The congruences of Clausen-von Staudt and Kummer for half-integral weight Eisenstein series

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In 1899 Hurwitz [4] defined analogues of the Bernoulli numbers for imaginary quadratic fields and proved that they satisfy a congruence similar to that of Clausen-von Staudt for the ordinary Bernoulli numbers. These so-called Bernoulli-Hurwitz numbers are essentially values of integral weight Eisenstein series on the full modular group and the congruences can be expressed entirely in terms of these series [3], [5]. In this note we would like to give the analogous congruences of Clausen-von Staudt and Kummer for half-integral weight Eisenstein series.

We consider only the simplest case: that of the series $\mathcal{H}_{k+\frac{1}{2}}$ ($k \geq 2$) introduced by Cohen [2], which are a linear combination of the two Eisenstein series of weigth $k+\frac{1}{2}$ on $\Gamma_0(4)$. They have the following q-expansions

$$\mathcal{H}_{k+\frac{1}{2}}(z) = \sum_{n=0}^{\infty} H(k,n)q^n, \qquad q = e^{2\pi i z}, \quad \Im(z) > 0, \tag{1}$$

where $H(k,n) = L(1-k, (-1)^k n)$ and $L(s,D)$ is an L-series defined as follows. For $D = 0$, $L(s,0) = \zeta(2s-1)$. For $D \neq 0$, $L(s,D)$ is identically zero unless D is a discriminant (i.e. $D \equiv 0$ or $1 \bmod 4$), and in that case write $D = D_0 f^2$, where D_0 is the discriminant of $\mathbf{Q}(\sqrt{D})$ (allowing also the split case $D_0 = 1$)

and $f \geq 1$, then

$$L(s,D) = \sum_{n=1}^{\infty} (\frac{D_0}{n}) n^{-s} \cdot \sum_{d|f} \mu(d) (\frac{D_0}{d}) d^{-s} \sigma_{1-2s}(\frac{f}{d}), \tag{2}$$

where μ is Moebius function, $(\frac{D_0}{\cdot})$ is the Kronecker symbol, and $\sigma_\nu(n) = \sum_{d|n} d^\nu$ ($\nu \in \mathbf{C}$).

We need to introduce some notation. Let

$$\theta = \sum_{n \in \mathbf{Z}} q^{n^2} = 1 + 2q + 2q^4 + \cdots, \qquad q = e^{2\pi i z}, \quad \Im(z) > 0,$$

$$F = \sum_{n \geq 1, \text{odd}} \sigma_1(n) q^n = q + 4q^3 + 6q^5 \cdots,$$

(forms on $\Gamma_0(4)$ of weights $\frac{1}{2}$ and 2 respectively), and

$$t = \frac{F}{\theta^4} = q - 8q^2 + 44q^3 - 192q^4 \cdots$$

(a Hauptmodul for $\Gamma_0(4)$; it is related to the more familiar parameter λ for $\Gamma(2)$ by $\lambda(2z+1) = 16t(z)$).

The forms θ and F generate a ring of modular forms on $\Gamma_0(4)$ containing all forms $\mathcal{H}_{k+\frac{1}{2}}$ with $k \geq 2$ (see [2]). Since the coefficients of $\mathcal{H}_{k+\frac{1}{2}}$ are rational there is a polynomial $\Phi_{k+\frac{1}{2}}$, with rational coefficients and degree at most $k/2$, such that

$$2k \cdot \frac{\mathcal{H}_{k+\frac{1}{2}}}{\theta^{2k+1}} = \Phi_{k+\frac{1}{2}}(t). \tag{3}$$

Finally, for odd primes p let

$$A_p(t) = \sum_{n=0}^{(p-1)/2} \left(\begin{array}{c} 2n \\ n \end{array} \right)^2 t^n. \tag{4}$$

This polynomial is related to Hasse invariants of elliptic curves modulo p. Precisely, for $\lambda \neq 0, 1$ in $\mathbf{Z}/p\mathbf{Z}$, the number of points of the elliptic curve $y^2 = x(x-1)(x-\lambda)$ over $\mathbf{Z}/p\mathbf{Z}$ is congruent, modulo p, to $1 + (-1)^{\frac{p-1}{2}} A_p(t)$, where $\lambda \equiv 16t \bmod p$.

We can now state the Clausen-von Staudt and Kummer congruences.

Theorem *Let p be an odd prime, $k \geq 2$ an integer, A_p as in (4), and $\Phi_{k+\frac{1}{2}} \in \mathbf{Q}[t]$ the polynomial defined by (3).*

(1) If $p-1$ divides $2k$ then the coefficients of $p \cdot \Phi_{k+\frac{1}{2}}$ are p-integral, and

$$p \cdot \Phi_{k+\frac{1}{2}} \equiv A_p^{\frac{k_p}{p-1}} \bmod p\mathbf{Z}_p[t],$$

where

$$k_p = \begin{cases} k & \text{if } k \equiv 0 \bmod p-1 \\ k - \frac{1}{2}(p-1) & \text{if } k \equiv \frac{p-1}{2} \bmod p-1 \end{cases}.$$

(2) If $p-1$ does not divide $2k$ then the coefficients of $\Phi_{k+\frac{1}{2}}/k$ are p-integral, and

$$\Phi_{k+p-\frac{1}{2}}/(k+p-1) \equiv A_p \cdot \Phi_{k+\frac{1}{2}}/k \bmod p\mathbf{Z}_p[t].$$

Remarks (1) By choosing $p=3$ in (1) above we see that $\Phi_{k+\frac{1}{2}}$ has degree $[k/2]$. As was pointed out to us by J. Sturm, this would also follow from analyzing the behaviour of $\mathcal{H}_{k+\frac{1}{2}}, \theta$ and F at the cusps.

(2) The Theorem gives no more p-adic information about the values of the L-series $L(s,D)$ than was put into it and as we will see next, this amounts to the classical Clausen-von Staudt and Kummer congruences.

(3) The function t is holomorphic on the upper-half plane and has p-integral coefficients at every cusp for each odd prime p. It follows that for a CM point z_0, $t(z_0)$ is an algebraic number, integral outside primes dividing 2. We can therefore translate the congruences of the Theorem to the values $2k \cdot \mathcal{H}_{k+\frac{1}{2}}(z_0)/\theta(z_0)^{2k+1}$ as in Hurwitz.

Proposition *Let $k \geq 2$ be an integer, p an odd prime, and $\mathcal{H}_{k+\frac{1}{2}}$ the modular form defined by (1).*

(1) If $p-1$ divides $2k$ then the coefficients of $p \cdot 2k \cdot \mathcal{H}_{k+\frac{1}{2}}$ are p integral and

$$p \cdot 2k \cdot \mathcal{H}_{k+\frac{1}{2}} \equiv \theta^{e_k} \bmod p\mathbf{Z}_p[[q]],$$

where

$$e_k = \begin{cases} 1 & \text{if } k \equiv 0 \bmod p-1 \\ p & \text{if } k \equiv \frac{p-1}{2} \bmod p-1 \end{cases}.$$

(2) If $p-1$ does not divide $2k$ then the coefficients of $\mathcal{H}_{k+\frac{1}{2}}$ are p-integral and

$$\mathcal{H}_{k+p-\frac{1}{2}} \equiv \mathcal{H}_{k+\frac{1}{2}} \bmod p\mathbf{Z}_p[[q]].$$

Remark More general statements like (2) above allow one to define p-adic limits of the forms $\mathcal{H}_{k+\frac{1}{2}}$, see [6].

Proof The classical Clausen-von Staudt and Kummer congruences for generalized Bernoulli numbers imply the following. Let D be a fundamental discriminant (allowing also $D=1$) or $D=0$, and $k \geq 1$. If $p-1$ divides $2k$ then $2k \cdot L(1-k, D)$ is p-integral unless: (i) $D = (\frac{-1}{p})p$ and $k \equiv \frac{1}{2}(p-1) \bmod p-1$, (ii) $D=1$ and $k \equiv 0 \bmod p-1$, or (iii) $D=0$. In those cases, $p \cdot 2k \cdot L(1-k, D)$ is p-integral and

$$\begin{array}{rrcllcl}
(i) & p \cdot 2k \cdot L(1-k, \left(\frac{-1}{p}\right)p) & \equiv & 2 \bmod p\mathbf{Z}_p, & k & \equiv & \frac{1}{2}(p-1) \bmod p-1 \\
(ii) & p \cdot 2k \cdot L(1-k, 1) & \equiv & 2 \bmod p\mathbf{Z}_p, & k & \equiv & 0 \bmod p-1 \\
(ii) & p \cdot 2k \cdot L(1-k, 0) & \equiv & 1 \bmod p\mathbf{Z}_p, & 2k & \equiv & 0 \bmod p-1.
\end{array}$$

If $p-1$ does not divide $2k$ then $L(1-k, D)$ is p integral and

$$L(1-(k+p-1), D) \equiv L(1-k, D) \bmod p\mathbf{Z}_p.$$

Therefore, according to the definition (2), all that remains to be proved is that for any $f \geq 1$, $k \geq 1$, and D_0 a fundamental discriminant the number

$$c_f(k) = \sum_{d|f} \mu(d)(\frac{D_0}{d})d^{k-1}\sigma_{2k-1}(\frac{f}{d}),$$

which is clearly an integer, is congruent to 1 modulo p for cases (i) and (ii) above, and that in general $c_f(k+p-1) \equiv c_f(k) \bmod p$. The last congruence is clear; to prove the first, consider the Dirichlet series

$$\sum_{f=1}^{\infty} c_f(k) f^{-s} = \prod_l \frac{(1-(\frac{D_0}{l})l^{k-1-s})}{(1-l^{-s})(1-l^{2k-1-s})}.$$

For cases (i) and (ii) it is easy to check that each Euler factor is formally congruent to $(1-l^{-s})^{-1}$ and hence $c_f(k) \equiv 1 \bmod p$ for every f. This concludes the proof. $\square$

Proof of the Theorem At this point, we could argue as in Katz [5]; we prefer to give a proof along more classical lines like [1] and [3].

Since θ is a unit in $\mathbf{Z}_p[[q]]$ the Proposition implies that

$$p \cdot \Phi_{k+\frac{1}{2}}(t) \equiv \theta^{-2k_p} \bmod p\mathbf{Z}_p[[q]], \qquad 2k \equiv 0 \bmod p-1, \tag{5}$$

and

$$\Phi_{k+p-\frac{1}{2}}/(k+p-1) \equiv \theta^{2(1-p)} \cdot \Phi_{k+\frac{1}{2}}/k \bmod p\mathbf{Z}_p[[q]], \qquad 2k \not\equiv 0 \bmod p-1. \tag{6}$$

Recall that $t = \frac{F}{\theta^4} = q - 8q^2 + 44q^3 \cdots$. Following Abel's advice we formally invert the relation between t and q and regard (5) and (6) as identities between power series in $\mathbf{Z}_p[[t]]$. We now use the following remarkable identity expressing θ^2 explicitly as a power series in t

$$\theta^2 = F(\frac{1}{2}, \frac{1}{2}; 1; 16t) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n = 1 + 4t + 36t^2 \cdots,$$

(here F is the standard hypergeometric function). This is a classical formula of Jacobi, see for example [7, p. 486].

Take $k = k_p = p-1$ in (5); we know that $\Phi_{p-\frac{1}{2}}$ is a polynomial of degree at most $(p-1)/2$ so

$$p \cdot \Phi_{p-\frac{1}{2}} \equiv \theta^{2(1-p)} \equiv \sum_{n=0}^{(p-1)/2} \binom{2n}{n}^2 t^n \bmod p \cdot \mathbf{Z}_p[[t]].$$

Therefore,

$$p \cdot \Phi_{k+\frac{1}{2}} \equiv A_p^{\frac{k_p}{p-1}} \bmod p \cdot \mathbf{Z}_p[t], \qquad 2k \equiv 0 \bmod p-1,$$

and

$$\Phi_{k+p-\frac{1}{2}}/(k+p-1) \equiv A_p \cdot \Phi_{k+\frac{1}{2}}/k \bmod p \cdot \mathbf{Z}_p[t], \qquad 2k \not\equiv 0 \bmod p-1,$$

which is what we wanted to prove. □

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