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SQUARE ROOT FORMULAS FOR CENTRAL VALUES OF HECKE L-SERIES II

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1. Introduction. This paper is a complement to [8]; we show how the results proved there can be extended to quadratic imaginary fields K with arbitrary odd discriminant $-d < -3$, hence fulfilling, in part, the promise made in [7]. We consider the central value of the L-series $L(\psi, s)$ associated to a Hecke character ψ of K that satisfies (see (1))

$$\psi((\alpha)) = \varepsilon(\alpha)\alpha^{2k-1}, \qquad \text{for integers } \alpha \text{ of } K \text{ prime to } d.$$

We show that one can associate to every such ψ a well-defined genus $\mathscr{G}_\psi$ and that the central value $L(\psi, k)$ equals $c|\mathscr{T}|^2$, where c is a simple normalizing factor and $\mathscr{T}$ is a linear combination of values of the $(k-1)$st nonholomorphic derivative of a classical half-integral weight theta series, at CM points corresponding to ideals in $\mathscr{G}_\psi$. This is our main result (14).

The essential new feature is the analysis of the genus theory involved. For nonprime discriminants the class number of K is even, and hence we cannot rely on having squares of ideals in every class as it happened in [8] and [7].

We have organized the paper as follows: a main part and an appendix. We also included a list of minor corrections to [8]. In the main part, we set things up in Section 2, deal with genus theory in Section 3, put together the final formula in Section 4, after recalling the key ingredients of [8], and present some numerical examples in Section 5. In the appendix we briefly discuss some ideas related to the factorization formula (12). This sheds some light on how such formulas work; for example, we show how in general they are truly identities of values holding only on a finite number of points. It also gives an indication of more general formulas, which we will discuss elsewhere.

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2. Basic setup. Let K be an imaginary quadratic field of discriminant $-d$, with $d > 3$, $d \equiv 3 \bmod 4$, viewed as a subfield of the complex numbers $\mathbf{C}$. We understand by $\sqrt{-d}$ the root with positive imaginary part.

Let $\mathcal{O}_K$ be the ring of integers of K; note that $\mathcal{O}_K^* = \{\pm 1\}$. Unless stated otherwise, by ideals we will always mean integral ideals. An ideal is primitive if it is not divisible by rational integers > 1. We let Cl be the class group of K, and $Cl_{(2)}$ the subgroup

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of classes annihilated by 2. Any primitive ideal $\mathscr{A}$ can be written as

$$\mathscr{A} = [a, (b + \sqrt{-d})/2],$$

where $a = \mathbf{N}(\mathscr{A})$ is the norm of $\mathscr{A}$ and b is an integer defined modulo $2a$, which satisfies $b^2 \equiv -d \bmod 4a$. Conversely, given any such pair of numbers a, b, the above formula gives a primitive ideal of norm a. We denote the class of $\mathscr{A}$ by $[\mathscr{A}]$.

Let ε be the quadratic character of K of conductor $(\sqrt{-d})$. Explicitly, we can write any $\mu \in \mathscr{O}_K$ as $\mu = (m + n\sqrt{-d})/2$, with m, n of the same parity, and then $\varepsilon(\mu) = (2m/d)$. Note that for rational integers r, $\varepsilon(r) = (-d/r)$. We define $\delta = 0$ or 1 by $(d + 1)/4 \equiv \delta \bmod 2$ or, equivalently, $(-1)^\delta = \varepsilon(2)$.

For any positive integer k we consider Hecke characters ψ of K with values in $\mathbf{C}$ satisfying

$$\psi((\alpha)) = \varepsilon(\alpha)\alpha^{2k-1}, \qquad \text{for } \alpha \in \mathscr{O}_K \text{ prime to } d. \tag{1}$$

We will say that ψ has weight $2k - 1$. Clearly, the number of such ψ is h, the class number of K, each one with conductor $(\sqrt{-d})$. We associate to ψ its L-series

$$L(\psi, s) = \sum_{\mathscr{A}} \frac{\psi(\mathscr{A})}{\mathbf{N}\mathscr{A}^s}.$$

This series converges only for $\Re(s) > k + 1/2$ but can be analytically continued to the whole s-plane and satisfies a functional equation under $s \mapsto 2k - s$, with root number $(-1)^{k+1+\delta}$. We are interested in its central value $L(\psi, k)$.

3. Genus theory. We recall some genus theory on K. Each element of $Cl_{(2)}$ is associated to a factorization of the discriminant $-d$. Given $U \in Cl_{(2)}$, there are ideals $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $(\sqrt{-d}) = \mathscr{D}_1 \cdot \mathscr{D}_2$ and $U = [\mathscr{D}_1] = [\mathscr{D}_2]$. We let $d_1 = \mathbf{N}(\mathscr{D}_1)$ and $d_2 = \mathbf{N}(\mathscr{D}_2)$, so that $d = d_1 \cdot d_2$. We will always choose $d_1 \equiv 1 \bmod 4$ and $d_2 \equiv 3 \bmod 4$, determining them uniquely. Notice that $d_1 > 0$, $d_2 > 0$, and $\gcd(d_1, d_2) = 1$.

We also associate a genus character to every such factorization, defining

$$\chi_U(C) = \left(\frac{\mathbf{N}(\mathscr{A})}{d_1}\right), \qquad \mathscr{A} \in C \text{ prime to } (\sqrt{-d}). \tag{2}$$

This map is a homomorphism in each variable. By a genus we understand a set of ideals of K whose classes lie in a fixed coset of Cl/Cl^2. Every character ϕ: $Cl_{(2)} \to \{\pm 1\}$ determines a unique genus consisting of those ideals whose classes C verify $\chi_U(C) \cdot \phi(U) = 1$ for every $U \in Cl_{(2)}$.

Let $U \in Cl_{(2)}$, $\mathscr{D}_1$, $\mathscr{D}_2$, and d_1, d_2 have the same meaning as above. Choose a primitive ideal $\mathscr{A}$ prime to $(\sqrt{-d})$ such that $\mathscr{A}\mathscr{D}_1 = (\mu)$, $\mu = (md_1 + n\sqrt{-d})/2$ with integers m, n of the same parity. Given a character ψ of weight $2k - 1$ as in (1), we define

$$\chi_\psi(U) = \left(\frac{2m}{d_2}\right)\left(\frac{n}{d_1}\right)\frac{(\mu/\sqrt{d_1})^{2k-1}}{\psi(\mathscr{A})}, \tag{3}$$

where $\sqrt{d_1} > 0$. Since $4\mathrm{N}(\mathscr{A}) = m^2 d_1 + n^2 d_2$ is prime to d, we have that m is prime to d_2 and n is prime to d_1 and therefore $\chi_\psi(U) \neq 0$. In fact, we have the following.

PROPOSITION A. *For any character ψ as in (1), the map*

$$\chi_\psi\colon Cl_{(2)} \to \{\pm 1\}$$

given by (3) is a well-defined homomorphism.

Proof. First notice that the right-hand side of (3) is not changed if we take $-\mu$ instead of μ. To show the map χ_ψ is well defined, consider the ideal $(\gamma)\mathscr{A}$ instead of $\mathscr{A}$, where (γ) is primitive prime to $d\bar{\mathscr{A}}$, and $\gamma = (r + s\sqrt{-d})/2$, $r \equiv s \bmod 2$. Then μ becomes $\mu\gamma = (1/2)(m'd_1 + n'\sqrt{-d})$ where $m' = (1/2)(rm - nsd_2)$ and $n' = (msd_1 + rn)/2$. We have $\psi((\gamma)\mathscr{A}) = \varepsilon(\gamma)\gamma^{2k-1}\psi(\mathscr{A})$, and so all we need to check is that $\varepsilon(\gamma)(2m'/d_2)(n'/d_1) = (2m/d_2)(n/d_1)$. Now $\varepsilon(\gamma) = (2r/d)$, $2m' \equiv rm \bmod d_2$, and $4n' \equiv 2rn \bmod d_1$ and we are done. This proves the definition is independent of the choice of ideal $\mathscr{A}$.

Let us show now that the map is a homomorphism. Suppose we have two factorizations of d as in the definition of χ_ψ, say d_j' and d_j'' $(j = 1, 2)$. We will denote by $*'$ and $*''$ the different quantities used above associated with each factorization. First assume that d_1' and d_1'' are relatively prime and choose $\mathscr{A}'$ and $\mathscr{A}''$ with relatively prime norms. Let $d_1 = d_1'd_1''$, $d_2 = d/d_1$ (which is integral because of our assumption of coprimality), $\mathscr{D}_1 = \mathscr{D}_1'\mathscr{D}_1''$, and $\mathscr{A} = \mathscr{A}'\mathscr{A}''$. We then have $\mathscr{A}\mathscr{D}_1 = (\mu'\mu'')$ and choose $\mu = \mu'\mu'' = (md_1 + n\sqrt{-d})/2$. A calculation shows that $m = (m'm'' - n'n''d_2)/2$ and $n = (m'n''d_1' + m''n'd_1'')/2$. We find that

$$\frac{\chi_\psi(U')\chi_\psi(U'')}{\chi_\psi(U'U'')} = \left(\frac{2m'}{d_2'}\right)\left(\frac{n'}{d_1'}\right)\left(\frac{2m''}{d_2''}\right)\left(\frac{n''}{d_1''}\right)\left(\frac{2m}{d_2}\right)\left(\frac{n}{d_1}\right). \tag{4}$$

We need, therefore, to check that the product of symbols on the right-hand side is equal to 1. From the above expressions for m and n, we find $(2m/d_2) = (m'm''/d_2)$ and $(n/d_1) = (2m'n''/d_1'')(2m''n'/d_1')$; so the right-hand side of (4) equals $(2m'/d_2'd_1''d_2)$ $(2m''/d_2''d_1'd_2) = 1$ since $d_2'd_1''d_2 = d_2'^2$ and $d_2''d_1'd_2 = d_2''^2$. This proves the claim under the assumption of coprimality, it is easy to see that the general case reduces to this one, completing the proof of the proposition. □

Remark. Note that, for any character ϕ of Cl, $\chi_{\phi\psi} = \phi\chi_\psi$, where ϕ here is restricted to $Cl_{(2)}$.

The proposition allows us to associate to every ψ a genus $\mathscr{G}_\psi$ as indicated at the beginning of the section.

$$\mathscr{G}_\psi = \{\mathscr{A}: \chi_U([\mathscr{A}])\cdot\chi_\psi([U]) = 1, \text{for every } U \in Cl_{(2)}\}. \tag{5}$$

4. Formula for the central value $L(\psi, k)$. We first recall notation and formulas of [8]. For every ideal $\mathscr{A}$ we let

$$\Theta_{\mathscr{A}}(z) = \frac{1}{2}\sum_{\mu\in\mathscr{A}} q^{\mathrm{N}(\mu)/\mathrm{N}(\mathscr{A})}, \qquad (\mathscr{I}(z) > 0, q = e^{2\pi i z}), \tag{6}$$

which is a modular form on $\Gamma_0(d)$ of weight one and character ε. Applying the differential operator

$$\partial^{k-1} = \partial_{2k-3}\circ\cdots\circ\partial_1, \tag{7}$$

where

$$\partial_r = \frac{1}{2\pi i}\frac{d}{dz} - \frac{r}{4\pi y} \qquad (z = x + iy, r \in \mathbf{R}),$$

to $\Theta_{\mathscr{A}}$, we get a modular form on the same group of weight $2k - 1$ and same character ε. The first formula we need is the first in (26) of [8], namely

$$L(\psi, k) = (-1)^\delta\frac{(2\pi/\sqrt{d})^k}{(k-1)!}\sum_{[\mathscr{A}],[\mathscr{A}']\in Cl}\overline{\psi(\mathscr{A})}^{-1}\partial^{k-1}\Theta_{\mathscr{A}'}(z_{\mathscr{A}}^{(d)}), \tag{8}$$

where

$$z_{\mathscr{A}}^{(d)} = \frac{b+\sqrt{-d}}{2ad}, \qquad \text{with } \mathscr{A} = [a, (b+\sqrt{-d})/2], \gcd(a, d) = 1, b \equiv 0 \bmod d. \tag{9}$$

(This number is in the upper half-plane and is determined modulo $d\mathbf{Z}$.)

Consider also the theta series

$$\theta_{p+1/2}(z) = \frac{1}{(2\pi y)^{p/2}}\sum_{\substack{n\geq 1\\ n \text{ odd}}}\left(\frac{-4}{n}\right)^p H_p(n\sqrt{\pi y/2})e^{\pi i n^2 z/4}, \qquad \mathfrak{I}(z) > 0, \tag{10}$$

where

$$H_p(x) = \sum_{0\leq j\leq p/2}\frac{p!}{j!(p-2j)!}(-1)^j(2x)^{p-2j}$$

is the pth Hermite polynomial. To be able to evaluate these series on CM points, we fix the choice, analogous to (9),

$$z^{(2)}_{\mathscr{A}} = \frac{b+\sqrt{-d}}{2a}, \qquad \text{with } \mathscr{A} = [a, (b+\sqrt{-d})/2],\ \gcd(a,2)=1,\ b \equiv 1 \bmod 16, \tag{11}$$

which is well defined modulo $8\mathbf{Z}$.

The second formula we need is the factorization formula (25) of [8] (with $h = 1$).

THEOREM. *Let $\mathscr{A}$, $\overline{\mathscr{A}_1}$ be two primitive coprime ideals prime to $2d$ and 2 respectively. Then for every positive integer k with $k \equiv 1 + \delta \bmod 2$, we have*

$$\partial^{k-1}\Theta_{\mathscr{A}\mathscr{A}_1}(z^{(d)}_{\mathscr{A}}) = (-1)^{\delta}\left(\frac{-4}{\mathbf{N}(\mathscr{A})}\right)^{\delta} \frac{d^{k-3/4}}{2^{2k-3}\mathbf{N}(\mathscr{A}_1)^{k-1/2}}\theta_{k-1/2}(z^{(2)}_{\mathscr{A}_1\mathscr{A}^2})\overline{\theta_{k-1/2}(z^{(2)}_{\mathscr{A}_1})}, \tag{12}$$

where Θ, θ, and ∂^{k-1} are defined in (6), (7), and (10) and $z^{(d)}_{\mathscr{A}}$ and $z^{(2)}_{\mathscr{A}}$ in (9) and (11) respectively.

Finally, we prove the following result, which is a direct consequence of a lemma of Gross and Zagier [2, p. 274].

PROPOSITION B. *Let $\mathscr{A}_0$, $\mathscr{A}$, $\mathscr{A}'$ be three primitive ideals prime to $(\sqrt{-d})$ with $\mathscr{A}$, $\overline{\mathscr{A}_0}$ relatively prime and $[\mathscr{A}_0] \in Cl_{(2)}$. Let ψ be a character of weight $2k-1$ as in (1). Then*

$$\overline{\psi(\mathscr{A}_0)}^{-1}\partial^{k-1}\Theta_{\mathscr{A}_0\mathscr{A}'}(z^{(d)}_{\mathscr{A}_0\mathscr{A}}) = \chi_{[\mathscr{A}_0]}([\mathscr{A}\mathscr{A}'])\cdot\chi_{\psi}([\mathscr{A}_0])\cdot\partial^{k-1}\Theta_{\mathscr{A}'}(z^{(d)}_{\mathscr{A}}),$$

where $\chi_{[*]}$, χ_{ψ}, $\Theta_{\mathscr{A}}$, ∂ are defined in (2), (3), (6), and (7), respectively.

Proof. Let $d = d_1\cdot d_2$, with $d_1, d_2 > 0$, relatively prime, $d_1 \equiv 1 \bmod 4$, and $d_2 \equiv 3 \bmod 4$, be the factorization associated to $[\mathscr{A}_0]$ as in Section 3. We let $\mathscr{D}_1$, $\mathscr{D}_2$ be the corresponding ideals dividing d_1 and d_2 respectively. Choose $b \in \mathbf{Z}$ such that $b \equiv 0 \bmod d$ and $(1/2)(b+\sqrt{-d}) \in \mathscr{A}_0 \cap \mathscr{A}$. Since $[\mathscr{A}_0] = [\mathscr{D}_1]$, there is a $\mu \in \mathscr{O}_K$ such that $(\mu)\mathscr{A} = \mathscr{A}\mathscr{A}_0\mathscr{D}_1$; notice that (μ) is primitive. It follows that there is a $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in Sl_2(\mathbf{Z})$, such that

$$\begin{cases}\frac{1}{2}(b+\sqrt{-d}) = \alpha\mu\frac{1}{2}(b+\sqrt{-d}) + \beta\mu a\\ a_0 a d_1 = \gamma\mu\frac{1}{2}(b+\sqrt{-d}) + \delta\mu a,\end{cases} \tag{13}$$

where $a = \mathbf{N}(\mathscr{A})$. Since d_1, d_2 are relatively prime, $d_2 | \beta$. We find that

$$\frac{b+\sqrt{-d}}{2a_0ad} = \begin{pmatrix}\alpha & \beta/d_2\\ \gamma d_2 & \delta\end{pmatrix}\circ\frac{b+\sqrt{-d}}{2ad_2},$$

where $\circ$ denotes the standard action of $Sl_2(\mathbf{Z})$ on the upper half-plane. Using (13), it is easy to check that $\gcd(\gamma, d_1) = 1$ so that $\gcd(\gamma d_2, d) = d_2$ and $d_1 | \delta$. Let $z = (b + \sqrt{-d})/2ad_2$; then we have

$$\frac{z + \gamma'\delta}{d_1} \equiv z_{\mathscr{A}}^{(d)} \bmod \mathbf{Z} \quad \text{and} \quad \begin{pmatrix} \alpha & \beta/d_2 \\ \gamma d_2 & \delta \end{pmatrix} \circ z \equiv z_{\mathscr{A}_0 \mathscr{A}}^{(d)} \bmod \mathbf{Z},$$

where $\gamma'\gamma \equiv 1 \bmod d_1$. Now the proposition follows from lemma (2.3) of [2, p. 274] for $k = 1$ and by applying ∂^{k-1} to this case in general. □

MAIN THEOREM. *Let k be a positive integer, ψ a character of weight $2k - 1$ as in (1), $\mathscr{G}_\psi$ its associated genus as defined in (5), $\mathscr{A}_1$ a primitive ideal in $\mathscr{G}_\psi$ prime to $(\sqrt{-d})$, and t the number of prime factors of d. Then*

$$L(\psi, k) = c_k \frac{2^{t-1}}{\mathbf{N}(\mathscr{A}_1)^{k-1/2}} \left| \sum_{[\mathscr{A}] \in Cl/Cl_{(2)}} \left(\frac{-4}{\mathbf{N}(\mathscr{A})}\right)^{\delta} \overline{\psi(\mathscr{A})}^{-1} \theta_{k-1/2}(z_{\mathscr{A}_1 \mathscr{A}^2}^{(2)}) \right|^2, \tag{14}$$

where

$$c_k = \frac{\pi^k d^{k/2-3/4}}{2^{k-3}(k-1)!}.$$

Proof. We start with the identity (8) and break the sum according to the genus of $[\mathscr{A}\mathscr{A}']$. Then by Propositions A and B the only nonzero sum is the one for the genus $\mathscr{G}_\psi$ and therefore

$$L(\psi, k) = \frac{(2\pi/\sqrt{d})^k}{(k-1)!}(-1)^\delta \sum_{\substack{[\mathscr{A}],[\mathscr{A}'] \in Cl \\ [\mathscr{A}\mathscr{A}'] \in \mathscr{G}_\psi}} \overline{\psi(\mathscr{A})}^{-1} \partial^{k-1} \Theta_{\mathscr{A}'}(z_{\mathscr{A}}^{(d)}).$$

We change indices in the sum, picking up a factor of $2^{t-1} = \# Cl_{(2)}$, and get

$$L(\psi, k) = 2^{t-1} \frac{(2\pi/\sqrt{d})^k}{(k-1)!}(-1)^\delta \sum_{[\mathscr{A}],[\mathscr{A}'] \in Cl/Cl_{(2)}} \psi(\bar{\mathscr{A}}\mathscr{A}')^{-1} \partial^{k-1} \Theta_{\mathscr{A}_1 \mathscr{A}\mathscr{A}'}(z_{\mathscr{A}\bar{\mathscr{A}'}}^{(d)}).$$

Finally we use the factorization formula (12), again changing indices in the sum, to prove our claim. □

Remarks. (1) Notice that the terms of the sum in the theorem are well defined by Proposition B, the factorization formula (12), and the definition of $\mathscr{G}_\psi$; also the right-hand side is independent of $\mathscr{A}_1$.

(2) It would be nice to have a canonical way of factoring the term $\mathbf{N}(\mathscr{A}_1)^{k-1/2}$; we would need something like "$\psi(\sqrt{\mathscr{A}_1})$" but were unable to define this satisfactorily. This issue is related to a question of Shimura [10, (B), p. 478].

(3) However, we can still make some choice of factorization of $\mathrm{N}(\mathscr{A}_1)^{k-1/2}$ and then use Shimura's reciprocity law to find algebraic and Galois properties of the terms as we did in [7]. Undoubtedly, the factor 2^{t-1} would then match factors corresponding to primes of bad reduction when writing down what the Birch–Swinnerton-Dyer conjecture predicts (see the next section for some numerical examples with $t = 2$).

COROLLARY. (i) *Let k be a positive integer, ψ of weight $2k - 1$ as in (1); then $L(\psi, k)$ is nonnegative.*

(ii) *Moreover, if $k = 1$ or $k = 2$ and 3 does not divide h, then $L(\psi, k) > 0$.*

Proof. The first statement is clear; to prove the second we sum $L(\psi, k)$ over all ψ and obtain

$$\sum_{\psi} L(\psi, k) = c \cdot \sum_{[\mathscr{A}]} \mathrm{N}(\mathscr{A})^{1/2-k} |\theta_{k-1/2}(z_{\mathscr{A}}^{(2)})|^2,$$

where c is a positive constant. First notice that under the assumptions of (ii), the different ψ's are Galois conjugates of each other, and therefore, by a theorem of Shimura [11], one value $L(\psi, k)$ is zero if and only if all of them are zero. On the other hand, $\theta_{k-1/2}(z)$ equals $\eta(2z)^2/\eta(z)$ for $k = 1$ and $\eta(z)^3$ for $k = 2$, and hence it does not vanish on the upper half-plane. This completes the proof of the corollary. □

Remarks. (1) Nonvanishing results for central values of more general L-series are known due to the work Greenberg and Rohrlich (see [1], [9] and their sequel).

(2) The condition that 3 not divide h in (ii) is necessary; otherwise the ψ's will form more than one orbit under Galois and the proof breaks down. Indeed, we found that, for $d = 59$, of class number 3, and ψ a character of weight one as in (1), $L(\psi^3, 2) = 0$. However, this seems to be the only such case for d prime congruent to 3 modulo 8.

5. Numerical examples. Let d be as in Section 2, with $d \equiv 7 \bmod 8$ (so $\delta = 0$) and Cl cyclic of order $h = 2r$, for some $r \geqslant 1$. Let ψ be as in (1), of weight 1 (so $k = 1$), and $\mathscr{P}$ a primitive ideal such that $[\mathscr{P}]$ generates Cl. We then have two genera: the squares $\mathscr{G}_0$ and the nonsquares $\mathscr{G}_1$. We fix two characters ψ_0 and ψ_1 such that $\mathscr{G}_{\psi_s} = \mathscr{G}_s$ for $s = 0, 1$ (see (5) for the definition of $\mathscr{G}_\psi$).

For $s = 0, 1$, we define

$$\mathscr{S}_s = \frac{i^s \prod_{m=0}^{r-1} \sum_{n=0}^{r-1} e^{2\pi i m n/r} \psi_s(\bar{\mathscr{P}})^{-n} \theta_{1/2}(z_{\mathscr{P}^{2n+s}}^{(2)})}{2^{r-1} \prod_{m=0}^{r-1} \psi_s(\bar{\mathscr{P}})^{-m} \theta_{1/2}(z_{\mathscr{P}^{2m+s}}^{(2)})},$$

and $\mathscr{S} = \mathscr{S}_0 \mathscr{S}_1$. These three numbers are real, and our choices define them up to a factor of ± 1.

Finally, we let

$$\Omega = \prod_{[\mathscr{A}] \in Cl} \frac{2\pi}{\sqrt[4]{d}} \frac{|\eta(z_{\mathscr{A}})|^2}{\sqrt{a}},$$

where η is Dedekind's eta function and for a primitive ideal $\mathscr{A} = [a, (b + \sqrt{-d})/2]$ with norm a, $z_{\mathscr{A}} = (b + \sqrt{-d})/2a$. This period can also be given in terms of values of the classical gamma function via the Chowla-Selberg formula

$$\Omega^2 = \left(\frac{2\pi}{d^{3/2}}\right)^h \prod_{n=1}^{d-1} \Gamma(n/d)^{(n/d)}.$$

It is not hard to verify that our main theorem implies the identity

$$\mathscr{S}^2 = \frac{\prod_\psi L(\psi, 1)}{2^{2h-4}\Omega},$$

where the product is taken over all characters ψ.

We have calculated the following examples.

d	15	39	55	95	111	183	295	407	471	559
h	2	4	4	8	8	8	8	16	16	16
$\lvert\mathscr{S}\rvert$	1	1	1	1	1	3	11	$3\cdot 73$	3^2	3^2

To give an idea of what these calculations involve, let us take for example $d = 407$. Let $\mathscr{P} = [13, (-3 + \sqrt{-407})/2]$; then $[\mathscr{P}]$ generates Cl, which is of order 16. We find that $\psi_0(\mathscr{P}^8) = 2489\sqrt{37} - 7302\sqrt{-11}$ (with the standard branch of square root), and hence we choose $\psi_0(\mathscr{P})$ to be some fixed 8th root of this number, and $\psi_1(\mathscr{P}) = e^{2\pi i/16}\psi_0(\mathscr{P})$. Finally, we may take

$$z^{(2)}_{\mathscr{P}j} = \frac{356297316751015 09 + \sqrt{-407}}{2\cdot 13^j} \qquad \text{for } 0 \leqslant j < 16.$$

We now have all the terms, and we can calculates $\mathscr{S}$.

Choosing $z^{(2)}_{\mathscr{P}j}$ all with the same b as we did is certainly not necessary but makes the bookkeeping easier. (b here is just a square root of -407 to order $O(13^{15})$ 13-adically, with $b \equiv -3 \bmod 13$ and $b \equiv 1 \bmod 16$.) In order to calculate $\theta_{1/2}$ at these points efficiently, however, we need a program for $\theta_{1/2}$ that takes into account its transformation formulas. In this way we will always sum a theta series for a z in the standard fundamental domain for $Sl_2(\mathbf{Z})$. (The theta series may be any one of the three weight 1/2 Jacobi theta functions, and the program must keep track of which one.) Also, the numerator in the expressions for $\mathscr{S}_s$ can be calculated using a resultant function, which would give the product without giving each factor.

Appendix

Factorization formulas and intersections. In this appendix we briefly outline an approach to the crucial factorization formula (12) different to the ones appearing in [7] and [8]. This was already mentioned in [8] and is related to intersection of Humbert surfaces in Siegel's threefold.

Let $\mathscr{H}_n$ be the Siegel upper half-plane of genus n and let Γ_n be the corresponding symplectic group over $\mathbf{Z}$. We will only need $n = 1$ and $n = 2$. We want to describe certain subvarieties of $\mathscr{H}_2/\Gamma_2$ first considered by Humbert [5]. Related curves on Hilbert modular surfaces have been studied extensively by Hirzebruch and Zagier [4], and this has been generalized to orthogonal groups of signature (p, q) by Kudla and Millson [6].

We will consider two simple examples of these varieties. First, the surface obtained by the diagonal map

$$\mathscr{H}_1 \times \mathscr{H}_1 \to \mathscr{H}_2$$

$$z_1, z_2 \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

Second, the curve obtained by the map

$$\mathscr{H}_1 \to \mathscr{H}_2$$

$$z \mapsto z \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where (a, b, c) is an integral positive definite binary form of discriminant $-d$, associated to the ideal $\mathscr{A} = [a, (b + \sqrt{-d})/2]$.

Now consider the theta function

$$\theta^{(n)}(u, Z) = \sum_{m \in \mathbf{N}^n} e^{\pi i m^t Z m} e^{2\pi i m^t u},$$

where $u \in \mathbf{C}^n$ and $Z \in \mathscr{H}_n$.

When we restrict the theta function $\theta^{(2)}$ to the above surface, we clearly get

$$\theta^{(1)}(u_1, z_1) \cdot \theta^{(1)}(u_2, z_2), \tag{15}$$

where $u = (u_1, u_2)$. On the other hand, if we restrict it to the curve and set $u = 0$, we get

$$2\Theta_{\mathscr{A}}(z) = \sum_{m, n \in \mathbf{N}} q^{am^2 + bmn + cn^2}, \qquad q = e^{2\pi i z}. \tag{16}$$

Since we know the action of Γ_2 on θ, it is clear that we will have an identity between the values of the two functions (15) and (16) wherever their corresponding varieties meet modulo Γ_2. This intersection consists in general of a finite number of points and certainly includes all those points for which the factorization formula was already known, giving a more conceptual proof. It turns out, however, that there are more points of intersection, giving more identities related to values of L-series, which we will explore in a later publication.

Corrections to [8]

Section 1. The first formula after (6) should read

$$2\sum_{n=1}^{\infty} a_n^{(k)} \left(\sum_{j=0}^{k-1} \frac{1}{j!}\left(\frac{2\pi n}{7}\right)^j\right) \frac{e^{-2\pi n/7}}{n^k}.$$

Section 2. The first formula which defines the differential operator D should read $D = (1/2\pi i)(d/dz) = q(d/dq)$.

Section 4. The term $(ay)^\alpha$ on the left-hand side of (23) should not be there, and the remark immediately following should read "... the right-hand side of (23) ...".

In formula (25) the right-hand side needs an extra factor of $(-1)^{(h-1)/2}$ and the hypothesis of the Theorem should start with "Let $\mathscr{A}, \bar{\mathscr{A}}_1 \ldots$".

Section 5. The terms in the sum on the right-hand side of the formula right above the Main Theorem should have a factor of $(-4/N(\mathscr{A}))^\delta$; also, here and in (27) the power of 2 on the right-hand side should be 2^{k-3}.

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