Hypergeometric Motives

Beeger Lecture

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Euler 1760

DE

SERIEBVS DIVERGENTIBVS.

Auctore LEON. EVLERO.

§. 1.

Cum ſeries conuergentes ita definiantur, vt conſtent terminis continuo decreſcentibus, qui tandem, ſi ſeries in infinitum proceſſerit penitus euaneſcant; facile intelligitur, quarum ſerierum termini infiniteſimi non in nihilum abeant, ſed vel finiti maneant, vel in infinitum excreſcant, eas, quia non ſunt conuergentes, ad claſſem ſerierum diuergentium referri oportere. Prout igitur termini ſeriei vltimi, ad quos progreſſione in infinitum continuata peruenitur, fuerint vel magnitudinis finitae, vel infinitae, duo habebuntur ſerierum diuergentium genera, quorum vtrumque porro in duas ſpecies ſubdiuiditur, prout vel omnes termini eodem ſint affecti ſigno, vel ſigna + et − alternatim ſe excipiant. Omnino ergo habebimus quatuor ſerierum diuergentium ſpecies, ex quibus maioris perſpicuitatis gratia aliquot exempla ſubiungam.

I. . . . $1 + 1 + 1 + 1 + 1 + 1 + \text{etc.}$

$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \text{etc.}$

II. . . . $1 - 1 + 1 - 1 + 1 - 1 + \text{etc.}$

$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} + \text{etc.}$

III. . . . $1 + 2 + 3 + 4 + 5 + 6 + \text{etc.}$

$1 + 2 + 4 + 8 + 16 + 32 + \text{etc.}$

IV. . . . $1 - 2 + 3 - 4 + 5 - 6 + \text{etc.}$

$1 - 2 + 4 - 8 + 16 - 32 + \text{etc.}$

Wallis 1685

§. 13. His praemiſſis neminem fore arbitror, qui me reprehendendum putet, quod in ſummam ſequentis ſeriei diligentius inquiſiuerim:

$$1 - 1 + 2 - 6 + 24 - 120 + 720 - 5040 + 40320 - \text{etc.}$$

quae eſt ſeries a Wallifio hypergeometrica dicta, ſignis alternantibus inſtructa. Haec ſeries autem eo magis notatu digna videtur, quod plures ſummandi methodos, quae mihi alias in huiusmodi negotio ingentem vſum praeſtiterunt, hic fruſtra tentauerim. Primo quidem dubitare licet, vtrum haec ſeries ſummam habeat finitam, nec ne? quia multo magis diuergit, quam vlla ſeries geometrica; ſummam autem geometricarum eſſe finitam, extra dubium eſt poſitum. Veruntamen cum in geometricis diuergentia non obſtet, quominus ſint ſummabiles, ita veriſimile videtur, et hanc ſeriem hypergeometricam ſummam habere finitam. Quaeritur ergo in numeris, proxime ſaltem, valor eius expreſſionis finitae, ex cuius euolutione ipſa ſeries propoſita naſcitur.

Hypergeometric series

- $$ {}_2F_1\left[\begin{matrix}\alpha & \beta \\ \gamma & \end{matrix} \mid t\right] := \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{t^n}{n!}, \qquad |t| < 1. $$

- The coefficients A_n satisfies the recursion

$$ (n+1)(n+\gamma)A_{n+1} = (n+\alpha)(n+\beta)A_n. $$

- Consequently, the series satisfies the second order differential equation

$$ t(1-t)\frac{d^2y}{dx^2} + (\gamma - (\alpha+\beta+1)t)\frac{dy}{dx} - \alpha\beta y = 0. $$

Integral representation

- For $\Re(\gamma) > \Re(\beta) > 0$

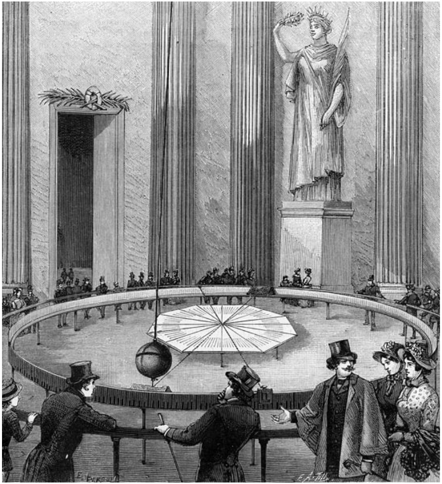
$$ {}_2F_1\left[\begin{matrix}\alpha & \beta \\ \gamma & \end{matrix} \mid t\right] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-tx $$

- For example

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1 & \end{matrix} \mid t\right] = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{x(1-x)(1-tx)}}\, dt $$

- This is an elliptic integral

Pendulum

- The period of a pendulum for arbitrary amplitudes involves elliptic integrals.
- 

Elliptic curves

- Concretely, take $t \in \mathbb{C}$ not equal to 0 or 1. Then

$$\varpi(t) := \pi \cdot {}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1 & \end{matrix} \;\middle|\; t\right]$$

is a period of the Legendre elliptic curve

$$E_t : \quad y^2 = x(1-x)(1-tx)$$

- Namely

$$\varpi(t) = \int_\gamma \omega,$$

- where $\omega := dx/y$ is a holomorphic differential and γ a closed cycle in $E_t(\mathbb{C})$.

Algebraic Geometry

- In general, the integral representation connects $_2F_1$ to algebraic geometry ($\alpha, \beta, \gamma \in \mathbb{Q}$ meeting some simple conditions).
- It shows it appears as a *period function*.
- I.e., as the integral of a holomorphic differential on a family of algebraic varieties (curves).
- Period functions satisfy linear differential equations (Picard-Fuchs), ultimately because cohomology is finite dimensional.
- Their singularities are regular.

Arithmetic

- Note

$$
{}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ 1 & \end{matrix} \,\middle|\, t\right] = \sum_{n\geq 0} \binom{2n}{n}^2 \left(\frac{t}{16}\right)^n
$$

- Take $t \in \mathbb{F}_p$ different from 0 and 1; here $\mathbb{F}_p$ is a finite field $p > 2$ prime.
- Then (Deuring)

$$
\#E_t(\mathbb{F}_p) \equiv p - A_p(t) + 1 \bmod p,
$$

- where

$$
A_p(t) := (-1)^{(p-1)/2} \sum_{n=0}^{(p-1)/2} \binom{2n}{n}^2 \left(\frac{t}{16}\right)^n .
$$

L-functions

- Fix now $t \in \mathbb{Q}$ different from 0 and 1.
- For $p \nmid N$ define a_p by

$$\#E(\mathbb{F}_p) =: p - a_p + 1$$

- With these form the L-function of E

$$\Lambda(E, s) := \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) \prod_p (1 - a_p p^{-s} + p^{-2s})^{-1},$$

- By modularity (Wiles et al) $\Lambda(s)$ extends to all s and satisfies

$$\Lambda(2 - s) = \pm\Lambda(s)$$

General (motivic) L-functions

- In general, for a pure motive M over $\mathbb{Q}$ of rank d we have

$$\Lambda(M, s) = N^{s/2} L_\infty(s) \prod_p L_p(M, p^{-s})^{-1},$$

- where $L_p(M, T)$ are polynomials of degree at most d (known as *Euler factors*)
- For $p \nmid N$ (the *conductor*)

$$L_p(M, T) = \prod_{i=1}^{d} (1 - \xi_i T)^{-1}, \qquad |\xi_i| = p^{w/2}$$

- the integer w is called the *weight* of M.

General hypergeometric series

- $$ {}_dF_{d-1}\begin{bmatrix}\alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1}\end{bmatrix} \mid t \Bigg] := \sum_{n\geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!}, $$
- Satisfies a linear differential equation of order d
- with regular singularities at $t = 0, 1, \infty$.
- Integral representation

$$ C \int_0^1 \cdots \int_0^1 \prod_{i=1}^{d-1} x_i^{\alpha_i - 1}(1-x_i)^{\beta_i-\alpha_i-1}(1-tx_1\cdots x_d)^{-\alpha_d}\, dx $$

Algebraic Geometry

- Take $\alpha := (\alpha_1, \ldots, \alpha_d), \beta := (\beta_1, \ldots, \beta_{d-1})$ multisets in $\mathbb{Q}$ disjoint modulo $\mathbb{Z}$.
- Then
$$ {}_dF_{d-1}\left[\begin{matrix}\alpha_1 & \ldots & \alpha_d \\ \beta_1 & \ldots & \beta_{d-1}\end{matrix} \mid t\right] $$
is a period function of a family of varieties
- (by the integral representation)
$$ y^m = \prod_{i=1}^{d-1} x_i^{a_i}(1-x_i)^{b_i}(1-tx_1\cdots x_d)^{a_d} $$
for appropriate integers $a_1, \ldots, a_d; b_1, \ldots, b_{d-1}$ and m.

Hypergeometric Motives

- *Conjecture:* There is a family of pure motives $\mathcal{H}(\alpha, \beta \mid t)$ associated to the data (α, β) defined over a cyclotomic field.

- Fix $t_0 \in \mathbb{Q}$ different from 0 and 1 and specialize, say $M := \mathcal{H}(\alpha, \beta \mid t_0)$

- The Euler factors $L_p(T)$ of M are computable in terms of finite analogues of

$$ {}_dF_{d-1}\left[\begin{matrix}\alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1}\end{matrix} \;\middle|\; t\right] $$

  for all but finitely many primes.

- Other ingredients of the L-function of M, weight w, gamma factor $L_\infty(s)$, etc. are computable (or

MAGMA implementation

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  $$M = \mathcal{H}((1/2, 1/2, 1/2, 1/2), (0, 0, 0, 0) \,|\, t_0), \qquad t_0 = 2$$

- Motive of rank 4 and weight 3. Euler factor at $p = 2$ (of good reduction: $N = 255$).

  $$L_2(T) = 64T^4 + 8T^3 + 6T^2 + T + 1$$

- `H := HypergeometricData([1/2,1/2,1/2,1/2],[0,0`
- `L := LSeries(H,t0 : BadPrimes:=[<2,0,L2>]);`
- `CFENew(L);`
- `0.00000000000000000000000000000000000`

MAGMA implementation

- $\alpha = [1/8, 1/3, 3/8, 5/8, 2/3, 7/8], \quad \beta = [0, 1/6, 1/2, 1/2, 1/2, 5/6]$
- $t = 1$ (a singular point)
- Degree drops: $d = 5, w = 2$
- Guess: $N = 2^7 \cdot 3^2, L_2 = 8T^3 - 4T^2 - 2T + 1, L_3 = -27T^3 - 3T^2 + T + 1$
- `> H := HypergeometricData([3,8],[1,2,2,2,6]);`
- `> L:=LSeries(H,1);`
- `> CFENew(L);`
- `> 0.000000000000000000000000000000000000`