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Hypergeometric families of Calabi-Yau manifolds

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Hypergeometric Weight Systems

By a *hypergeometric weight system* we will mean a formal linear combination

$$\gamma = \sum_{\nu \geq 1} \gamma_\nu [\nu], \tag{0.1}$$

where $\gamma_\nu \in \mathbf{Z}$ are zero for all but finitely many ν, satisfying the following two conditions

$$\begin{aligned} &(i) \qquad \sum_{\nu \geq 1} \nu \gamma_\nu = 0 \\ &(ii) \qquad d = d(\gamma) := -\sum_{\nu \geq 1} \gamma_\nu > 0 \end{aligned} \tag{0.2}$$

We denote by Γ the set of all such γ. Note that Γ is a cone, i.e. if $\gamma, \gamma' \in \Gamma$ then $\gamma + \gamma' \in \Gamma$. We call d the *dimension* of the weight system γ.

To $\gamma \in \Gamma$ we associate the hypergeometric function

$$u(\lambda) := \sum_{n \geq 0} u_n \lambda^n \tag{0.3}$$

where

$$u_n = \prod_{\nu \geq 1} (\nu n)!^{\gamma_\nu} .$$

It is easy to check that for some minimal r we have

$$u(\lambda) = {}_rF_{r-1} \left(\begin{matrix} \alpha_1 & \cdots & \alpha_r \\ \beta_1 & \cdots & \beta_r \end{matrix} \middle| \begin{matrix} \lambda \\ \lambda_0 \end{matrix} \right) ,$$

where

$$\lambda_0^{-1} := \prod_{\nu \geq 1} \nu^{\nu \gamma_\nu} ,$$

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and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_r < 1$ and $0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_r < 1$ are two sets of r rational numbers. We call $\lambda_0 = \lambda_0(\gamma)$ and $r = r(\gamma)$, respectively, the *special point* and the *rank* of γ.

The condition 0.2 (i) precisely guarantees that the linear differential equation (of order r) satisfied by u has only regular singularities.

By Stirling

$$u_n \sim \frac{\sqrt{D}}{(2\pi n)^{d/2}}\ \lambda_0^{-n}$$

where

$$D := \prod_{\nu \ge 1} \nu^{\gamma_\nu}\ .$$

It will be convenient to introduce

$$P(t) := \prod_{\nu \ge 1} (1 - t^\nu)^{\gamma_\nu}$$

the *Poincaré series* of γ.

Note that

$$d = \text{order of pole of } P \text{ at } t = 1 \tag{0.4}$$

and

$$D = \prod_{\nu \ge 1} \left(\frac{1 - t^\nu}{1 - t}\right)^{\gamma_\nu}\Bigg|_{t=1} = (-1)^d\ \left[(t-1)^d P(t)\right]\big|_{t=1}\,,$$

is $(-1)^d$ times its leading coefficient at $t = 1$. If we write

$$P(t) = (-1)^d \frac{A(t)}{B(t)}$$

in lowest terms, with $A, B \in \mathbf{Z}[t]$ monic and relatively prime then $r = \deg A = \deg B$ and

$$A(t) = \prod_{j=1}^{r} (t - e^{2\pi i \alpha_j}), \qquad B(t) = \prod_{j=1}^{r} (t - e^{2\pi i \beta_j}),$$

with the $\alpha' s$ and $\beta' s$ as above.

1 Integrality

Our goal is to obtain $\gamma \in \Gamma$ such that the truncation of u

$$\sum_{n=0}^{p-1} u_n \lambda^n \bmod p \tag{1.1}$$

for a prime p is related to the number of points over $\mathbf{F}_p$ of some family of varieties X_λ. In order to even make sense of this expression we need some integrality assumption on u_n. The simplest such assumption would be

$$u_n \in \mathbf{Z} \qquad \text{for all } n = 0, 1, 2, \ldots \tag{1.2}$$

and as we will see below this is in fact necessary if we want the truncation of u modulo p for all sufficiently large primes p.

Let us denote by $\Gamma_{\text{int}} \subset \Gamma$ all weight systems satisfying (1.2), which we will call *integral*; they clearly form a sub-cone of Γ. There is an obvious collection of elements in Γ_{int}, namely, the *multinomials*

$$\gamma = [w_0 + \cdots + w_d] - [w_0] - \cdots - [w_d], \qquad w_0, \cdots, w_d \in \mathbf{N} \ .$$

Let Γ_{mon} be the cone spanned by the multinomials. We have then

$$\Gamma_{\text{mon}} \subset \Gamma_{\text{int}} \ .$$

It is interesting and perhaps at first surprising that these two cones are not in fact the same. A classical example

$$\gamma = [30] + [1] - [6] - [10] - [15] \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$$

was used by Chebychev in his work on the distribution of prime numbers.

There is a beautiful criterion due to Landau [8] for checking whether $\gamma \in \Gamma_{\text{int}}$. Let

$$\mathcal{L}(x) = \mathcal{L}_\gamma(x) := -\sum_{\nu \geq 1} \gamma_\nu \{\nu x\}, \qquad x \in \mathbf{R}$$

where $\{x\}$ denotes the fractional part of the real number x. This *Landau function* is periodic of period 1.

Proposition 1 *We have*

$$u_n \in \mathbf{Z} \qquad \textit{for all } n = 0, 1, 2, \cdots$$

if and only if

$$\mathcal{L}(x) \geq 0 \qquad \textit{for all } x \in \mathbf{R} \ .$$

Proof Fix a prime p and let v_p denote the corresponding valuation. By the well know formula for $v_p(n!)$ we have

$$v_p(u_n) = \sum_{\nu \geq 1} \gamma_\nu \sum_{k \geq 1} \left[\frac{\nu n}{p^k} \right] .$$

Combining $x = [x] + \{x\}$ with 0.2 (i) we find that

$$v_p(u_n) = \sum_{k \geq 1} \mathcal{L}\left(\frac{n}{p^k} \right) . \tag{1.3}$$

It follows that if $\mathcal{L}(x) \geq 0$ then $v_p(u_n) \geq 0$ for all p and hence $u_n \in \mathbf{Z}$.

Foor some $\delta > 0$ we have that $\mathcal{L}(x) = 0$ for $x \in [0, \delta)$ (proposition 3) ; hence, for all primes p such that $p\delta > 1$, $v_p(u_n) = \mathcal{L}(n/p)$ for $n < p$. Since $\mathcal{L}$ is right continuous it follows that $u_n \in \mathbf{Z}$ implies the non-negativity of $\mathcal{L}$. □

We can be a bit more precise about the way in which the u_n may fail to be integral.

Proposition 2 *If γ is not integral then for all p sufficiently large there exists an $0 \leq n < p$ such $v_p(u_n) < 0$.*

Proof If γ is not integral the by the previous proposition $\mathcal{L}$ is negative at some point. Since $\mathcal{L}$ is locally constant (proposition 3) it follows that $\mathcal{L}$ is negative in some interval of length say $\alpha > 0$. As in the proof of the previous proposition, for all $p\delta > 1$ we have $v_p(u_n) = \mathcal{L}(n/p)$ for $0 \leq n < p$. It follows that for all p with $p\delta > 1$ and $p\alpha > 1$ we have $v_p(u_n) < 0$ for some $0 \leq n < p$. □

We summarize in the next proposition a number of simple properties of $\mathcal{L}$; we leave the proofs to the reader (more details will appear in [10]). We call the *support* $\mathcal{N}$ of $\gamma \in \Gamma$ those $\nu \in \mathbf{N}$ for which $\gamma_\nu \neq 0$. For $z \in \mathbf{C}$ with $|z| < 1$ we let $\log(1-z)$ be the standard branch of the logarithm vanishing at $z = 0$ and define

$$\log(P(z)) = \sum_{\nu \geq 1} \gamma_\nu \log(1 - z^\nu)$$

Proposition 3 1. *The regularity condition 0.2 (i) is equivalent to $\mathcal{L}$ being locally constant.*
2. *For all x*

$$\mathcal{L}(x) = \frac{d}{2} - \lim_{t \to 1^-} \tfrac{1}{\pi} \Im \left[\log P(te^{2\pi i x}) \right].$$

3. *$\mathcal{L}$ is right continuous with discontinuity points exactly at $x \equiv \alpha_j \bmod 1$ or $x \equiv \beta_j \bmod 1$ for some $j = 1, \ldots, r$.*
4. *More precisely,*

$$\mathcal{L}(x) = \#\{j \mid \alpha_j \leq x\} - \#\{j \mid 0 < \beta_j \leq x\}.$$

5. *$\mathcal{L}$ takes only integer values.*
6.

$$\int_0^1 \mathcal{L}(x)\, dx = \tfrac{1}{2} d, \qquad \lim_{x \to 1^-} \mathcal{L}(x) = d, \qquad \lim_{x \to 0^+} \mathcal{L}(x) = 0. \tag{1.4}$$

In particular, for a general non-zero $\gamma = \sum_{\nu \geq 1} \gamma_\nu [\nu]$, the conditions 0.2 (i) and $\mathcal{L}(x) \geq 0$ imply 0.2 (ii); i.e., integrality implies positive dimension.
7. *If we fix the support $\mathcal{N}$ of $\gamma \in \Gamma$ the finitely many conditions*

$$\mathcal{L}\left(\frac{k}{N}\right) \geq 0, \qquad k = 0, 1, \ldots, N-1, \tag{1.5}$$

where N is the lcm *of the numbers in $\mathcal{N}$, is equivalent to $\gamma \in \Gamma_{\text{int}}$.*
8. *Away from the discontinuity points of $\mathcal{L}$ we have*

$$\mathcal{L}(-x) = d - \mathcal{L}(x) \tag{1.6}$$

and, in particular, for all x

$$\mathcal{L}(x) \leq d, \qquad \textit{if } \gamma \in \Gamma_{\text{int}}. \tag{1.7}$$

1.1 Examples of $\gamma \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$. (i) A computer search reveals some simple examples of $\gamma \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$

$$[6] + [1] - 2[2] - [3], \qquad [9] + [6] - [1] - [3] - [4] - [7], \qquad \text{etc.}$$

Note that by (1.5) integrality can be checked in finite time; in fact, if we fix the support $\mathcal{N}$ of γ the condition $\gamma \in \Gamma_{\text{int}}$ is defined by finitely many inequalities on the γ_ν for $\nu \in \mathcal{N}$. We can solve this system completely using, for example, the computer package PORTA `www.zib.de/Optimization/Software/Porta`.

Here is a list of 22 generators of the cone of integral γ with support $\mathcal{N} = 1, 2, 3, 5, 6, 10, 15, 30$.

1	2	3	5	6	10	15	30
0	1	−2	−1	−1	0	1	0
0	0	−1	0	−2	0	1	0
1	0	−2	−2	0	0	1	0
−1	0	1	0	−2	1	−2	1
−2	0	1	1	−1	0	−2	1
0	0	0	0	0	0	−2	1
0	0	0	−2	0	1	0	0
0	0	−2	0	1	0	0	0
−2	1	0	0	0	0	0	0
2	−2	−1	−2	0	0	1	0
0	0	0	−1	0	−1	1	0
0	−1	−1	1	0	0	0	0
−2	0	1	2	−1	−2	−1	1
−1	0	0	−1	1	0	0	0
−1	−1	1	0	0	0	0	0
0	0	1	0	−3	0	−1	1
1	−2	−1	0	1	0	0	0
0	0	0	1	0	−2	−1	1
1	0	0	0	−1	−1	−1	1
0	−1	1	−1	−1	1	0	0
−1	1	0	0	−1	−1	1	0
−1	1	0	−1	−1	1	0	0

(The fourth vector from the bottom is Chebychev's example.)

(ii) We may also exhibit an infinite sequence $\gamma(n) \in \Gamma_{\text{int}} \setminus \Gamma_{\text{mon}}$ for $n = 3, 4, \dots$ of dimension $n - 2$. Define recursively

$$\begin{aligned} p_1 :=& 2 \\ p_k :=& p_{k-1} \cdots p_2 p_1 + 1, \qquad k = 1, \cdots, n-1 \\ p_n :=& p_{n-1} \cdots p_2 p_1 - 1 \end{aligned}$$

and set

$$\gamma(n) := [p_1 \cdots p_n] + [1] - \sum_{j=1}^{n} [p_1 \cdots \overline{p_j} \cdots p_n],$$

where a bar indicates that that argument should be omitted. For $n = 3$ we have $p_1 = 2, p_2 = 3, p_3 = 5$ so that $\gamma(3)$ is Chebychev's example.

It is easy to show by induction that

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1 + \frac{1}{p_1 \cdots p_n} \tag{1.8}$$

It is clear that $\gamma(n)$ is not in $\Gamma_{\rm mon}$; to check that indeed $\gamma(n) \in \Gamma_{\rm int}$ we use Landau's criterion. By (1.8) we have

$$\mathcal{L}\left(\frac{k}{p_1\cdots p_n}\right) = k - \sum_{j=1}^{n}\left[\frac{k}{p_j}\right] \tag{1.9}$$

We need only check that the value of $\mathcal{L}$ in (1.9) is non-negative for all $0 < k < p_1 \cdots p_n$. First note that we cannot have $\{\frac{k}{p_j}\} = 0$ for all $j = 1, 2, \ldots, n$. Otherwise, k would be divisible by p_j for all $j = 1, 2, \ldots, n$ and therefore divisible by their product $p_1 \cdots p_n$, since the p_j's are clearly pairwise relatively prime, contradicting that $k < p_1 \cdots p_n$. Say $1 \le l \le n$ is such that $\{\frac{k}{p_l}\} > 0$. Then by (1.9) and (1.8)

$$\mathcal{L}\left(\frac{k}{p_1\cdots p_n}\right) \ge k - \sum_{j=1}^{n}\frac{k}{p_j} + \frac{1}{p_l} = \frac{1}{p_l} - \frac{1}{p_1\cdots p_n} \ge 0$$

(iii) We may use the following theorem of Eisenstein (see [7] for a modern treatment and further references). If

$$f(t) = \sum_{n\ge 0} f_n t^n \in \mathbf{Q}[[t]]$$

is the Taylor expansion of an algebraic function then there exist an $N \in \mathbf{N}$ such that $N^n f_n \in \mathbf{Z}$ for all $n = 0, 1, \ldots$

When is a hypergeometric function algebraic? It is possible to describe the hypergeometric differential equations for which all of its solutions are algebraic. Schwartz famously did this for the case of rank $r = 2$. More recently Beukers and Heckman [4] did the general case. Scanning their list we find a number of examples of hypergeometric functions of the type (0.3) we are considering (these correspond to the monodromy group being defined over $\mathbf{Q}$), including, once again, Chebychev's case; in fact, we discover that for Chebychev's example the monodromy group of the corresponding differential equation is the Weyl group of the E_8 lattice!

We will return to the algebraic hypergeometric functions in a later publication [10] but let us mention one of the results. A general converse of Eisenstein's theorem is blatantly false; nevertheless, we have the following.

Theorem 1 *Let $\gamma \in \Gamma$ be a hypergeometric weight system. Then the associated hypergeometric function (0.3) is algebraic if and only if γ is integral of dimension $d = 1$.*

2 Cases with dimension equal rank

By (0.4) we have

$$d \le r.$$

We will consider the $\gamma \in \Gamma$ were equality holds, which precisely means that $\lambda = 0$ is a point of maximal unipotency for the hypergeometric equation satisfied by u.

If $d = r$ then

$$P(t) = \frac{A(t)}{(t-1)^d} \tag{2.1}$$

where

$$A(t) = \prod_{n\ge 2} \Phi_n(t)^{e_n}$$

with Φ_n the n-th cyclotomic polynomial and e_n non-negative integers zero for all but finitely many n.

It follows that

$$d = r = \sum_{n \geq 2} e_n \phi(n) \tag{2.2}$$

where ϕ is Euler's phi-function.

On the other hand we have

$$\Phi_n(t) = \prod_{m|n} (t^m - 1)^{\mu(n/m)},$$

where μ is the Möbius function, and therefore any solution to (2.2) gives rise via (2.1) to the Poincaré series of some $\gamma \in \Gamma$ with $d = r$.

It follows that the $\gamma \in \Gamma$ with $d = r$ form a cone Γ_{uni} generated by

$$\phi(n)[1] - \sum_{m|n} \mu\left(\frac{n}{m}\right)[m], \qquad n = 2, \cdots$$

It easy to verify using Landau's criterion (proposition 3, 4.) that $\Gamma_{\text{uni}} \subset \Gamma_{\text{int}}$.

Since there are only finitely many n's with $\phi(n)$ less than a given bound we see that there are finitely many $\gamma \in \Gamma_{\text{uni}}$ of fixed dimension. For small d these are easy to enumerate. For example, $n = 2, 3, 4, 5, 6, 8, 10, 12$ are all $n > 1$ with $\phi(n) \leq 4$, the respective values of ϕ being $1, 2, 2, 4, 2, 4, 4, 4$. We now describe all cases with $d = r \leq 4$

For $\underline{d = 1}$ we only have one case

$$\gamma = [2] - 2[1], \qquad P(t) = \frac{t+1}{t-1} = \frac{t^2-1}{(t-1)^2}$$

corresponding to

$$u_n = \binom{2n}{n}, \qquad u(\lambda) = (1 - 4\lambda)^{-\frac{1}{2}}.$$

For $\underline{d = 2}$ we obtain

γ	a	b	λ_0^{-1}
$2[2] - 4[1]$	$1/2$	$1/2$	16
$[3] - 3[1]$	$1/3$	$2/3$	27
$[4] - [2] - 2[1]$	$1/4$	$3/4$	4
$[6] - [3] - [2] - [1]$	$1/6$	$5/6$	432

where

$$u(\lambda) = {}_2F_1\left(\begin{matrix} a & b \\ 1 & 1 \end{matrix} \middle| \frac{\lambda}{\lambda_0}\right)$$

Note that since $\phi(n)$ is even for all $n > 2$ any solution of (2.2) for d odd arises from a solution for $d - 1$ by adding an extra $1 = \phi(2)$. In terms of the weight systems, if γ has $d = r$ odd then

$$\gamma = \gamma_0 + [2] - 2[1],$$

where $\gamma_0 \in \Gamma_{\text{uni}}$ has dimension and rank $d - 1$. In particular, we get a description of all $\gamma \in \Gamma_{\text{uni}}$ with $d = r = 3$ from those with $d = r = 2$.

There are 14 cases with $\underline{d = 4}$ and these are listed in the first column of the table at the end of the paper.

As it happens all cases of $\gamma \in \Gamma_{\text{uni}}$ with $d \leq 4$ are also in Γ_{mon} (this is not true for general d; for example, $[30]+[5]+[3]+[2]-[15]-10]-[6]-9[1]$ is integral with $d=r=8$ but is not monomial). We can associate to them a one-parameter family of CY hypersurfaces in a weighted projective space of dimension d with $u(\lambda)$ as one of its periods. More precisely, for $d=2,3,4$ we obtain families of elliptic curves, K3 surfaces, and CY threefolds respectively. The 14 families of CY threefolds, except for the last in the table, are discussed in a paper of Batyrev and Straten [2].

3 The case of threefolds

Let X_λ be one of the families of threefolds associated via toric geometry to a $\gamma \in \Gamma_{\text{uni}}$ of dimension 4. Corresponding to the Picard-Fuchs equation for u there is a factor $R_0(t)$ (see [6] for more details) of the numerator of the zeta function of X_λ. For $\lambda \neq \lambda_0$ this factor is of degree 4 but at the special point $\lambda = \lambda_0$, where X_λ becomes singular, we have

$$R_0(t) = (1 - \left(\frac{D}{p}\right) pt)(1 - a_p t + p^3 t^2).$$

One can verify, the same way that Schoen [11] did it for the standard quintic $[5]-5[1]$ using powerful results of Faltings and Serre, that a_p is the p-th coefficient of a certain modular form of weight 4 for a congruence subgroup of $SL_2(\mathbf{Z})$. (This feature of *rigid CY threefolds* has been discussed by several people, including N. Yui and H. Verrill.)

We identified these modular forms by computing a_p for several p's using the p-adic formulas of [5], [6] and then comparing with the tables of W. Stein `www.math.harvard edu/~was` (Actually, some of the forms had bigger level than those tabulated there and we had to generate the modular forms with an a-priori guess for the level.) The resulting data is tabulated at the end of the paper (N is the level of the modular form, the last columns list a_p for $p = 2, 3, \cdots, 11$).

In the process something interesting emerged: the congruence between the truncation (1.1) of the period u at $\lambda = \lambda_0$ actually appears to hold $\bmod p^3$ rather than just $\bmod p$. This phenomenon of *super-congruence* was first observed by Beukers [3] in connection with the numbers Apéry used in his proof of the irrationality of $\zeta(3)$. There is by now a quite extensive literature on questions of this kind (see for example [1]).

Precisely, we find (numerically) that for all primes p not diving λ_0^{-1}

$$\sum_{n=0}^{p-1} u_n \lambda_0^{-n} \equiv a_p \bmod p^3.$$

The super-congruences also appear to hold for smaller dimensions. For example, for the case $\gamma = 2[2]-4[1]$ with $d=r=2$ we find (again numerically) that for odd p

$$\sum_{n=0}^{p-1} \binom{2n}{n}^2 16^{-n} \equiv \left(\frac{-4}{p}\right) \bmod p^2.$$

These have been now proved by E. Mortenson [9].

γ	λ_0^{-1}	D	N	2	3	5	7	11
$4[2]-8[1]$	2^8	2^4	$8=2^3$	0	-4	-2	24	-44
$[4]+[3]-[2]-5[1]$	$2^6\cdot 3^3$	$2\cdot 3$	$9=3^2$	0	0	0	20	0
$[4]+[2]-6[1]$	2^{10}	2^3	$16=2^4$	0	4	-2	-24	44
$[5]-5[1]$	5^5	5	$25=5^2$	1	7	0	6	-43
$2[3]-6[1]$	3^6	3^2	$27=3^3$	-3	0	-15	-25	15
$2[4]-2[2]-4[1]$	2^{12}	2^2	$32=2^5$	0	-8	-10	-16	40
$[3]+2[2]-7[1]$	$2^4\cdot 3^3$	$2^2\cdot 3$	$36=2^2\cdot 3^2$	0	0	18	8	-36
$[6]+[2]-[3]-5[1]$	$2^8\cdot 3^3$	2^2	$72=2^3\cdot 3^2$	0	0	-14	-24	28
$[6]-[2]-4[1]$	$2^4\cdot 3^6$	3	$108=2^2\cdot 3^3$	0	0	-9	-1	-63
$[8]-[4]-4[1]$	2^{16}	2	$128=2^7$	0	-2	6	-20	-14
$[6]+[4]-[3]-2[2]-3[1]$	$2^{10}\cdot 3^3$	2	$144=2^4\cdot 3^2$	0	0	16	12	-64
$[10]-[5]-[2]-3[1]$	$2^8\cdot 5^5$	1	$200=2^3\cdot 5^2$	0	1	0	-6	-19
$2[6]-2[3]-2[2]-2[1]$	$2^8\cdot 3^6$	1	$216=2^3\cdot 3^3$	0	0	1	-9	-17
$[12]+[2]-[6]-[4]-4[1]$	$2^{12}\cdot 3^6$	1	$864=2^5\cdot 3^3$	0	0	-19	-13	-65

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