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ON THE ZEROS OF CERTAIN POLYNOMIALS

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ABSTRACT. We prove that certain naturally arising polynomials have all of their roots on a vertical line.

1. The statement

Given a rational function of the form

$$P(t) = \frac{U(t)}{(1-t)^d}\ , \qquad U(1) \neq 0\ , \tag{1}$$

where $d \in \mathbb{N}$ and $U \in \mathbb{C}[x]$ is of degree e, there is a polynomial $H \in \mathbb{C}[x]$ of degree $d-1$ such that if

$$P(t) = \sum_{n=0}^{\infty} h_n t^n, \tag{2}$$

then

$$h_n = H(n), \qquad \text{for all } \ n \geq \max\{0, e-d+1\}\ .$$

For $a \in \mathbb{Z}$ let

$$h_a(x) = \begin{cases} (x+1)(x+2)\cdots(x+a), & a \geq 1, \\ 1, & a < 1, \end{cases} \tag{3}$$

and

(4)

$$S_a = \{p \in \mathbb{C}[x] \setminus \{0\} \,|\, p = h_a v,\ v \in \mathbb{C}[x] \text{ has all its roots on } \Re(x) = -(a+1)/2\}$$

(this includes the case that $v \in \mathbb{C}^*$).

The purpose of this note is to prove the following result.

Theorem. *Let the notation be as above. Suppose that all the roots of U are on the unit circle. Then $H \in S_{d-1-e}$.*

Remarks. 1) Borrowing terminology from commutative algebra, we will call P the *Poincaré series* and H the *Hilbert polynomial*. What follows owes much to the work of Stanley [St], [St1], [PS] on the interplay between commutative algebra and combinatorics.

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The Poincaré series of a finitely generated graded algebra A over a field k is defined as

$$P_A(t) = \sum_{n=0}^{\infty} \dim_k(A_n) t^n,$$

where the A_n consist of the elements of A of degree n. If A is standard, i.e. generated by A_1, with $A_0 = k$, of Krull dimension d, and a complete intersection, then

$$P_A(t) = \frac{\prod_{j=1}^{s}(1 + t + \cdots + t^{n_j})}{(1-t)^d}$$

for some positive integers $n_1, \ldots, n_s$ [St, 3.4]. The theorem implies that the corresponding Hilbert polynomial H_A is in S_{d-1-e}, where $e = n_1 + \cdots + n_s$.

2) As pointed out by the referee, if $P_A(t) = U(t)/(1-t)^d$ is the Poincaré series of a finitely-generated connected standard graded k-algebra such that all zeros of $U(t)$ are on the unit circle, then these zeros ζ are roots of unity and U is a product of cyclotomic polynomials. Indeed, since A is connected, $h_0 = 1$ and hence $1/\zeta$ is an algebraic integer, but $|\zeta| = 1$ and the result follows.

3) Our original motivation to study the roots of Hilbert polynomials arose from reading [BChKV]. Among other things they prove (Theorem 6) the following facts about the Ehrhart polynomial of the n-dimensional octahedron. Let

$$f_n(s) = \sum_{k=0}^{n} 2^k \binom{n}{n-k} \binom{-s}{k},$$

so that $f_n(-m)$ equals the number of integral points $(x_1, \ldots, x_n)$ satisfying $\sum_j |x_j| \leq m$. Then

$$f_n(s) = (-1)^n f_n(1-s)$$

and all of its roots are on the vertical line $\mathrm{Re}\,(s) = 1/2$. Their proof of this last fact uses the orthogonality of the functions $f_n(1/2 + it)$ with respect to a certain positive measure. It is also a consequence of our theorem, as it is not hard to check that the Hilbert polynomial corresponding to $P(t) = (1+t)^n/(1-t)^{n+1}$ is given by $f(-1-x)$. We have not seen a way to extend their approach to all the cases covered by our theorem.

4) Though the particular statement in the above theorem appears not to have been noticed before, results of this nature (with analogous proofs) go back a long while (see Lemma 9.13 of [PS1], problem 196.1 of [PS], and [O]).

5) The converse of the theorem is visibly false; adding an arbitrary polynomial of the form $(1-t)^d F(t)$ to $U(t)$ does not change H, but clearly destroys the property of having all zeros on the unit circle. The converse is not even true if $e < d$ since, for example, $U = t^3 + 23t^2 + 23t + 1$ does not have all of its roots on the unit circle whereas the Hilbert polynomial for $P = U(t)/(1-t)^4$ is $H = (2x+1)^3 \in S_0$.

2. The proof

We will use the following lemma (compare with [PS1], Lemma 9.13).

Lemma. *Let $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, and $f \in S_a$ for some $a \in \mathbb{Z}$. Then*

$$f(x-1) - \alpha f(x) \in S_{a-1}. \tag{5}$$

Proof. Note that h_{a-1} divides both $f(x)$ and $f(x-1)$. Let

$$f(x) = h_{a-1}(x)r(x), \quad f(x-1) = h_{a-1}(x)s(x), \quad g(x) = s(x) - \alpha r(x) \ . \tag{6}$$

We need to prove that g has all its roots on the line $\Re(x) = -a/2$. Let $\beta \in \mathbb{C}$ be a root of g. Then

$$|r(\beta)| = |s(\beta)| \ . \tag{7}$$

Every root μ of r lies on the left half plane $\Re(x) < -a/2$ and (counted with the appropriate multiplicity) corresponds to a unique root ν of s located symmetrically on the the right half plane $\Re(x) > -a/2$. Conversely, every root ν of s lies on $\Re(x) > -a/2$ and corresponds to a unique root μ of r on $\Re(x) < -a/2$.

It follows that if $\Re(\beta) > -a/2$ say, then

$$|\beta - \mu| < |\beta - \nu| \ , \tag{8}$$

which on taking the product over all roots contradicts (7) since clearly r and s have the same leading coefficient. An analogous argument shows that we cannot have $\Re(\beta) < -a/2$ either, proving the lemma. □

Proof of the Theorem. We proceed by induction on e. For $e = 1$ it easy to check that $H(x) = \binom{x+d-1}{d-1}$, which belongs to S_{d-1}. Now if $H \in S_{d-1-e}$ is the Hilbert polynomial corresponding to U, then a simple calculation shows that the Hilbert polynomial corresponding to $(t-\alpha)U(t)$ is $H_1(x) = H(x-1) - \alpha H(x)$. By hypothesis $|\alpha| = 1$, and hence the lemma implies that $H_1 \in S_{d-e-2}$, and the theorem is proved. □

3. Functional equations

We recall a special case of a theorem of Popoviciu ([St, 4.7]).

Proposition. *Let the notation be as in §1 and assume $e < d$. Then*

$$P(1/t) = (-1)^d t^{d-e} P(t) \tag{9}$$

if and only if

$$H(-1) = \cdots = H(-d+e+1) = 0 \tag{10}$$

and

$$H(x) = (-1)^{d-1} H(-d+e-x) \ . \tag{11}$$

Remark. If P is the Poincaré series of a Gorenstein graded algebra A of Krull dimension d, then P satisfies (9) [St, 4.1]. It is a non-trivial result of Stanley that the converse (i.e. that if the Poincaré series P of an algebra A satisfies (9), then A is Gorenstein) is true if A is a Cohen-Macaulay integral domain [St, 4.4].

We record the following special case of the proposition combined with the theorem of §1 for future reference.

Corollary. *Assume $e < d$, U has all of its roots in the unit circle, and $U \in \mathbb{R}[t]$. Then H satisfies* (10) *and* (11) *and all roots of H, except $x = -1, -2, \ldots, -d+e-1$ listed in* (10)*, are on the vertical line $\Re(x) = -(d-e-1)/2$, the line of symmetry of the functional equation* (11).

4. Some speculations

It is hard to resist the temptation to speculate about the possible connections of the above discussion with L-functions. Let U be of degree $e = d - 1$, with real coefficients and such that all of its roots are on the unit circle but $U(1) \neq 0$. Let H be the corresponding Hilbert polynomial and let

$$z(s) = H(-s)\ . \tag{12}$$

Then

$$z(1-s) = (-1)^{d-1} z(s) \tag{13}$$

and all of its roots satisfy $\Re(s) = 1/2$. Of course, this is what we would like to be able to prove for the Riemann zeta function $\zeta(s)$.

We may ask rather vaguely whether it is possible to interpret, in some sense, $\zeta(-s)$ as the Hilbert function of some natural infinite dimensional (Gorenstein?) graded algebra; perhaps an algebra associated to some infinite dimensional reflexive polytope?, or constructed via Lichtenbaum's conjecture?, or via the work of Harer and Zagier [HZ]? What would be the appropriate Poincaré series?

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