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Lattice polytopes, Hecke operators, and the Ehrhart polynomial

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Abstract. Let P be a simple lattice polytope. We define an action of the Hecke operators on $E(P)$, the Ehrhart polynomial of P, and describe their effect on the coefficients of $E(P)$. We also describe how the Brion–Vergne formula for $E(P)$ transforms under the Hecke operators for nonsingular lattice polytopes P.

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1. Introduction

1.1. Let L be a rank n lattice, embedded in a real n-dimensional vector space V. Let $\mathscr{P}(L)$ be the set of n-dimensional convex polytopes in V with vertices in L. For any $P \in \mathscr{P}(L)$, and for any nonnegative integer t, let tP be P scaled by the factor t. Then by a result of Ehrhart [11], the function $t \mapsto \#(tP \cap L)$ is a degree n polynomial with rational coefficients, called the *Ehrhart polynomial* of P. Hence one can think of the Ehrhart polynomial as giving a map E from $\mathscr{P}(L)$ to the polynomial ring $\mathbb{Q}[t]$.

Write $E(P) = \sum_{l=0}^{n} c_l t^l$. Formulas for the coefficients c_l, in various settings and with varying degrees of generality, have been given by several authors [5–7, 10, 14, 16, 20, 21]. Some coefficients are easy to understand, for example

$$c_0 = 1, \quad c_n = \operatorname{Vol} P, \quad \text{and} \quad c_{n-1} = \operatorname{Vol}(\partial P)/2. \tag{1}$$

Here $\operatorname{Vol} P$ is taken with respect to the measure that gives a fundamental domain of L volume 1; if a polytope has dimension less than n, we compute its volume with respect to the lattice obtained by intersecting its affine hull with L. For a general lattice polytope, expressions for the Ehrhart coefficients involve not only volumes but also subtle arithmetic information, namely *higher-dimensional Dedekind sums* as studied by Carlitz and Zagier [8, 23].

1.2. The Ehrhart polynomial depends not just on the combinatorial type of P, but rather on the pair (P, L). Hence it is natural to consider how $E(P)$ changes as L is varied. The theory of automorphic forms provides a powerful machine to accomplish this, namely the technique of *Hecke operators.*

Thus let p be a prime, and let $k \le n$ be a positive integer. Given a lattice polytope P with Ehrhart polynomial $E(P)$, we define a new polynomial $T(p,k)E(P)$ as follows. Let $p^{-1}L$ be the canonical superlattice of L of coindex p^n. We have $p^{-1}L/L \simeq \mathbb{F}_p^n$, and any lattice M satisfying $p^{-1}L \supsetneq M \supsetneq L$ determines a subspace $\overline{M} \subset \mathbb{F}_p^n$. Let $\mathscr{L}_k$ be the set of such lattices with $\dim \overline{M} = k$. Then we define

$$T(p,k)E(P) = \sum_{M \in \mathscr{L}_k} E(P_M), \tag{2}$$

where $P_M \in \mathscr{P}(M)$ denotes the lattice polytope with vertices in M canonically determined by P.

1.3. In this paper we consider the relationship between $T(p,k)E(P)$ and $E(P)$. To state our results, we require more notation. For any nonnegative integer $l \le n$, fix an l-dimensional subspace U of $\mathbb{F}_p^n$, and define

$$\nu_{n,k,l}(p) = \sum_{\substack{W \subset \mathbb{F}_p^n \\ \dim W = k}} p^{\dim W \cap U}. \tag{3}$$

Note that this value is independent of the choice of U. Finally, for any polynomial $f \in \mathbb{Q}[t]$ let $c_l(f)$ be the coefficient of t^l in f. Then our first result can be stated as follows:

Theorem 1.4. *For each triple (n,k,l), there is a polynomial with positive coefficients*

$$\Phi_{n,k,l}(t) \in \mathbb{Z}[t], \tag{4}$$

independent of p, such that $\Phi_{n,k,l}(p) = \nu_{n,k,l}(p)$. Moreover,

$$c_l(T(p,k)E(P)) = \nu_{n,k,l}(p)c_l(E(P)), \tag{5}$$

independently of P. The ratios ν satisfy

$$\nu_{n,k,l}(p)/\nu_{n,n-k,n-l}(p) = p^{k+l-n}.$$

The sum (3) can be viewed as a sum of p-powers over a certain geometrically-defined stratification of the finite Grassmannian $\mathrm{Gr}(k,n)(\mathbb{F}_p)$; however, the existence of Φ, as well as the statement that it has positive coefficients, does not follow immediately from (3) since the number of terms in the sum grows with p and since the strata are only locally closed.

As an example of Theorem 1.4, if $l = 0$, then $c_0(E(P)) = 1$ for any P. Hence the ratio on the left of (5) is the number of terms in (2). It is well known that this is the cardinality of $\mathrm{Gr}(k,n)(\mathbb{F}_p)$ (cf. Lemma 2.5), which equals $\nu_{n,k,0}(p)$. For further examples, Table 1 shows the Hecke eigenvalues that arise for the Ehrhart coefficients of 4-dimensional polytopes.

Table 1. Eigenvalues for $n = 4$

	$T(p,1)$	$T(p,2)$	$T(p,3)$
c_4	$p^4+p^3+p^2+p$	$p^6+p^5+2p^4+p^3+p^2$	$p^6+p^5+p^4+p^3$
c_3	$2p^3+p^2+p$	$p^5+2p^4+2p^3+p^2$	$p^5+p^4+2p^3$
c_2	p^3+2p^2+p	$2p^4+2p^3+2p^2$	$p^4+2p^3+p^2$
c_1	p^3+p^2+2p	$p^4+2p^3+2p^2+p$	$2p^3+p^2+p$
c_0	p^3+p^2+p+1	$p^4+p^3+2p^2+p+1$	p^3+p^2+p+1

1.5. A geometric interpretation of the eigenvalue (3) is the following. Consider the map

$$\mathrm{Vol}_l\colon \mathscr{P}(L) \to \mathbb{Q}$$

taking P to the sum of the volumes of all faces of dimension l. Then we can define an action of the Hecke operators on Vol_l as in (2), and one can show that $T(p,k)\,\mathrm{Vol}_l = \nu_{n,k,l}(p)\,\mathrm{Vol}_l$ (Proposition 2.8). Hence Theorem 1.4 says that the lth coefficient of the Ehrhart polynomial transforms under the Hecke operators exactly as the volumes of the l-dimensional faces do. For another interpretation, in terms of counting the number of $\mathbb{F}_p$-points on certain varieties, see Remark 3.4.

1.6. Recall that an n-dimensional lattice polytope is called *simple* if every vertex meets exactly n edges, and is called *nonsingular* if for any vertex v, the primitive lattice vectors parallel to the edges emanating from v form a $\mathbb{Z}$-basis of L. Our next result concerns how the Hecke operators interact with certain formulas for the coefficients of the Ehrhart polynomial in the special case that P is simple.

Let $\mathscr{F}(n-1)$ be the set of facets of P, and let $h = (h_F)_{F\in\mathscr{F}(n-1)}$ be a real multivariable indexed by the facets of P. Let $P(h)$ be the convex region obtained by parallel translation of the facets of P by the parameter h, normalized by $P(0) = P$ (§4.1). For small h the region $P(h)$ is bounded, and the volume $\mathrm{Vol}\,P(h)$ is a polynomial function of h.

Let Σ be the normal fan to P (§2.2). Then the polytope P determines a differential operator $\mathrm{Td}(\Sigma, \partial/\partial h)$, called the *Todd operator* (§4.7). In the special case that P is nonsingular, this operator is defined as follows. Let $\mathrm{Td}(x)$ be the power series expansion of $x/(1-e^{-x})$, i.e.

$$\mathrm{Td}(x) = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j,$$

where B_j are the Bernoulli numbers. For each h_F let $\mathrm{Td}(\partial/\partial h_F)$ be the differential operator obtained by formally replacing x with $\partial/\partial h_F$ in $\mathrm{Td}(x)$. Then $\mathrm{Td}(\Sigma, \partial/\partial h)$ is defined to be the product

$$\mathrm{Td}(\Sigma, \partial/\partial h) = \prod_{F\in\mathscr{F}(n-1)} \mathrm{Td}(\partial/\partial h_F). \tag{6}$$

Note that product may be taken in any order, since the derivatives mutually commute. This is an infinite-degree differential operator, and we denote by

$$\mathrm{Td}_l(\Sigma, \partial/\partial h)$$

the homogeneous terms of degree l. By Khovanskiĭ–Pukhlikov [21] one has

$$c_{n-l}(E(P)) = \mathrm{Td}_l(\Sigma, \partial/\partial h)\,\mathrm{Vol}\,P(h)\big|_{h=0}.$$

If the polytope P is simple and not nonsingular, then one must enlarge (6) with additional terms involving higher-dimensional Dedekind sums; the corresponding formula is due to Brion–Vergne [5].

1.7. Let f be a face of P of codimension $\leq l$, and let $\pi = (\pi(F))_{F \supset f}$ be an ordered partition of l into positive parts indexed by the facets containing f. The pair (f, π) determines a differential operator

$$\partial_f^\pi = \prod_{F \supset f} (\partial/\partial h_F)^{\pi(F)},$$

and we can collect common terms in (6) to write

$$\mathrm{Td}_l(\Sigma, \partial/\partial h) = \sum_{(f,\pi)} A(f, \pi)\partial_f^\pi. \tag{7}$$

The coefficient $A(f, \pi)$ is rational, and for simple P it is essentially a rank l Dedekind sum. Our next result shows that if P is nonsingular, then these individual terms transform under the Hecke operators exactly as the coefficients of $E(P)$ do:

Theorem 1.8. *Let $P \in \mathscr{P}(L)$ be a nonsingular lattice polytope. For any superlattice $M \supset L$, let f_M be the face f, thought of as a face of P_M. Then for each degree l term $A(f, \pi)\partial_f^\pi \in \mathrm{Td}_l(\Sigma, \partial/\partial h)$ in the Brion–Vergne formula, we have*

$$\sum_{M \in \mathscr{L}_k} A(f_M, \pi)\partial_{f_M}^\pi \,\mathrm{Vol}\,P_M(h)\big|_{h=0} = \nu_{n,k,n-l}(p) A(f, \pi)\partial_f^\pi \,\mathrm{Vol}\,P(h)\big|_{h=0}. \tag{8}$$

Note that the Hecke images P_M in (8) are in general singular, even if P is nonsingular. Also, the proof of Theorem 1.8 is independent of that of Theorem 1.4, and hence provides another proof of Theorem 1.4 for nonsingular lattice polytopes.

1.9. We comment briefly on the proofs of Theorems 1.4 and 1.8. The proof of Theorem 1.4 is a counting argument. The new lattice points appearing in P in the sum (2) all lie in the superlattice $p^{-1}L$, and to compute $T(p, k)E(P)$ one keeps track of which lattice points appear in a given Hecke image. This gives an expression for $T(p, k)E(P)$ in terms of $E(P)(t)$, $E(P)(pt)$, and the cardinalities of some finite Grassmannians. An additional argument shows that this expression implies (5).

The proof of Theorem 1.8 is more complicated. At the heart of (8) are certain "distribution relations" of Dedekind sums, essentially coming from a distribution relation satisfied by the Hurwitz zeta function (§6.2). In the proof of Theorem 1.8, these relations appear in identities involving Dedekind sums and the cardinalities of strata in certain stratifications of finite Grassmannians.[1]

Rather than proving these identities directly, we show that they occur in the computation of the constant term of $T(p, j)E(P')$ for lower-dimensional polytopes P' and for $j \leq k$. Since these constant terms are always 1, by appropriately choosing P' we show that our identities hold. Then we use induction to complete the argument.

1.10. Here is a fanciful interpretation of Theorem 1.4. The Ehrhart polynomial is clearly invariant under the action of $\mathrm{GL}(L)$, the linear automorphisms of V preserving L. One can think of $\mathscr{P}(L)$ as being like the upper halfplane $\mathfrak{H}$, and the equivalence class of $P \in \mathscr{P}(L)$ as being a point on the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$. Then the lth coefficient c_l, thought of as a function $\mathrm{GL}(L)\backslash\mathscr{P}(L) \to \mathbb{Q}$, plays the role of a weight l modular form, and Theorem 1.4 says that c_l is a "weight l Hecke eigenform of level 1." Indeed, the analogy between coefficients of E and modular forms was our original motivation to consider this problem.

One can say more about this analogy. Suppose one has an action of the Hecke operators in a finite-dimensional complex vector space W. For instance, W could be the space of holomorphic modular forms for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, or the cohomology of fixed degree of an arithmetic subgroup of $\mathrm{GL}_n(\mathbb{Z})$, or even more general spaces of automorphic forms. Then under certain conditions ("algebraic type at infinity"), the Langlands philosophy predicts that Hecke eigenclasses $\beta \in W$ should correspond to families of ℓ-adic representations of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The connection is revealed through the Hecke eigenvalues.

Suppose that for a prime p the Hecke operator $T(p, k)$ acts on β with eigenvalue $a(p, k)$. In saying this we assume that p is not a "bad prime" for β; at this heuristic level we cannot make this more precise, except to say that in the example of modular forms these are the primes dividing the level. Hence for our analogy there are no bad primes since the level is 1. Using these eigenvalues we can form the Hecke polynomial

$$H_p(\beta) = \sum_{k=0}^{n} (-1)^k a(p, k) p^{k(k-1)/2} X^k \in \mathbb{C}[X].$$

Now let $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(Q_\ell)$ be a continuous semisimple Galois representation unramified away from p. We also assume ρ is unramified away from "bad primes" of β. Choose an isomorphism $\iota\colon \mathbb{Q}_\ell \to \mathbb{C}$. Then one says β is *attached* to ρ if for all good primes p we have

$$H_p(\beta) = \det(I - \rho(\mathrm{Frob}_p)X),$$

[1]We remark that the action of the Hecke operators on Dedekind sums has been studied in the literature, starting with Dedekind himself [9], and more recently by Knopp [17], Nagasaka [19], Zheng [24], and Beck [3].

where $\mathrm{Frob}_p \subset G_{\mathbb{Q}}$ is the Frobenius conjugacy class, and where we used ι to compare the two sides.

Now let β_l be the eigenclass corresponding to the lth Ehrhart coefficient c_l, with the Hecke eigenvalues $\nu_{n,k,l}(p)$ from Theorem 1.4. We claim that indeed β_l is attached to a Galois representation. To see this, let ϵ be the ℓ-adic cyclotomic character, which satisfies $\epsilon(\mathrm{Frob}_p) = p$ for all $p \neq \ell$. Let ω be the Galois representation $\epsilon \oplus \epsilon^2 \oplus \cdots \oplus \epsilon^{n-1}$.[2] Then we claim β_l is attached to the representation $\epsilon^l \oplus \omega$. Checking this amounts to verifying the identity

$$\sum_{k=0}^{n} (-1)^k \nu_{n,k,l}(p) p^{k(k-1)/2} X^k = (1 - p^l X) \prod_{j=1}^{n-1} (1 - p^j X),$$

a pleasant exercise. For $l = 0$, when the eigenvalues are the cardinalities of finite Grassmannians, this is a special case of the q-binomial theorem.

1.11. The paper is organized as follows. Section 2 recalls background about lattice polytopes and their normal fans, and discusses the connection between Hecke operators and finite Grassmannians. Section 3 gives the proof of Theorem 1.4. Section 4 discusses the computation of the Ehrhart polynomial using the Todd operator, and Section 5 gives the proof of Theorem 1.8. Section 6 discusses explicit examples of Theorem 1.8 for three-dimensional polytopes, and relates the identities occurring in the proof of Theorem 1.8 to Dedekind sums and the Hurwitz zeta function. Finally, Section 7 addresses the problem of computing the average Ehrhart polynomial as one varies over a family of superlattices.

2. Hecke operators and finite Grassmannians

2.1. Let P be a simple lattice polytope in the vector space V with vertices in the lattice L. For convenience we fix a nondegenerate bilinear form $\langle\ ,\ \rangle$ and use it to identify V with its dual. We also assume that L is self-dual with respect to this form.

Let $\mathscr{F}$ be the set of faces of P, and for any l let $\mathscr{F}(l)$ be the subset of faces of dimension l. Let $F \in \mathscr{F}(n-1)$ be a facet of P. Then F is the intersection of P with an affine hyperplane

$$H_F = \{x \mid \langle x, u_F \rangle + \lambda_F = 0\},$$

where the normal vector u_F is taken to be a primitive vector in L, and points into the interior of P.

[2] Galois representations that are direct sums of powers of the cyclotomic character are sometimes called *punctual* [1].

2.2. Let $f \in \mathscr{F}(n-l)$ be a face of codimension l, and let H_f be the affine subspace spanned by f. Since P is simple, there are exactly l hyperplanes in $\{H_F \mid F \in \mathscr{F}(n-1)\}$ whose intersection is H_f. Let $\sigma_f \subset V$ be the convex cone generated by the corresponding normal vectors $\{u_F\}$. The cone σ_f is called the *normal cone* to f.

The set $\Sigma = \{\sigma_f \mid f \in \mathscr{F}\}$ of all normal cones forms an acute rational polyhedral fan in V. This means the following:

(1) Each $\sigma \in \Sigma$ contains no nontrivial linear subspace.
(2) If σ' is a face of $\sigma \in \Sigma$, then $\sigma' \in \Sigma$.
(3) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of each.
(4) Given $\sigma \in \Sigma$, there exists a finite set $S \subset L$ such that any point in σ can be written as $\sum \rho_s s$, where $s \in S$ and $\rho_s \geq 0$.

Moreover, P simple implies Σ is simplicial, which means that in (4) we can take $\#S = \dim \sigma$ for all σ. The fan Σ is called the *normal fan* to P.

2.3. Let $\rho \in \Sigma$ be a 1-dimensional cone. Then ρ contains a unique normal vector u_F, which we call the *spanning point* of ρ. For any cone σ, we denote by $\sigma(1)$ the set of spanning points of all 1-dimensional faces of σ, and write

$$\Sigma(1) = \bigcup_{\sigma \in \Sigma} \sigma(1).$$

There is bijection between $\Sigma(1)$ and $\mathscr{F}(n-1)$.

For any rational cone σ, let $U(\sigma)$ be the sublattice of L generated by the spanning points of σ. Put $L(\sigma) = L \cap (U(\sigma) \otimes \mathbb{Q})$, and let $\operatorname{Ind} \sigma = [L(\sigma) : U(\sigma)]$. If $\operatorname{Ind} \sigma = 1$, then σ is called *unimodular*. Then P is nonsingular if and only if all its normal cones are unimodular.

2.4. Now we recall some basic facts about Hecke operators for the linear group GL_n. Let p be a prime, and let $\overline{V} \simeq \mathbb{F}_p^n$ be the quotient $p^{-1}L/L$. For any lattice M satisfying $p^{-1}L \supsetneq M \supsetneq L$, let $\overline{M} \subset \overline{V}$ be the corresponding subspace. More generally, for any rational subspace $W \subseteq V$ we let $\overline{W} \subseteq \overline{V}$ be the corresponding subspace. Fix a positive integer $k \leq n$, and let $\mathrm{Gr}(k,n)$ be the Grassmannian of k-dimensional subspaces of an n-dimensional vector space.

Lemma 2.5. *The set $\mathscr{L}_k$ of superlattices $p^{-1}L \supsetneq M \supsetneq L$ of coindex p^k is in bijection with the set $\mathscr{T}$ of upper triangular matrices of the form*

$$\begin{pmatrix} p^{e_1} & & a_{ij} \\ & \ddots & \\ & & p^{e_n} \end{pmatrix},$$

where

- $e_i \in \{0,1\}$, *and exactly k of the e_i are equal to* 0,
- $a_{ij} = 0$ *unless $e_i = 0$ and $e_j = 1$, in which case a_{ij} satisfies* $0 \leq a_{ij} < p$.

Moreover, the map $M \mapsto \overline{M}$ induces a bijection between $\mathscr{L}_k$ and $\mathrm{Gr}(k,n)(\mathbb{F}_p)$.

Proof. It is well known that the set of *sublattices* $L \supsetneq N \supsetneq pL$ of index p^{n-k} is in bijection with $\mathscr{T}$ [18, Prop. 7.2]. To realize this bijection, we take $L = \mathbb{Z}^n$, and then any N is constructed as the sublattice generated by the rows of some $A \in \mathscr{T}$. The sublattice N determines a subspace $\overline{N} \subset \overline{V}$, generated by the k rows with diagonal entry 1. It is clear that we obtain all k-dimensional subspaces of $\overline{V}$ in this way, for example by considering the decomposition of $\mathrm{Gr}(k,n)(\mathbb{F}_p)$ into Schubert cells [13, p. 147]. Finally, both statements of the lemma follow from the isomorphism $p^{-1}L/L \simeq L/pL$ given by scaling by p, and from the fact that a sublattice has coindex p^k if and only if it has index p^{n-k}. □

2.6. Let $f \in \mathscr{F}$ be a face of P, and let σ_f be the normal cone to f. Let $V_f \subset V$ be the linear subspace parallel to H_f, and let C_f be the linear span of σ_f. The subspace C_f contains the distinguished 1-dimensional subspaces $\{C_\rho \mid \rho \in \sigma_f(1)\}$.

Proposition 2.7. *Let $M \in \mathscr{L}_k$, and for any $f \in \mathscr{F}$, let f_M be the corresponding face of P_M. Then*

(1) $\operatorname{Vol} f_M = p^{\dim(\overline{M} \cap \overline{V}_f)} \operatorname{Vol} f$,
(2) $\operatorname{Ind} \sigma_{f_M} = p^{\dim(\overline{M} \cap \overline{C}_f) - r} \operatorname{Ind} \sigma$,

where $r = \#\{\overline{C}_\rho \mid \rho \in \sigma_f(1)$ and $\overline{C}_\rho \subset \overline{M}\}$.

Proof. Choose a $\mathbb{Z}$-basis B of L such that $B \cap V_f$ is a $\mathbb{Z}$-basis for $L \cap V_f$. By Lemma 2.5, with respect to B any $M \in \mathscr{L}_k$ is spanned by the rows of $p^{-1}A$ for some $A \in \mathscr{T}$. Each row of A with diagonal entry 1 contributes a factor of p to $\operatorname{Vol} f_M / \operatorname{Vol} f$, which proves (1).

For C_f we argue similarly. The only difference is that each row of A with diagonal entry 1 contributes a factor of p to $\operatorname{Ind} \sigma_{f_M} / \operatorname{Ind} \sigma_f$, unless the diagonal entry is the only nonzero entry in the row. This situation corresponds to some subspace $\overline{C}_\rho$ being contained in $\overline{M}$, and (2) follows. □

Proposition 2.7 allows us to give a geometric interpretation for the eigenvalue $\nu(p)$.

Proposition 2.8. *Fix nonnegative integers $k, l \leq n$, and let p be a prime. Let $\mathrm{Vol}_l \colon \mathscr{P}(L) \to \mathbb{Q}$ be the function*

$$\mathrm{Vol}_l(P) = \sum_{f \in \mathscr{F}(l)} \mathrm{Vol}(f),$$

and define

$$T(p,k) \mathrm{Vol}_l(P) = \sum_{M \in \mathscr{L}_k} \mathrm{Vol}_l(P_M).$$

Then $T(p,k)\mathrm{Vol}_l(P) = \nu_{n,k,l}(p)\mathrm{Vol}_l(P)$.

Proof. Suppose $f \in \mathscr{F}(l)$. According to Proposition 2.7, we have

$$\sum_{M \in \mathscr{L}_k} \operatorname{Vol} f_M = \sum_{M \in \mathscr{L}_k} p^{\dim(\overline{M} \cap \overline{V}_f)} \operatorname{Vol} f. \tag{9}$$

The right of (9) equals $\nu_{n,k,l}(p) \operatorname{Vol} f$, and the statement follows immediately. □

3. Proof of Theorem 1.4

3.1. Throughout this section we allow P to be a general lattice polytope. Let $U \subset \overline{V}$ be a fixed subspace of dimension l as in §1.3, and recall

$$\nu_{n,k,l}(p) = \sum_{\substack{W \subset \mathbb{F}_p^n \\ \dim W = k}} p^{\dim W \cap U}.$$

Let $G_{k,n}$ be the cardinality of the finite Grassmannian $\mathrm{Gr}(k,n)(\mathbb{F}_p)$. It is well known that

$$G_{k,n} = \frac{[n]_p!}{[k]_p![n-k]_p!}, \tag{10}$$

where $[n]_p = (p^n - 1)/(p-1)$, and $[n]_p! = \prod_{i=1}^n [i]_p$.

Lemma 3.2. *Let $E = E(t)$ be the Ehrhart polynomial of P. Then*

$$T(p,k)E(t) = G_{k-1,n-1}E(pt) + (G_{n,k} - G_{k-1,n-1})E(t). \tag{11}$$

In particular,

$$c_l(T(p,k)E)/c_l(E) = G_{k,n} + (p^l - 1)G_{k-1,n-1}. \tag{12}$$

Proof. We have

$$\bigcup_{M \in \mathscr{L}_k} M = p^{-1}L, \tag{13}$$

and since counting points in $p^{-1}L \cap P$ is done by $E(pt)$, we must count how often a point $x \in p^{-1}L$ appears in the union (13). There are two separate cases, namely (i) $x \in p^{-1}L \smallsetminus L$, and (ii) $x \in L$. The former contribute to $E(pt)$, and the latter to $E(t)$.

For (i), note that the point x determines a line $\Lambda_x \in \overline{V}$, and the number of k-dimensional subspaces containing Λ_x is $G_{k-1,n-1}$. For (ii), each $x \in L$ will appear in every Hecke image, which gives $G_{k,n}$ in total. However, such points are also counted in the sublattices contributing to (i). When these contributions are subtracted, we obtain (11). This proves the first statement.

Finally, (12) follows easily from (11), since $c_l(E(pt)) = p^l c_l(E(t))$. □

Lemma 3.3. *We have*

$$\nu_{n,k,l}(p) = G_{k,n} + (p^l - 1)G_{k-1,n-1}. \tag{14}$$

Moreover,

$$\nu_{n,k,l}(p)/\nu_{n,n-k,n-l}(p) = p^{k+l-n}. \tag{15}$$

Proof. We treat the case $k \geq l$; the case $k < l$ is similar.

For $j = 0, \ldots, l$, let Y_j be the locally closed subvariety of $\mathrm{Gr}(k,n)(\mathbb{F}_p)$ defined by

$$Y_j = \{W \mid \dim W = k,\ \dim(W \cap U) = j\},$$

and let $y_j = \#Y_j$. Note that $\sum_{j\geq 0} y_j = G_{k,n}$, and that $\nu_{n,k,l}(p) = \sum_{j\geq 0} y_j p^j$. Since $y_0 = G_{k,n} - \sum_{j\geq 1} y_j$, it follows that

$$\nu_{n,k,l}(p) = G_{k,n} + \sum_{j\geq 1} y_j(p^j - 1). \tag{16}$$

We prove the lemma by showing

$$[l]_p G_{k-1,n-1} = \sum_{j\geq 1} [j]_p y_j, \tag{17}$$

which is equivalent to (14) and (16) taken together. To do this, we explicitly describe Y_j recursively in terms of $\{Y_i \mid i > j\}$, and show that the right of (17) telescopes to the left of (17).

Consider first Y_l. Any point in Y_l is given by choosing a k-dimensional subspace W in $\overline{V}$ containing U. Such subspaces are in bijection with $(k-l)$-dimensional subspaces of $\overline{V}/U$, and thus $y_l = G_{k-l,n-l}$.

Next, any point in Y_{l-1} is given by choosing an $(l-1)$-dimensional subspace S of U, and then choosing a k-dimensional subspace W of $\overline{V}$ with $W \cap U = S$. The subvariety of those W with $W \cap U \supset S$ gives $G_{l-1,l} G_{k-(l-1),n-(l-1)}$ points; this is not y_{l-1} since for each S we have included those W that contain U, instead of just meeting U in a subspace of codimension 1. The correct value of y_{l-1} is given by subtracting the contributions corresponding to points in Y_l, which gives

$$y_{l-1} = G_{l-1,l}(G_{k-(l-1),n-(l-1)} - G_{k-l,n-l}).$$

For the general Y_j similar considerations apply. We summarize the results as follows. For $j = 1, \ldots, l$ let $U_j \subset \mathbb{F}_p^{n-j}$ be a fixed subspace of dimension $l - j$, and let Z_l be the subvariety of the Grassmanian $\mathrm{Gr}(k-j, n-j)(\mathbb{F}_p)$ of all $(k-j)$-dimensional subspaces W such that $W \cap U_j = \{0\}$. Putting $z_j = \#Z_j$, we have

$$z_j = \begin{cases} G_{k-l,n-l}, & j = l, \\ G_{k-j,n-j} - \displaystyle\sum_{i=1}^{l-j} G_{i,l-j} z_{i+j}, & j < l. \end{cases}$$

Then

$$y_j = G_{j,l} z_j, \quad j = 1, \ldots, l,$$

and in particular

$$y_1 = G_{1,l}(G_{k-1,n-1} - G_{1,l-1} z_2 - G_{2,l-1} z_3 - \cdots - G_{l-1,l-1} z_l). \tag{18}$$

Finally, using (10) we see

$$[1]_p G_{1,l} G_{k-1,n-1} = [l]_p G_{k-1,n-1}, \tag{19}$$

and

$$[j]_p G_{j,l} = [1]_p G_{1,l} G_{j-1,l-1}. \tag{20}$$

Using (19) and (20) with (18) shows that the right of (17) telescopes to the left of (17), which proves (14). A simple computation yields (15) from (14), and Lemma 3.3 is proved. □

Lemmas 3.2 and 3.3 imply almost all of Theorem 1.4. Equations (12) and (14) imply (5), and the existence of the polynomial $\Phi_{n,k,l}$ from (4) is clear by (10) and (14). The only remaining statement is the positivity of the coefficients of $\Phi_{n,k,l}$. To see this, fix a complete flag

$$\{0\} = U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n = \overline{V},$$

where $\dim U_j = j$. We define a polynomial $\widehat{\Phi}_{n,k} \in \mathbb{Z}[x_0, \ldots, x_n]$ by

$$\widehat{\Phi}_{n,k} = \sum_{\substack{W \subset \overline{V} \\ \dim W = k}} \prod x_j^{\dim W \cap U_j}. \tag{21}$$

Clearly $\Phi_{n,k,l}(p)$ is obtained from $\widehat{\Phi}_{n,k}$ by the substitutions $x_l = p$ and $x_j = 1$ if $j \neq l$. We claim $\widehat{\Phi}_{n,k}$ is a polynomial with positive coefficients. Indeed, the distinct monomials $x^\alpha := \prod x_j^{\alpha_j}$ in (21) correspond to the different possibilities of intersections of W with the fixed flag, which correspond to the decomposition of $\mathrm{Gr}(n,k)(\mathbb{F}_p)$ into Schubert cells S_α [13]. Thus we can rewrite (21) as

$$\widehat{\Phi}_{n,k} = \sum_\alpha \#S_\alpha(\mathbb{F}_p) x^\alpha.$$

But each Schubert cell is isomorphic to an affine space, and hence the coefficients $\#S_\alpha(\mathbb{F}_p)$ are pure p-powers. This completes the proof of Theorem 1.4.

Remark 3.4. We have the following additional geometric interpretation of the eigenvalue $\nu_{n,k,l}(p)$. Let T be the total space of the rank n trivial bundle over $G(k,n)(\mathbb{F}_p)$, and let $T_l \subset T$ be the subbundle corresponding to a fixed l-dimensional subspace. Let B be the total space of the tautological bundle over $G(k,n)(\mathbb{F}_p)$, i.e. for any $x \in G(k,n)(\mathbb{F}_p)$ the fiber B_x over x is the k-dimensional subspace corresponding to x. Then

$$\upsilon_{n,k,l}(p) = \#(B \cap T_l).$$

4. The Todd operator

4.1. In this section we describe the Todd operator $\mathrm{Td}(\Sigma, \partial/\partial h)$ and how it can be used to compute the Ehrhart polynomial of a simple lattice polytope P. We closely follow [5].

Recall that $\mathscr{F}$ is the set of faces of P, and that each facet $F \in \mathscr{F}(n-1)$ determines an affine hyperplane

$$H_F = \{x \mid \langle x, u_F \rangle + \lambda_F = 0\},$$

where the normal vector $u_F \in L$ is a primitive vector pointing into the interior of P.

Let $h = (h_F)_{F \in \mathscr{F}(n-1)}$ be a real multivariable indexed by the facets of P, and let $P(h)$ be the convex region determined by the inequalities

$$\{\langle x, u_F \rangle + \lambda_F + h_F \geq 0 \mid F \in \mathscr{F}(n-1)\}. \tag{22}$$

Note that $P(0) = P$. Then $P(h)$ is isomorphic to P for small h, and thus for small h one can consider the volume $\operatorname{Vol} P(h)$. The following examples will play an important role in the proof of Theorem 1.8.

Example 4.2. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{R}^n$, and let $e_0 = 0$. Let $P = \Delta_n$ be the convex hull of the vectors $\{e_0, \dots, e_n\}$. Then Δ_n is the n-dimensional simplex. Let h_i be the parameter attached to the facet obtained by deleting the vertex e_i. It is easy to check that

$$\operatorname{Vol} \Delta_n(h) = \Big(1 + \sum_{i=0}^{n} h_i\Big)^n / n!.$$

Example 4.3. Let P and P' be two lattice polytopes, and let h and h' be multi-variables indexed by their facets. Then

$$\operatorname{Vol}(P \times P')(h, h') = \operatorname{Vol} P(h) \operatorname{Vol} P'(h').$$

In particular, for the unit n-cube $P = (\Delta_1)^n$ we obtain

$$\operatorname{Vol} P(h) = \prod_{i=1}^{n} (1 + h_i + h_i').$$

4.4. Let Σ be the normal fan to P. For any $\sigma \in \Sigma$, define

$$Q(\sigma) = \Big\{ \sum_{s \in \sigma(1)} \rho_s s \;\Big|\; 0 \le \rho_s < 1 \Big\}.$$

Note that $\operatorname{Vol} Q(\sigma) = \operatorname{Ind} \sigma$, and $Q(\sigma) \cap U(\sigma) = \{0\}$ if and only if σ is unimodular. Put

$$\Gamma_\Sigma = \bigcup_{f \in \mathscr{F}} Q(\sigma_f) \cap L.$$

We have $\Gamma_\Sigma = \{0\}$ if and only if P is nonsingular.

4.5. For each $F \in \mathscr{F}(n-1)$, let $\xi_F \colon V \to \mathbb{R}$ be the unique piecewise-linear continuous function defined by

- $\xi_F(s) = 1$ if $s \in \Sigma(1)$ is the spanning point corresponding to F,
- $\xi_F(s') = 0$ for all other $s' \in \Sigma(1)$,
- ξ_F is linear on all the cones of Σ.

Put $a_F(x) = \exp(2\pi i \xi_F(x))$ for all $x \in V$.

Suppose $g \in \Gamma_\Sigma \cap \sigma$. Then the pair (g, σ) determines a tuple of roots of unity as follows. If $s_1, \dots, s_l$ are the spanning points of σ, and $F_1, \dots, F_l$ are the corresponding facets, then we can attach to (g, σ) the tuple $(a_1(g), \dots, a_l(g))$, where we have written a_i for a_{F_i}.

4.6. Let a be a complex number and x a real variable. We define $\mathrm{Td}(a, \partial/\partial x)$ to be the differential operator given formally by the power series

$$\frac{\partial/\partial x}{1 - a\exp(-\partial/\partial x)} = \sum_{k=0}^{\infty} c(a,k)\left(\frac{\partial}{\partial x}\right)^k.$$

Note that $c(1,k) = B_k/k!$, where B_k is the kth Bernoulli number.[3] If $a \neq 1$, then $c(a,k)$ is a rational function in a of degree -1 closely related to the kth *circle function of Euler* (§6.2). Table 2 gives some examples of the $c(a,k)$.

TABLE 2. The coefficients $c(a,k)$

k	$c(a,k)$
1	$-1/(a-1)$
2	$-a/(a^2-2a+1)$
3	$-(a^2+a)/(2a^3-6a^2+6a-2)$
4	$-(a^3+4a^2+a)/(6a^4-24a^3+36a^2-24a+6)$

4.7. Now let h be a multivariable with components h_F indexed by the facets of P. Let $g \in \Gamma_\Sigma$, and define

$$\mathrm{Td}(g, \partial/\partial h) = \prod_{F \in \mathscr{F}(n-1)} \mathrm{Td}(a_F(g), \partial/\partial h_F)$$

and

$$\mathrm{Td}(\Sigma, \partial/\partial h) = \sum_{g \in \Gamma_\Sigma} \mathrm{Td}(g, \partial/\partial h). \tag{23}$$

We have the following theorem, proved by Khovanskiĭ–Pukhlikov if P is nonsingular, and by Brion–Vergne for general simple lattice polytopes.

Theorem 4.8 ([5,21]). *Suppose P is a simple lattice polytope. Then the coefficients of the Ehrhart polynomial $E_P(t) = \sum_{i=0}^{n} c_i t^i$ are given by*

$$c_{n-l} = \mathrm{Td}_l(\Sigma, \partial/\partial h)\,\mathrm{Vol}\,P(h)\big|_{h=0},$$

where $\mathrm{Td}_l(\Sigma, \partial/\partial h)$ is the degree l part of $\mathrm{Td}(\Sigma, \partial/\partial h)$.

For the connection between coefficients of the Todd operator and higher-dimensional Dedekind sums, we refer to [2, §9].

[3] With our conventions the Bernoulli numbers are $B_1 = 1/2$, $B_2 = 1/6$, $B_4 = -1/30$, …, and $B_{2k-1} = 0$ for $k > 1$. Note that for many authors $B_1 = -1/2$, cf. §6.2.

5. Proof of Theorem 1.8

5.1. We recall some notation from §1.7. Let $f \in \mathscr{F}$ be a face of codimension $\le l$, and let $\pi = (\pi(F))_{F \supset f}$ be an ordered partition of l indexed by the facets containing f. We expand (23) as a sum over pairs

$$\mathrm{Td}_l(\Sigma, \partial/\partial h) = \sum_{(f,\pi)} A(f,\pi)\partial_f^\pi,$$

where

$$\partial_f^\pi = \prod_{F \supset f} (\partial/\partial h_F)^{\pi(F)}$$

and

$$A(f,\pi) = \sum_{g \in \Gamma \cap \sigma_f} \prod_{F \supset f} c(a_F(g), \pi(F)). \tag{24}$$

Note that if σ_f is unimodular, then

$$A(f,\pi) = \prod_{F \supset f} \frac{B_{\pi(F)}}{\pi(F)!}. \tag{25}$$

5.2. Now fix a total ordering on (unordered) partitions of l by using the *lexicographic* order. In other words, let $\pi = \{\pi_1, \dots, \pi_j\}$ and $\pi' = \{\pi'_1, \dots, \pi'_k\}$ be two partitions of l with parts arranged in *nonincreasing* order. Then we have $\pi < \pi'$ if and only if there exists an index m with $\pi_i = \pi'_i$ for $i < m$ and $\pi_i < \pi'_i$ for $i \ge m$. For example, if $l = 6$, then in increasing order (and in obvious notation) the partitions are

$$1^6,\ 21^4,\ 2^21^2,\ 31^3,\ 321,\ 3^2,\ 41^2,\ 42,\ 51,\ 6.$$

5.3. We say the pair (f,π) is *squarefree* if $\pi(F) = 1$ for all $F \supset f$, and we write $\pi = \mathbf{1}$. We begin with two lemmas. Lemma 5.4 gives a geometric interpretation of the squarefree terms, and Lemma 5.6 allows us to compute nonsquarefree terms using squarefree terms.

Lemma 5.4. *Let P be simple. For any face $f \in \mathscr{F}$, we have*

$$\partial_f^{\mathbf{1}} \operatorname{Vol} P(h)\big|_{h=0} = \frac{\operatorname{Vol} f}{\operatorname{Ind} \sigma_f}.$$

In particular, if P is nonsingular and f has codimension l, then

$$A(f,\mathbf{1})\partial_f^{\mathbf{1}} \operatorname{Vol} P(h)\big|_{h=0} = \frac{\operatorname{Vol} f}{2^l}.$$

Proof. The first statement is Lemma 4.7 in [5]. The second statment follows from (25) since the Bernoulli number B_1 is $1/2$, and $\operatorname{Ind} \sigma_f = 1$ if P is nonsingular. □

5.5. The following result is well known to experts, and is stated (for nonsingular P) in [21, Theorem, p. 795]. For the convenience of the reader we present a proof for P simple. For unexplained concepts from toric geometry, we refer to [12]. What we will need from Lemma 5.6 is (26).

Lemma 5.6 ([4]). *Let X be the projective toric variety associated to the simple lattice polytope P. Then the rational Chow ring $H^*(X, \mathbb{Q})$ is isomorphic to the quotient of*

$$\mathbb{Q}\left[\partial/\partial h_F \mid F \in \mathscr{F}(n-1)\right]$$

by the ideal I of differential operators that annihilate the function $\operatorname{Vol} P(h)$.

Proof. The rational Chow ring $H^*(X, \mathbb{Q})$ has generators the classes of the divisors $[D_F]$, $F \in \mathscr{F}(n-1)$, and the following relations:

- square-free monomial relations $\prod_{F\in I}[D_F] = 0$ unless the facets in I intersect transversally along a face of P, and
- linear relations $\sum_F \langle w, u_F\rangle [D_F] = 0$, where $w \in L$.

But the analogous relations hold for $\mathbb{Q}[\partial/\partial h_F \mid F \in \mathscr{F}(n-1)]$ applied to $\operatorname{Vol} P(h)$; for example, the linear relations express invariance of volume under translation. Thus, we obtain a surjective homomorphism of graded rings

$$H^*(X, \mathbb{Q}) \to \mathbb{Q}[\partial/\partial h_F \mid F \in \mathscr{F}(n-1)]/I, \quad [D_F] \mapsto \partial/\partial h_F,$$

where the $\partial/\partial h_F$ have degree 2. To show its injectivity, it is enough (by Poincaré duality) to show that all intersection numbers of the form $[D_{F_1}]\cdots[D_{F_n}]$ can be read off from the images of the D_F. But this follows from the formula

$$\Bigl(\sum_F (\lambda_F + h_F)[D_F]\Bigr)^n = \operatorname{Vol} P(h),$$

where the λ_F come from the inequalities (22) determining $P(h)$. Indeed, since the h_F are independent variables, any monomial of degree n in the $[D_F]$ can be expressed in terms of partial derivatives of $\operatorname{Vol} P(h)$. □

5.7. Let $w \in L$. Then by Lemma 5.6 the differential operator

$$\sum_{F\in\mathscr{F}(n-1)} \langle w, u_F\rangle \partial/\partial h_F \tag{26}$$

annihilates $\operatorname{Vol} P(h)$. Hence if $\pi > \mathbf{1}$, by repeatedly applying (26) we can write

$$\partial_f^\pi \operatorname{Vol} P(h) = \varepsilon_f(\pi) \sum_{f'} \Bigl(\prod_{w\in W(f')} \langle w, u_w\rangle \Bigr) \partial_{f'}^{\mathbf{1}} \operatorname{Vol} P(h), \tag{27}$$

where the quantities in (27) satisfy the following:

- The integer $\varepsilon_f(\pi) \in \{\pm 1\}$ depends only on the pair (f, π).
- The sum ranges over a finite set of codimension l faces f', each of which is contained in f.

- For each f', the set $W(f') \subset L \otimes \mathbb{Q}$ satisfies
  - $\langle w, u \rangle = 1$ for some $u \in \sigma_f(1)$,
  - $\langle w, v \rangle = 0$ for all $v \in \sigma_f(1) \smallsetminus \{u\}$.
- The $\{u_w\} \subset \Sigma(1)$ are such that for each f', we have

$$\sigma_{f'}(1) = \sigma_f(1) \cup \{u_w\}_{w \in W(f')}.$$

- The sets $W(f')$ are ordered and

$$\langle w', u_w \rangle = 0 \quad \text{for all } w < w'.$$

We choose and fix an expression of the form (27) for each pair (f, π).

5.8. We are now ready to prove Theorem 1.8. Our goal is to show

$$\sum_{M \in \mathscr{L}_k} A(f_M, \pi) \partial_f^\pi \operatorname{Vol} P_M(h) = \nu_{n,k,n-l}(p) A(f, \pi) \operatorname{Vol} P(h). \tag{28}$$

Let $\nu(p) = \nu_{n,k,n-l}(p)$. Applying (27) in (28) and using Lemma 5.4, we see that it suffices to verify

$$\sum_{M \in \mathscr{L}_k} A(f_M, \pi) \sum_{f'_M} \Big(\prod_{w \in W(f'_M)} \langle w, u_w \rangle \Big) \frac{\operatorname{Vol} f'_M}{\operatorname{Ind} \sigma_{f'_M}} = \nu(p) A(f, \pi) \sum_{f'} \Big(\prod_{w \in W(f')} \langle w, u_w \rangle \Big) \operatorname{Vol}(f'). \tag{29}$$

Since the faces f' appearing in (27) are independent of the lattice M, we can interchange the sum over $\mathscr{L}_k$ and the sum over f'_M, and focus on a single f'. Furthermore, Lemma 2.5 implies that the sum over M in (29) is really a sum over $\operatorname{Gr}(k, n)(\mathbb{F}_p)$. We construct a stratification $\{X_{ij}\}$ of $\operatorname{Gr}(k, n)(\mathbb{F}_p)$ by defining

$$X_{ij} = \{W \subset \overline{V} \mid \dim W = k,\ \dim W \cap \overline{V}_f = i,\ \dim W \cap \overline{C}_f = j\}, \tag{30}$$

and the left of (29) becomes

$$\sum_{i,j} \sum_{\overline{M} \in X_{ij}} A(f_M, \pi) \Big(\prod_{w \in W(f'_M)} \langle w, u_w \rangle \Big) \frac{\operatorname{Vol} f'_M}{\operatorname{Ind} \sigma_{f'_M}}.$$

Now let $S_j \subset \overline{C}_f$ be a fixed subspace of dimension j, and put

$$m_{ij} = \#\{\overline{M} \in X_{ij} \mid \overline{M} \supset S_j\}. \tag{31}$$

The number m_{ij} is independent of the choice of S_j. If $\overline{M} \in X_{ij}$, then

$$\operatorname{Vol} f'_M = p^i \operatorname{Vol} f', \tag{32}$$

and equation (29) becomes

$$\sum_{i,j} p^i m_{ij} \sum_{\substack{S\subset\overline{C}_{f'}\\ \dim S=j}} A(f_S,\pi)(\operatorname{Ind}\sigma_{f'_S})^{-1} \prod_{w\in W(f'_S)} \langle w,u_w\rangle = \nu(p)A(f,\pi) \prod_{w\in W(f')} \langle w,u_w\rangle, \tag{33}$$

where we have written

$$A(f_S,\pi) = \sum_{g\in\Gamma\cap\sigma_{f_S}} \prod_{F\supset f} c(a_F(g),\pi(F)). \tag{34}$$

Note that it makes sense to replace the subscript M with S in (33) and (34), since $\operatorname{Ind}\sigma_{f'_M}$ (respectively $\Gamma\cap\sigma_{f_M}$) depends only on $S=\overline{M}\cap\overline{C}_{f'}$ (resp., $\overline{C}_f$). The notation $W(f'_S)$ also makes sense, because all points in $W(f'_M)$ are multiples of points in $W(f')$ (in fact they differ at most by a factor of p), and which multiples we take depends only on S.

To verify (33), we show that for each j the identity

$$\sum_{\substack{S\subset\overline{C}_{f'}\\ \dim S=j}} A(f_S,\pi) \prod_{w\in W(f'_S)} \langle w,u_w\rangle(\operatorname{Ind}\sigma_{f'_S})^{-1} = G_{j,l}A(f,\pi)\sum_{f'}\Big(\prod_{w\in W(f')} \langle w,u_w\rangle\Big) \tag{35}$$

holds. This will complete the proof of the theorem, since

$$\sum_{i,j} p^i m_{ij} G_{j,l} = \nu(p).$$

We verify (35) by induction on the partition order; the main idea is to show that (35) appears in the computation of the constant term of $T(p,j)E(P)$ for some easily understood polytope P. Since we know how the constant terms transform under the Hecke operators, our identity is forced to hold. In particular, let

$$P = \prod_{F\supset f} \Delta_{\pi(F)},$$

where in the product the facets F are ordered so that π has nonincreasing parts. Using Examples 4.2 and 4.3, we see that the highest order terms contributing to $E(P)$ and $T(p,j)E(P)$ are those of type (f,π), where f is a vertex. Now assume that all weight l terms of type (f,π') with $\pi'<\pi$ satisfy (35). Since the constant term of $T(p,j)E(P)$ equals $G_{j,l}$, and since each vertex of P contributes equally to the constant term, this implies (35).

Hence to complete the proof, we must check (35) in the case $\pi=\mathbf{1}$. In this case we do not need to apply (27), since the terms are already squarefree. In view of (25), the identity to be proved is

$$\sum_{\substack{S\subset\overline{C}_f\\ \dim S=j}} (\operatorname{Ind}\sigma_{f_S})^{-1}A(f,\mathbf{1}) = \frac{G_{j,l}}{2^l}. \tag{36}$$

To prove (36), we let $P=(\Delta_1)^l$ and consider the action of $T(p,j)$ on the constant term of its Ehrhart polynomial. By Example 4.3, we have

$$\operatorname{Vol}P(h) = \prod_{i=1}^{l}(1+h_i+h_i').$$

We see from applying Td_l to $\operatorname{Vol}P(h)$ that only squarefree terms contribute to the constant term of $E(P)$, and that this contribution is the same for all vertices of P (in fact, it is 2^{-l}). Moreover, using the matrices given in Lemma 2.5, it is easy to see that only squarefree terms contribute to the constant term of $T(p,j)E(P)$, and that the contribution for any vertex f is equal to

$$\sum_{M\in\mathscr{L}_j} (\operatorname{Ind}\sigma_{f_M})^{-1}A(f_M,\mathbf{1}). \tag{37}$$

But under $T(p,j)$ the constant term of $E(P)$ is multiplied by $G_{j,l}$, and because the contribution of each vertex is the same, we see that (37) equals $G_{j,l}/2^l$. This completes the proof of (36), and the proof of Theorem 1.8.

Remark 5.9. We expect that Theorem 1.8 holds if P is replaced by a general simple lattice polytope, although the argument presented here does not prove this. In fact, Theorem 1.4 suggests that the analogous result for a general lattice polytope should hold, and indeed for the vector partition functions studied in [6].

Remark 5.10. The role of the polytopes $\prod_{F\supset f}\Delta_{\pi(F)}$ in the proof of Theorem 1.8 is very similar to the role of "basis sequences" in the theory of characteristic classes and genera (cf. [15, p. 79]). This is not a coincidence, since the machine behind the computation of c_l in Theorem 4.8 is the Hirzebruch–Kawasaki–Riemann–Roch theorem.

6. Examples of distribution relations

6.1. In this final section, we give examples of the identities appearing in the proof of Theorem 1.8, and directly prove them by exhibiting their connection with special values of the Hurwitz zeta function.

6.2. Let u be a real number, and let k be a positive integer. Consider the special value of the (symmetrized) Hurwitz zeta function

$$\zeta(k,u) = \sum_{m\in\mathbb{Z}}{}' \frac{1}{(m+u)^k}.$$

Here the prime next to the summation means to omit the meaningless term that arises when $u \in \mathbb{Z}$. The series is absolutely convergent unless $k = 1$, in which case we define the value of $\zeta(1, u)$ to be the limit of the partial sums with $|m| < C$ as $C \to \infty$. Define the *circle functions* $\theta_k(u)$ by the series expansion

$$\frac{z}{\exp(z - 2\pi i u) - 1} = \sum_{k=0}^{\infty} \theta_k(u) \frac{z^k}{k!}.$$

If $u > 0$ and $k > 1$, then $\theta_k(0) = B_k$, the kth Bernoulli number as in §4.6. However, note that $c_1(0) = -B_1$.

By a result of Euler, for all u we have

$$\zeta(k, u) = \begin{cases} -\dfrac{(2\pi i)^k}{k!} \theta_k(u), & k > 1, \\ -\dfrac{(2\pi i)^k}{k!} (\theta_k(u) + 1/2), & k = 1. \end{cases} \tag{38}$$

6.3. Now fix a positive integer n, and suppose $k > 1$. It is easy to see that

$$\sum_{j=0}^{n-1} \zeta(k, j/n) = n^k \zeta(k, 0).$$

Using (38), this becomes

$$\sum_{j=1}^{n-1} \theta_k(j/n) = (n^k - 1) B_k. \tag{39}$$

Comparing the definition of $c(a, k)$ from §4.6 yields

$$c(a, k) = \frac{(-1)^k}{k!} \theta_k(u), \qquad a = \exp(-2\pi i u),$$

which in (39) gives

$$\sum_{j=1}^{n-1} c(\omega^j, k) = \frac{n^k - 1}{k!} B_k, \qquad k > 1. \tag{40}$$

Here we have written $\omega = \exp(2\pi i/n)$ and used the fact that the sum on the left of (40) is real. In fact, (40) remains true if we take $k = 1$.

6.4. Let now P be a 3-dimensional nonsingular lattice polytope; we investigate the computation of $T(p, 1)$ on c_1. We focus on the squarefree case, since no Dedekind sums arise in the nonsquarefree case.

So let f be an edge of P. The key identity (28) becomes

$$\sum_{M \in \mathscr{L}_1} A(f_M, \mathbf{1}) \frac{\operatorname{Vol} f_M}{\operatorname{Ind} \sigma_{f_M}} = \frac{p^2 + 2p}{4} \operatorname{Vol} f. \tag{41}$$

We break the coefficient $A = A(f_M, \mathbf{1})$ into two parts,

$$A = A_{\text{ns}} + A_{\text{s}},$$

where A_{ns} corresponds to $g = 0$ in (24), and A_{s} corresponds to $g \neq 0$. The latter term appears only if $\operatorname{Ind} \sigma_{f_M} \neq 1$. Note that $A_{\mathrm{ns}} = 1/4$.

To analyze the left of (41), we use Proposition 2.7. Figure 1 shows $\overline{V}$ with the two subspaces $\overline{V}_f$ and $\overline{C}_f$. The subspaces $\overline{C}_1$ and $\overline{C}_2$ are the 1-dimensional subspaces corresponding to the two facets containing f. For simplicity, we draw these subspaces, and the subspaces that follow, by drawing their images in $\mathbb{P}(\overline{V}) = \mathbb{P}^2(\mathbb{F}_p)$. By abuse of notation, we denote a subspace of $\overline{V}$ and the subspace it induces in $\mathbb{P}(\overline{V})$ by the same symbol.

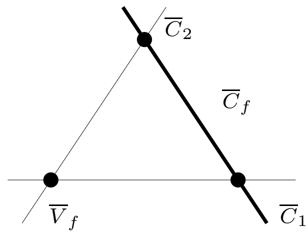

FIG. 1. Subspaces in $\overline{V}$ for an edge in a 3-dimensional polytope.

Each $M \in \mathscr{L}_1$ corresponds to a point $\overline{M} \in \mathbb{P}(\overline{V})$. By Proposition 2.7, we have $\operatorname{Vol} f_M = \operatorname{Vol} f$ unless $\overline{M} = \overline{V}_f$, in which case $\operatorname{Vol} f_M = p \operatorname{Vol} f$. Also $A_{\mathrm{s}} = 0$ unless $\overline{M}$ meets $\overline{C}_f \smallsetminus \{\overline{C}_1 \cup \overline{C}_2\}$. Hence there are $p-1$ nonzero A_{s}, and since $c(a,1) = 1/(1-a)$, each nonzero A_{s} has the form

$$A_{\mathrm{s}}(\alpha,\beta) = \sum_{i=1}^{p-1} \frac{1}{(1-\omega^{\alpha j})(1-\omega^{\beta j})}, \quad \omega = \exp(2\pi i/p),$$

for some nonzero integers $1 \le \alpha, \beta \le p-1$. The value of $A_{\mathrm{s}}(\alpha,\beta)$ depends only on the point $[\alpha : \beta] \in \mathbb{P}^1(\mathbb{F}_p)$. See Figure 2 for the four nonzero $A_{\mathrm{s}}(\alpha,\beta)$ when $p = 5$. The pairs (α,β) are given below each lattice, and the four terms in $A_{\mathrm{s}}(\alpha,\beta)$ correspond to the four grey dots.

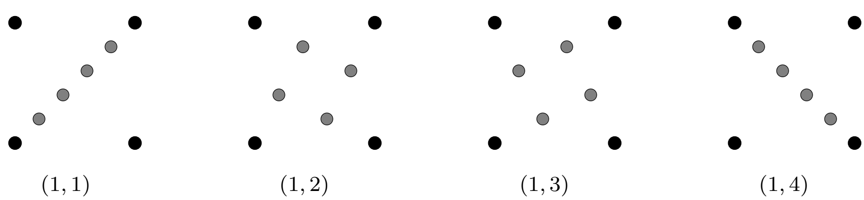

FIG. 2. Four superlattices giving a nonzero $A_{\mathrm{s}}(\alpha,\beta)$.

By (40), the contribution from the singular Hecke images is

$$\sum_{\substack{[\alpha:\beta]\in\mathbb{P}^1(\mathbb{F}_p)\\ [\alpha:\beta]\neq 0,\infty}} A_{\mathrm{s}}(\alpha,\beta) = \sum_{i,j=1}^{p-1} \frac{1}{(1-\omega^i)(1-\omega^j)} = \frac{(p-1)^2}{4}.$$

With this in hand it is easy to complete the analysis of (41). We break $\mathbb{P}(\overline{V})$ into four disjoint subsets,

$$\mathbb{P}(\overline{V}) = S_1 \cup S_2 \cup S_3 \cup S_4,$$

where

- $S_1 = \overline{V}_f$,
- $S_2 = \overline{C}_1 \cup \overline{C}_2$,
- $S_3 = \overline{C}_f \smallsetminus S_2$,
- $S_4 = \mathbb{P}(\overline{V}) \smallsetminus \{S_1 \cup S_2 \cup S_3\}$.

The relevant contributions are given in Table 3, and one easily sees that (41) holds.

TABLE 3. Summary of $T(p,1)$ on c_1 for a 3-dimensional polytope

S_i	$\#S_i$	$\mathrm{Vol}\, f_M / \mathrm{Vol}\, f$	$\mathrm{Ind}\, \sigma_f / \mathrm{Ind}\, \sigma_{f_M}$	$\sum_{\overline{M}\in S_i} A(f_M, \mathbf{1})$
S_1	1	p	1	$1/4$
S_2	2	1	1	$1/2$
S_3	p^2-1	1	1	$(p^2-1)/4$
S_4	$p-1$	1	$1/p$	$(p^2-1+p-1)/(4p)$

6.5. The computation of $T(p,2)$ on c_1 is similar. The only difference is that the sum over M corresponds to a sum over lines in $\mathbb{P}(\overline{V})$, and that we obtain a nonzero A_{s} exactly when a line meets S_3 in a point. For example, in Figure 3 a nonzero $A_{\mathrm{s}}(\alpha,\beta)$ arises from the solid triangle. Hence each nonzero $A_{\mathrm{s}}(\alpha,\beta)$ occurs with

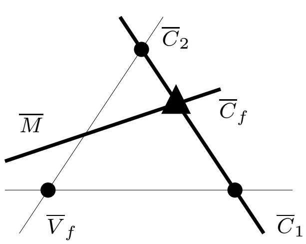

FIG. 3. Computing $T(p,2)$ on c_1.

multiplicity p. Taking this into account, as well as which lines meet $\overline{V}_f$, yields

$$\sum_{M\in\mathscr{L}_2} A(f_M,\mathbf{1})\frac{\operatorname{Vol} f_M}{\operatorname{Ind}\sigma_{f_M}} = \frac{2p^2+p}{4}\operatorname{Vol} f.$$

7. The regularized Ehrhart polynomial on average

7.1. Let P be a fixed n-dimensional lattice polytope respect to the lattice L. We can define a "regularized" version $\widetilde{E}(P)$ of $E(P)$ by

$$\widetilde{E}(P)(t) := E(P)(t) - \operatorname{Vol}(P)t^n.$$

Suppose $\mathscr{M}$ is a finite set of superlattices of L of finite coindex. We can define the average regularized Ehrhart polynomial of P with respect to the family $\mathscr{M}$ by

$$\widetilde{E}_{\mathrm{avg}}(P,\mathscr{M}) = \frac{1}{\#\mathscr{M}}\sum_{M\in\mathscr{M}}\widetilde{E}(P_M).$$

Our goal in this section is to show how Theorem 1.4 can be used to derive limiting formulas for $\widetilde{E}_{\mathrm{avg}}(P,\mathscr{M})$ as $\mathscr{M}$ ranges over families of superlattices satisfying certain arithmetical conditions.

7.2. As a first example, fix a prime p, and suppose $\mathscr{M} = \mathscr{L}_1(p)$ consists of all superlattices of L of coindex p. Then by definition

$$\widetilde{E}_{\mathrm{avg}}(P,\mathscr{M}) = G_{1,n}^{-1}\sum_{l=0}^{n-1} T(p,1)c_l t^l = G_{1,n}^{-1}\sum_{l=0}^{n-1}\nu_{n,1,l}(p)c_l t^l.$$

By Lemma 3.2, we have

$$\nu_{n,1,l}(p) = G_{1,n} + p^l - 1 = p^{n-1} + \cdots + p^{l+1} + 2p^l + p^{l-1} + \cdots + p.$$

This implies the following result:

Proposition 7.3.

$$\lim_{\substack{p\to\infty\\ p\ prime}} \widetilde{E}_{\mathrm{avg}}(P,\mathscr{L}_1(p)) = 2c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \cdots + c_1 t + 1.$$

7.4. We can use the relations in the Hecke algebra to derive similar results for more general sets of superlattices. Let $T_p(n,k)$ be the operator $T(n,k)$ at the prime p, and write $T(N)$ for the operator that associates to any lattice L the set of superlattices of coindex N. Suppose N has prime factorization $\prod p_j^{e_j}$. Then, in the algebra $\mathscr{H}$ generated by the $T_p(n,k)$ as p ranges over all primes, we have [22, Theorem 3.21]

$$T(N) = \prod T(p_j^{e_j}),$$

and the operators $T(p^e)$ satisfy the (formal) identity

$$\sum_{e=0}^{\infty} T(p^e)X^e = \Big(\sum_{i=0}^{n}(-1)^i p^{i(i-1)/2}T_p(n,k)X^i\Big)^{-1}.$$

As an example of this, suppose $\mathscr{M}(p^2)$ is the set of all superlattices of L of coindex p^2. Note that $\mathscr{M}(p^2) \neq \mathscr{L}_2$, i.e. $T(p^2) \neq T_p(n,2)$. In fact in $\mathscr{H}$ we have the relation

$$T(p^2) = T_p(n,1)^2 - pT_p(n,2).$$

One can easily show

$$\#\mathscr{M}(p^2) = G_{1,n}^2 - pG_{2,n} = G_{2,n+1},$$

and then from Lemma 3.2 we find the following:

Proposition 7.5.

$$\lim_{\substack{p\to\infty \\ p\ prime}} \widetilde{E}_{\mathrm{avg}}(P, \mathscr{M}(p^2)) = 3c_{n-1}t^{n-1} + c_{n-2}t^{n-2} + \cdots + c_1 t + 1.$$

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