger's theorem; this theorem relates the central values $L(f, D, 1)$ to the $|D|$-th Fourier coefficient of weight 3/2 modular forms in Shimura correspondence with f.

Concretely, the formulas we use have the basic form

$$L(f, D, 1) = \star\kappa_{\pm} \frac{|c_{\pm}(|D|)|^2}{\sqrt{|D|}}, \qquad \text{sign}(D) = \pm, \tag{1.1}$$

Key words and phrases. Waldspurger correspondence, Half integral weight forms, Special values of L-functions.

where $\star = 1$ if $p \nmid D$, $\star = 2$ if $p \mid D$, $\kappa_\pm > 0$ is a constant independent of D and $c_\pm(|D|)$ is $|D|$-th Fourier coefficient of a certain modular form $g_\pm$ of weight $3/2$.

Gross [Gr] proves such a formula, and gives an explicit construction of the corresponding form g_-, in the case that $L(f,1) \neq 0$ (which holds for about half of the cases). The purpose of this paper is to extend Gross's work to all cases. Specifically, we give an explicit construction of both g_- and g_+, regardless of the value of $L(f,1)$, together with the corresponding values of $k_\pm$ in (1.1). The proof of the validity of this construction will be given in a later publication and relies partly in the results of [B-M].

The construction gives $g_\pm$ as a linear combination of (generalized) theta series associated to positive definite ternary quadratic forms. Computing the Fourier coefficients of these theta series up to x is tantamount to running over all lattice points in ellipsoid of volume proportional to $x^{3/2}$. Doing this takes time roughly proportional to $x^{3/2}$ which yields our claim above.

This approach to computing $L(f,D,1)$ has several other advantages over the standard method. First, the numbers $c(|D|)$ are algebraic integers and are computed with exact arithmetic. Once $c(|D|)$ is know it is trivial to compute $L(f,D,1)$ to any desired precision. Second, the $c(D)$'s have extra information; if f has coefficients in $\mathbb{Z}$, for example, (1.1) gives a specific square root of $L(f,D,1)$ (if non-zero), whose sign remains a mystery.

Moreover, the actual running time of our method vs. the standard method is, in practice, significantly better even for small x.

2. Construction of $g_\pm(z)$

2.1. $g_-(z)$: when $L(f,1) \neq 0$. We recall Gross's construction of the map θ_1.

Let B be the quaternion algebra over $\mathbb{Q}$ ramified precisely at ∞ and p. Let R be a fixed maximal order in B. A right ideal I of R is a lattice in B that is stable under right multiplication by R. Two right ideals I and J are in the same class if $J = bI$ with $b \in B^\times$. The set of right ideal class is finite; we denote its order by n and let $\{I_1, \dots, I_n\}$ be the representatives.

Let $R_i = \{b \in B : bI_i \subset I_i\}$ be the left order of I_i. Then R_i are also maximal orders in B and each conjugacy class of maximal orders has a representative R_i for some i. Let $2w_i$ be the number of units in $R_i^\times$, then Eichler's mass formula states $\sum_{i=1}^n \frac{1}{w_i} = \frac{p-1}{12}$.

For $b \in B$, we use $\mathcal{N}\, b$ to denote the reduced norm of b. Let $\mathcal{N}\, I_i$ be the positive greatest common divisor of $\{\mathcal{N}\, b \,:\, b \in I_i\}$.

Let $S_i := \mathbb{Z} + 2R_i$, a suborder of index 8 in R_i. Let S_i^0 be the subset of S_i consisting of trace 0 elements. Define

$$h_i(z) = \frac{1}{2} \sum_{b \in S_i^0} q^{\mathcal{N}\, b} = \frac{1}{2} \sum_{m \geq 0} c_i(m) q^m.$$

Then $h_i(z)$ is a weight $3/2$ form with level $4p$ and satisfying $c_i(m) = 0$ whenever $m \equiv 1, 2 \bmod 4$.

As mentioned before e_f is a function on the ideal classes I_i. Let $a_i = e_f(I_i)$, then

$$g_-(z) = \theta_1(e_f) := \sum_i a_i h_i(z). \tag{2.1}$$

2.2. $g_-(z)$ **and weight functions:** $L(f,1) = 0$ **case.** When $L(f,1) = 0$, we construct $g_-(z)$ as follows:

1. Find a prime $l \neq p$ such that $l \equiv 1 \pmod 4$ and $L(f,l,1) \neq 0$; in particular, $\left(\frac{l}{p}\right)$ has to be equal to the sign of the functional equation for $L(f,s)$. From [B-F-H], there are infinitely many such l.
2. Fix a normalized weight function ω_l on R, as defined below.
3. Transport ω_l to weight functions $\omega_l(I_i, \cdot)$ on R_i, as explained below.
4. Define

$$h_i(z) := \frac{1}{2} \sum_{b \in S_i^0} \omega_l(I_i, b)\, q^{\mathcal{N}\, b/l}.$$

5. Let $a_i = e_f(I_i)$, then

$$g_-(z) = \theta_l(e_f) := \sum_i a_i h_i(z).$$

Definition 2.1. *Let R be a maximal order, and fix a prime $l \neq p$. A* weight function *ω_l on R is a nonzero function defined on $R^0(\mathbb{Z}_l)$ (where R^0 is the subset of trace zero elements) satisfying the following equations:*

$$\omega_l(a^{-1}ba) = \left(\frac{\mathcal{N}\,a}{l}\right)\omega_l(b),\ a \in R^\times(\mathbb{Z}_l),\ b \in R^0(\mathbb{Z}_l); \tag{2.2}$$

$$\omega_l(kb) = \left(\frac{k}{l}\right)\omega_l(b),\quad k \in \mathbb{Z}_l^\times,\ b \in R^0(\mathbb{Z}_l); \tag{2.3}$$

$$\omega_l(y) = \sigma \int_{R^0(\mathbb{Z}_l)} \omega_l(x)\psi(\mathrm{Tr}(xy)/l)\,dx. \tag{2.4}$$

Here $\psi(x) = q^{\iota(x)}$ where $\iota(x) \in \mathbb{Q}$ satisfies $x - \iota(x) \in \mathbb{Z}_l$; the measures are normalized so that $\mathbb{Z}_l$ has volume 1, and

$$\sigma = l^2 \int_{\mathbb{Z}_l^\times} \left(\frac{a}{l}\right)\psi(a/l)\,da.$$

We say that ω_l is normalized *if $\omega_l(b) \in \{0, \pm 1\}$ for all $b \in R^0$. A normalized weight function ω_l exists and is unique up to sign.*

Fix $b_0 \in R^0$ such that $l \mid \mathcal{N}\,b_0$ and $b_0 \notin lR^0$. Then $\omega_l(b_0) \neq 0$ for any weight function $\omega_l \neq 0$ on R. We fix ω_l to be the unique weight function on R such that $\omega_l(b_0) = 1$. Then ω_l can be computed by using Algorithm 2.2 below, applied to (R, b_0).

Let $x_i \in I_i$ be a generator of $I_i \otimes \mathbb{Z}_l$, so that $x_i^{-1}R_i^0(\mathbb{Z}_l)x_i = R^0(\mathbb{Z}_l)$ and $l \nmid n_i := \mathcal{N}\,x/\mathcal{N}\,I_i \in \mathbb{Z}$. If $b \in R_i^0(\mathbb{Z}_l)$, we set

$$\omega_l(I_i, b) := \left(\frac{n_i}{l}\right)\omega_l(x_i^{-1}\,b\,x_i).$$

This determines a weight function $\omega_l(I_i, \cdot)$ on R_i. Note that we can always assume $I_i \subseteq R$ and $\left(\frac{\mathcal{N}\,I_i}{l}\right) = 1$, in which case we would have $R_i^0(\mathbb{Z}_l) = R^0(\mathbb{Z}_l)$ and $\omega_l(I_i, \cdot) = \omega_l$.

In any case, $b_{0,i} := n_i\, x_i\, b_0\, x_i^{-1} \in R_i^0$ is such that $\omega_l(I_i, b_{0,i}) = \omega_l(b_0) = 1$; thus $\omega_l(I_i, \cdot)$ can also be computed by Algorithm 2.2 applied to $(R_i, b_{0,i})$,

Algorithm 2.2. Given a pair (R, b_0), where R is a maximal order and $b_0 \in R^0$ is such that $l \mid \mathcal{N}\, b_0$, but $b_0 \notin lR^0$, this algorithm computes the unique weight function ω_l on R^0 determined by $\omega_l(b_0) = 1$.

Input: $b \in R_0$.

Output: $\omega_l(b)$.

1. If $l \nmid \mathcal{N}\, b$, return 0.
2. If $l \nmid \mathcal{N}(b + b_0)$, return $\left(\frac{\mathcal{N}(b+b_0)}{l}\right)$.
3. Otherwise, there is some $k \in \mathbb{Z}$ is such that $b - k\, b_0 \in lR^0$.
   Find such a k, and return $\left(\frac{k}{l}\right)$.

2.3. $g_+(z)$ **and weight function.** The construction of $g_+(z)$ can be done as follows:

1. Identify a prime $l \neq p$ such that $l \equiv 3 \mod 4$ and $L(f, -l, 1) \neq 0$; in particular, $-\left(\frac{-l}{p}\right)$ has to be equal to the sign of the functional equation for $L(f, s)$. From [B-F-H], there are infinitely many such l.

2. Fix a normalized weight function ω_l on R and transport it to weight functions $\omega_l(I_i, \cdot)$ on R_i as in the previous section. Define another weight function ω_p on $B^0(\mathbb{Z}_p)$. As $S_i^0 \mapsto S_i^0 \otimes \mathbb{Q}_p \subset B^0(\mathbb{Z}_p)$ for all i, ω_p can be regarded as a function on S_i^0.

3. Define

$$h_i(z) = \frac{1}{2} \sum_{b \in S_i^0} q^{\mathcal{N}\, b/l} \omega_l(b) \omega_p(b).$$

4. Let $a_i = e_f(I_i)$, then

$$g_+(z) = \theta_{-l}(e_f) := \sum_i a_i h_i(z). \tag{2.5}$$

The weight function $\omega_p(b)$ is a function satisfying:

(1) ω_p is constant $\mod p\mathbb{Z}_p$.

(2) $\omega_p(a^{-1}ba) = [\mathcal{N}\, a, l]_p \omega_p(b)$ for all $a \in B(\mathbb{Q}_p)$ and $b \in B^0(\mathbb{Z}_p)$.

(3) $\omega_p(kb) = \chi_p(k)\omega_p(b)$ for $k \in \mathbb{Z}^\times$, and χ_p is any fixed odd character of $(\mathbb{Z}/p)^\times$ considered as a character on $\mathbb{Z}_p^\times$, ("odd" means $\chi(-1) = -1$).

When χ_p is fixed, there is a unique (up to scalar multiple) function satisfying the above conditions. Recall [P]

$$B^0(\mathbb{Z}_p) = \{b = \alpha I + \beta J + \gamma IJ \ : \ \alpha, \beta, \gamma \in \mathbb{Z}_p\}$$

where $I^2 = a$ and $J^2 = b$, $IJ = -JI$; a, b are negative integers satisfying $[a, b]_p = -1$ and $[a, b]_l = 1$ for all primes $l \neq p$. We may assume a is a unit in $\mathbb{Z}_p$ and b generates the prime ideal in $\mathbb{Z}_p$. Then ω_p can be defined as follows:

(1) when α is not a unit in $\mathbb{Z}_p$, $\omega_p(b) = 0$.

(2) when α is a unit in $\mathbb{Z}_p$, $\omega_p(b) = \chi_p(\alpha)$.

3. An explicit formula

The construction of section 3 singles out, for each l, an explicit form of weight $3/2$ for the formula of Theorem 2.2. This follows the construction of Gross for $l = 1$, for which there is a formula with an explicit constant [Gr, Proposition 13.5]. We can extend it to the case of $l \equiv 1 \pmod 4$.

Proposition 3.1. *Let $l \neq p$ be a prime such that $l \equiv 1 \pmod 4$, and let $-d < 0$ be a fundamental discriminant. Then*

$$L(f, -d, 1)\, L(f, l, 1) = \star\, \frac{\langle f, f\rangle}{\sqrt{dl}} \frac{|c_l(d)|^2}{\langle e_f, e_f\rangle},$$

where $\theta_l(e_f) = \sum_{n=1}^\infty c_l(n) q^n$, and where $\star = 1$ if $p \nmid d$, $\star = 2$ if $p \mid d$.

Proof. We deal here with the case $\left(\frac{-dl}{p}\right) = -1$ and $l \nmid d$. When $\left(\frac{-dl}{p}\right) = +1$, both sides are trivially 0. The remaining cases follow from similar methods, or from Theorem 2.2.

Consider the divisor

$$c := \sum_i \frac{1}{2w_i} \sum_{\substack{b \in S_i^0 \\ \mathcal{N} b = dl}} \omega_l(I_i, b)\, [I_i].$$

It is clear that $c_l(d) = \langle c, e_f\rangle$, since $\langle [I_i], [I_i]\rangle = w_i$ by definition of the height pairing. The pairs (I_i, b) where $b \in S_i^0$ is such that $\mathcal{N}\, b = dl$, modulo conjugation by the units of R_i, give a set of representatives for the special points of discriminant $-dl$. Thus

$$c = \frac{1}{2}\sum_x \omega_l(x)\,[x],$$

where the sum is over the special points $x = (I_i, b)$ of discriminant $-dl$, and where $[x] := [I_i]$.

Let $\mathcal{O}$ be the quadratic order of discriminant $-d\cdot l$, and let χ_l be the genus character of $\mathrm{Pic}(\mathcal{O})$ corresponding to this discriminant factorization; namely, if $A \in \mathrm{Pic}(\mathcal{O})$, then $\chi_l(A) := \left(\frac{\mathcal{N}\,\mathfrak{a}}{l}\right)$, where $\mathfrak{a} \in A$ is choosen so that $l \nmid \mathcal{N}\,\mathfrak{a}$.

Recall that $\mathrm{Pic}(\mathcal{O})$ acts freely on the special points of discriminant $-dl$, and since $\left(\frac{-dl}{p}\right) = -1$ there are exactly two orbits that are permuted by complex conjugation (see [Gr,§3]). It follows from the definitions that, for $A \in \mathrm{Pic}(\mathcal{O})$

$$\omega_l(A\cdot x) = \chi_l(A)\,\omega_l(x),$$

and also that $\omega_l(\overline{x}) = \omega_l(x)$ and $[\overline{x}] = [x]$. Consequently, we can write

$$c = \omega_l(x_0)\sum_{A\in\mathrm{Pic}(\mathcal{O})}\chi_l(A)\,[A\cdot x_0],$$

where x_0 is a fixed special point of discriminant $-dl$. Since $l \nmid d$ and ω_l is normalized, we have $\omega_l(x_0) = \pm 1$.

Apply now [Gr, Proposition 11.2] to χ_l, obtaining the formula

$$L(f,\chi_l,1) = \frac{\langle f, f\rangle}{\sqrt{dl}}\,\frac{\langle c, e_f\rangle^2}{\langle e_f, e_f\rangle}.$$

Since χ_l is a genus character, we have a decomposition

$$L(f,\chi_l,1) = L(f,-d,1)\,L(f,l,1),$$

and the result follows. □

4. Examples

4.1. **37A, imaginary twists.** Let $f = f_{37A}$, the modular form of level 37 and rank 1. Let $B = B(-2, -37)$, the quaternion algebra ramified precisely at ∞ and 37. A maximal order, and representatives for its right ideal classes, are given by

$$\begin{aligned} R = I_1 &= \left\langle 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right\rangle && \text{with } \mathcal{N}\, I_1 = 1, \\ I_2 &= \left\langle 2, 2i, \frac{1+3i+j}{2}, \frac{6+3i+k}{4} \right\rangle && \text{with } \mathcal{N}\, I_2 = 2, \\ I_3 &= \left\langle 4, 2i, \frac{3+3i+j}{2}, \frac{6+i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_3 = 4. \end{aligned}$$

By computing the Brandt matrices, we find a vector

$$e_f = \frac{[I_3] - [I_2]}{2}$$

of height $\langle e_f, e_f \rangle = 1/2$ corresponding to f. Since $L(f, 1) = 0$ we know that $2\theta_1(e_f) = \theta_1([I_3]) - \theta_1([I_2]) = 0$. Indeed, one checks that R_2 and R_3 are conjugate, which explains the identity $\theta_1([I_2]) = \theta_1([I_3])$.

Let now $l = 5$. One can compute $L(f, 5, 1) \approx 5.3548616$, and thus we expect $\theta_5(e_f)$ to be nonzero. We note that, by the same reason that the orders are conjugate, we have $\theta_5([I_3]) = -\theta_5([I_2])$, except now there's an extra sign, ultimately coming from the fact that $\left(\frac{37}{5}\right) = -1$. Thus, $\theta_5(e_f) = \theta_5([I_3])$. A basis for S_3^0 is given by

$$S_3^0 = \left\langle b_1 = \frac{3i+2j+k}{4},\ b_2 = \frac{7i-2j+k}{4},\ b_3 = \frac{3i-k}{2} \right\rangle,$$

with the norm in this basis (denoting $x_1b_1 + x_2b_2 + x_3b_3$ by (x_1, x_2, x_3))

$$\mathcal{N}_3\,(x_1, x_2, x_3) = 15x_1^2 + 20x_2^2 + 23x_3^2 - 8x_2x_3 - 14x_1x_3 - 4x_1x_2.$$

d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$
3	1	2.830621	95	0	0.000000	139	0	0.000000
4	1	2.451389	104	0	0.000000	148	-3	7.254107
7	-1	1.853076	107	0	0.000000	151	-2	1.595930
11	1	1.478243	111	1	0.930702	152	-2	1.590671
40	2	3.100790	115	-6	16.458713	155	2	1.575203
47	-1	0.715144	120	-2	1.790242	159	1	0.388816
67	6	21.562911	123	3	3.978618	164	-1	0.382843
71	1	0.581853	127	1	0.435051	184	0	0.000000
83	-1	0.538150	132	3	3.840589	195	2	1.404381
84	-1	0.534937	136	4	6.726557			

TABLE 1. Coefficients of $\theta_5(e_f)$ and central values for $f = f_{37A}$

Choose ω_5 so that $\omega_5(b_1) = +1$. Then $\omega_5(I_3, \cdot) = \omega_5$ can be computed as

$$\omega_5(I_3,(x_1,x_2,x_3)) = \begin{cases} 0 & \text{if } 5 \nmid \mathcal{N}_3\,(x_1,x_2,x_3), \\ \left(\frac{x_2+x_3}{5}\right) & \text{if } x_2+x_3 \not\equiv 0 \pmod 5, \\ \left(\frac{x_1}{5}\right) & \text{otherwise.} \end{cases}$$

Table 1 shows the values of $c_5(d)$ and $L(f,-d,1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{37}\right) \neq -1$. The formula

$$L(f,-d,1) = k_5\,\frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{37}\right) = +1, \\ 2 & \text{if } \left(\frac{-d}{37}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{37}\right) = -1, \end{cases}$$

is satisfied, where

$$k_5 = \frac{2(f,f)}{L(f,5,1)\sqrt{5}} \approx 4.90277876397358012170844966 3733...$$

Note that in the case $\left(\frac{-d}{37}\right) = -1$ it is trivial that $c_5(d) = L(f,-d,1) = 0$.

4.2. **43A, imaginary twists.** Let $f = f_{43A}$, the modular form of level 43 and rank 1. Let $B = B(-1, -43)$, the quaternion algebra ramified precisely at ∞ and 43. A maximal order, and representatives for its right ideals classes, are given by

$$\begin{aligned}
R = I_1 &= \left\langle 1, i, \frac{1+j}{2}, \frac{i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_1 = 1,\\
I_2 &= \left\langle 2, 2i, \frac{1+2i+j}{2}, \frac{2+3i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_2 = 2,\\
I_3 &= \left\langle 3, 3i, \frac{1+2i+j}{2}, \frac{2+5i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_3 = 3,\\
I_4 &= \left\langle 3, 3i, \frac{1+4i+j}{2}, \frac{4+5i+k}{2} \right\rangle && \text{with } \mathcal{N}\, I_4 = 3.
\end{aligned}$$

By computing the Brandt matrices, we find a vector

$$e_f = \frac{[I_4] - [I_3]}{2}$$

of height $\langle e_f, e_f \rangle = {}^1\!/_2$ corresponding to f. We can use $l = 5$, since $L(f, 5, 1) \approx 4.8913446$ is nonzero. Again, we find $\theta_5(e_f) = \theta_5([I_4])$; in a convenient basis of S_4^0, the norm is

$$\mathcal{N}_4\,(x_1, x_2, x_3) = 15x_1^2 + 23x_2^2 + 24x_3^2 + 12x_2x_3 + 8x_1x_3 + 2x_1x_2,$$

and $\omega_5(I_4, \cdot) = -\omega_5$ can be computed by

$$\omega_5(I_4, (x_1, x_2, x_3)) = \begin{cases} 0 & \text{if } 5 \nmid \mathcal{N}_4\,(x_1, x_2, x_3),\\ \left(\frac{2x_2+3x_3}{5}\right) & \text{if } x_2 \not\equiv x_3 \pmod 5,\\ \left(\frac{x_1}{5}\right) & \text{otherwise.} \end{cases}$$

d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$
3	1	3.148135	91	-1	0.571601	151	-1	0.443737
7	1	2.060938	104	1	0.534684	155	-1	0.437974
8	-1	1.927831	115	-3	4.576227	159	1	0.432430
19	2	5.003768	116	-1	0.506273	163	7	20.927447
20	-1	1.219267	119	-1	0.499851	168	-2	1.682749
39	-1	0.873136	120	0	0.000000	179	-1	0.407556
43	2	6.652268	123	-5	12.291402	184	-3	3.617825
51	1	0.763535	131	0	0.000000	191	0	0.000000
55	1	0.735246	132	3	4.271393	199	0	0.000000
71	0	0.000000	136	-1	0.467568			
88	3	5.231366	148	-4	7.171386			

TABLE 2. Coefficients of $\theta_5(e_f)$ and central values for $f = f_{43A}$

Table 2 shows the values of $c_5(d)$ and $L(f,-d,1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{43}\right) \neq -1$. The formula

$$L(f,-d,1) = k_5\,\frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{43}\right) = +1, \\ 2 & \text{if } \left(\frac{-d}{43}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{43}\right) = -1, \end{cases}$$

is satisfied, where

$$k_5 = \frac{2(f,f)}{L(f,5,1)\sqrt{5}} \approx 5.45272967268173438557072278528 3...$$

Note that in the case $\left(\frac{-d}{43}\right) = -1$ it is trivial that $c_5(d) = L(f,-d,1) = 0$.

4.3. **389A, imaginary twist.** Let $f = f_{389A}$, the modular form of level 389 and rank 2. Let $B = B(-2,-389)$, the quaternion algebra ramified precisely at ∞ and 389. A maximal order, with 33 ideal classes, is given by

$$R = \left\langle 1, i, \frac{1+i+j}{2}, \frac{2+3i+k}{4} \right\rangle.$$

i	a_i	$\mathcal{N}_i$	$b_{0,i}$
1	1/2	15, 107, 416, -100, -8, -14	2, 4, 0
2	-1/2	15, 104, 415, 104, 2, 4	0, 4, 1
3	-1/2	23, 136, 203, 68, 2, 8	2, 1, 4
4	1/2	23, 72, 407, 72, 10, 20	1, 1, 0
5	-1/2	31, 51, 407, -46, -26, -10	1, 2, 0
6	1/2	31, 103, 204, 56, 20, 18	2, 0, 3
7	1/2	39, 128, 160, -116, -8, -36	1, 1, 4
8	-1/2	39, 40, 399, 40, 2, 4	1, 0, 1
9	1/2	40, 47, 399, 18, 40, 36	4, 3, 0
10	-1/2	47, 107, 135, 42, 22, 38	4, 3, 1
11	-1/2	56, 84, 139, 56, 4, 12	3, 1, 4
12	1/2	56, 92, 151, 76, 52, 44	4, 2, 3
13	1/2	71, 83, 132, -16, -12, -70	2, 3, 4
14	-1/2	71, 103, 124, -36, -64, -66	4, 0, 2

TABLE 3. Coefficients of the ternary forms and of $b_{0,i}$

There is a vector e_f of height $\langle e_f, e_f\rangle = 5/2$ corresponding to f. We can use $l = 5$, since $L(f, 5, 1) \approx 8.9092552$.

We have omitted the 33 ideal classes; however, the computation of $\theta_l(e_f)$ involves only 14 distinct theta series. In table 3 we give the value of e_f and the coefficients of the norm form $\mathcal{N}_i$ and of $b_{0,i}$ on choosen bases of S_i^0.

Each row in the table allows one to compute an individual theta series

$$h_i(z) = \frac{1}{2} \sum_{b \in \mathbb{Z}^3} w_5(I_i, b) q^{\mathcal{N}_i(b)/5}.$$

The ternary form corresponding to a sextuple $(A_1, A_2, A_3, A_{23}, A_{13}, A_{12})$ is

$$\mathcal{N}_i(x_1, x_2, x_3) = A_1 x_1^2 + A_2 x_2^2 + A_3 x_3^2 + A_{23} x_2 x_3 + A_{13} x_1 x_3 + A_{12} x_1 x_2,$$

and the weight function $\omega_5(I_i, \cdot)$ is computed as per Algorithm 2.2 applied to $(\mathbb{Z}^3, b_{0,i})$. As an example, we show how to compute $h_1(z)$. First, we have

$$\mathcal{N}_1\left(x_1, x_2, x_3\right) = 15x_1^2 + 107x_2^2 + 416x_3^2 - 100x_2x_3 - 8x_1x_3 - 14x_1x_2.$$

A simple calculation shows that

$$\mathcal{N}_1\left(x_1 + 2, x_2 + 4, x_3 + 0\right) \equiv 4x_1 + 3x_2 + 4x_3 \pmod 5,$$

provided that $\mathcal{N}_1\left(x_1, x_2, x_3\right) \equiv 0 \pmod 5$. Thus, ω_5 can be computed as

$$\omega_5(I_1, (x_1, x_2, x_3)) = \begin{cases} 0 & \text{if } 5 \nmid \mathcal{N}_1\left(x_1, x_2, x_3\right), \\ \left(\frac{4x_1+3x_2+4x_3}{5}\right) & \text{if } 4x_1 + 3x_2 + 4x_3 \not\equiv 0 \pmod 5, \\ \left(\frac{x_2}{5}\right) & \text{otherwise,} \end{cases}$$

and we have

$$h_1(z) = q^3 - q^{12} - q^{27} + q^{39} + q^{40} + q^{48} - q^{83} - 2q^{92} + O(q^{100}).$$

Finally, we combine all of the theta series in

$$\theta_5(e_f) = \sum_{i=1}^{14} a_i h_i(z)$$

Table 4 shows the values of $c_5(d)$ and $L(f, -d, 1)$, where $0 < -d < 200$ is a fundamental discriminant such that $\left(\frac{-d}{389}\right) \neq +1$. The formula

$$L(f, -d, 1) = k_5 \frac{c_5(d)^2}{\sqrt{d}} \cdot \begin{cases} 1 & \text{if } \left(\frac{-d}{389}\right) = -1, \\ 2 & \text{if } \left(\frac{-d}{389}\right) = 0, \\ 0 & \text{if } \left(\frac{-d}{389}\right) = +1, \end{cases}$$

is satisfied, where

$$k_5 = \frac{2(f, f)}{5L(f, 5, 1)\sqrt{5}} \approx 7.886950806206592817689630792605...$$

Note that in the case $\left(\frac{-d}{389}\right) = +1$ it is trivial that $c_5(d) = L(f, -d, 1) = 0$.

d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$	d	$c_5(d)$	$L(f,-d,1)$
3	1	4.553533	83	-1	0.865705	139	-1	0.668962
8	-1	2.788458	84	1	0.860537	148	6	23.338921
15	-1	2.036402	88	-4	13.452028	151	2	2.567324
23	1	1.644543	103	0	0.000000	152	-1	0.639716
31	1	1.416538	104	-1	0.773379	155	3	5.701456
39	1	1.262923	107	0	0.000000	163	8	39.536232
40	1	1.247036	115	-1	0.735462	167	-1	0.610311
43	-3	10.824738	116	-2	2.929140	191	1	0.570680
47	0	0.000000	123	3	6.400282	195	1	0.564796
51	-2	4.417576	131	1	0.689086	199	-1	0.559091
56	1	1.053938	132	-2	2.745884			
71	1	0.936009	136	-2	2.705202			

TABLE 4. Coefficients of $\theta_5(e_f)$ and central values for $f = f_{389A}$

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