Hypergeometric Motives

Fernando Rodriguez Villegas

ICTP

June 2018

Collaborators

H. Cohen, Ph. Candelas, S. Baig, F. Beukers, X. de la Ossa,
A. Mellit, P. Molin, D. Roberts, M. Vashenko, M. Watkins

Motivic L-functions

- $$\Lambda(s) = N^{s/2} L_\infty(s) \prod_p L_p(p^{-s})^{-1}, \qquad \Re(s) > \sigma_0$$
- **Conductor:** N, positive integer
- **Euler factors:** $L_p(T)$, polynomials in $Z[T]$
- **Degree:** d, degree of L_p (generically)
- **Weight:** w, an integer

$$L_p(T) = \prod_{i=1}^{d} (1 - \xi_i T), \qquad |\xi_i| = p^{w/2}, \qquad p \nmid N$$

- **Infinity factor:** $L_\infty(s)$, product of gamma factors
- **Functional equation:** (expected)

$$\Lambda(w + 1 - s) = \epsilon \Lambda(s), \qquad \epsilon = \pm 1$$

Hodge numbers

- Refinement of the rank, determines $L_\infty(s)$.
- $$h^{p,q} \in \mathbb{Z}_{\geq 0}, \qquad p + q = w$$
- $$h^{p,q} = h^{q,p}, \qquad \sum_{p,q} h^{p,q} = d$$
- Hodge vector (up to Tate twists $w \mapsto w \pm 2r$)

$$\mathbf{h} := (h^{w,0}, h^{w-1,1}, \ldots, h^{0,w}), \qquad h^{w,0} \neq 0$$

- $$h^{p,p} = h^{p,p}_{+} + h^{p,p}_{-}$$

Gamma factors

- (Serre)

$$L_\infty(s) = \prod_p \Gamma_{\mathbb{R}}(s-p)^{h_+^{p,p}} \Gamma_{\mathbb{R}}(s-p+1)^{h_-^{p,p}} \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}}$$

- $$\Gamma_{\mathbb{R}}(s) := (2\pi)^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$$

Question

How are Hodge vectors distributed among all motives?

Source of L-functions

- Automorphic Forms.
- Cohomology of algebraic varieties.
- Typically appear as a piece of a bigger object cut out by endomorphisms.

Automorphic Forms

- Hard to deal with $h^{p,q} > 1$.
- Usual modular forms

k	$\mathbf{h}$
1	(2)
2	$(1,1)$
3	$(1,0,1)$
4	$(1,0,0,1)$

- Hard to compute L_p in general.

Algebraic Varieties

- Griffiths transversality $\to$ no gaps in $\mathbf{h}$.
- Example: quintic threefold

$$X : F(x_1, \ldots, x_5) = 0$$

- $$H := H^3(X, \mathbb{Q}), \qquad d = \dim H = 204, \qquad w = 3$$
- Dwork pencil

$$X_\psi : x_1^5 + \cdots x_5^5 - 5\psi x_1 \cdots x_5 = 0$$

- $$A \subseteq \mathrm{Aut}(X_\psi), \qquad x_i \mapsto \zeta_i x_i, \qquad \zeta_1^5 = \cdots = \zeta_5^5 = \zeta_1 \cdots \zeta_5 = 1$$
- $$V := H^A, \qquad d = \dim V = 4, \qquad \mathbf{h} = (1, 1, 1, 1), \qquad w = 3$$

Hypergeometric Motives

- $q_0, q_\infty \in \mathbb{Z}[T]$, coprime, same degree d, roots are roots of unity.
- Get associated family of motives $\mathcal{H}(t)$ with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.
- $\mathcal{H}(t)$ has rank d and a computable weight w in terms of q_0, q_∞.
- More precisely, can compute Hodge numbers hence $L_\infty(s)$
- For fixed $t \in \mathbb{Q}$: formula for $L_p(T)$ for $p \notin S$
- Katz's hypergeometric trace

Examples

- Belyi polynomials $c := a + b$

$$\mathbb{Q}[x]/(B(a,b;t)), \qquad B(a,b;t) := x^a(1-x)^b - \frac{a^a b^b}{c^c} t$$

- $$\frac{q_\infty}{q_0} = \frac{T^c - 1}{(T^a - 1)(T^b - 1)}$$

- Legendre family of elliptic curves: $H^1(E_t)$

$$E_t : \quad y^2 = x(x-1)(x-t)$$

- $$\frac{q_\infty}{q_0} = \frac{(T+1)^2}{(T-1)^2}$$

- Dwork pencil piece: V

$$\frac{q_\infty}{q_0} = \frac{T^5 - 1}{(T-1)^5}$$

Hypergeometric series

- Hypergeometric series $|t| < 1$ (classically $\beta_d = 1$)

$$u(t) = {}_dF_{d-1}\begin{bmatrix} \alpha_1 & \dots & \alpha_d \\ \beta_1 & \dots & \beta_{d-1} \end{bmatrix} \mid t \Bigg] := \sum_{n \geq 0} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{t^n}{n!},$$

- $$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$$

  is the Pochhammer symbol.

- Satisfies linear differential equation of order d with regular singularities at $t = 0, 1, \infty$.

- Gives rise to a monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \mathrm{GL}(V)$$

- $V :=$ space of local solutions of the DE at $z = t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

Integral representation

- In general ($b_d := 1$)

$$C \int_0^1 \cdots \int_0^1 \prod_{i=1}^{d-1} x_i^{\alpha_i - 1} (1 - x_i)^{\beta_i - \alpha_i - 1} (1 - t x_1 \cdots x_{d-1})^{-\alpha_d} \, dx_1 \cdots dx_{d-1}$$

- Our motive is a piece of the middle cohomology of

$$X_t : \quad y^m = \prod_{i=1}^{d-1} x_i^{a_i} (1 - x_i)^{b_i} (1 - t x_1 \cdots x_{d-1})^{a_d}$$

  cut out by automorphisms $y \mapsto \zeta_m y$ (up to twist by a Hecke character),

- for appropriate a_i, b_i with m a common denominator of α, β
- Note that $\dim X_t = d - 1$ whereas w could be much smaller

Chebyshev example

- Interlacing roots

$$q_\infty = \Phi_{30}, \qquad q_0 = \Phi_1\Phi_2\Phi_3\Phi_5,$$

- $$\begin{aligned} \alpha &= 1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30 \\ \beta &= 1, 1/2, 1/3, 2/3, 1/5, 2/5, 3/5, 4/5 \end{aligned}$$

- $$\frac{q_\infty}{q_0} = \frac{(T^{30}-1)(T-1)}{(T^{15}-1)(T^{10}-1)(T^6-1)}.$$

- $$u(t) := \sum_{n\geq 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!}\left(\frac{t}{M}\right)^n, \qquad M := \frac{30^{30}}{15^{15}\cdot 10^{10}\cdot 6^6}.$$

- $$\frac{(30n)!n!}{(15n)!(10n)!(6n)!} = 1, 77636318760, 53837289804317953893960, \cdots$$

  are integral for every n.

Chebyshev example (cont'd)

- Monodromy group is finite.
- Series $u(t)$: Taylor expansion of an algebraic function of t.
- Degree over $\bar{\mathbb{Q}}(t)$: $483,840$.
- $\mathcal{H}(t)$: Artin representation of degree 8
- $$|W(E_8)| = 696729600 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$$

MAGMA computation I

- $\alpha = (1/4, 3/4, 1/4, 3/4, 1/2, 1/2), \beta = (1/8, 3/8, 5/8, 7/8, 1, 1)$
- $d = 6, w = 3, \mathbf{h} = (1, 2, 2, 1)$
-

```
> H:=HypergeometricData([4,4,2,2],[8,1,1]);

> L:=LSeries(H,-1 : BadPrimes:=[<2,17,1+2*x+8*x^2>], Precision:=10);

> time CFENew(L);
Time: 1.980s
0.0000000000
```

MAGMA computation II

```
> L2:=1+2^2*x+3*2^5*x^2 + 2^9*x^3 + 2^14*x^4;
> H := HypergeometricData([4,4,4,4,2,2,2,2],[8,8,1,1,1,1]);
> L:=LSeries(H,1:BadPrimes:=[<2,18,L2>],Precision:=prec[10]);
> time [CFENew(L),Evaluate(L,4),Evaluate(L,4:Derivative:=1)];
[ 0.0000000000, 0.0000000000, 0.5789920870 ]
Time: 105.030
```

$$d = 10, \quad w = 7, \quad \mathbf{h} = (1,1,2,1,1,2,1,1)$$

MAGMA computation II (cont'd)

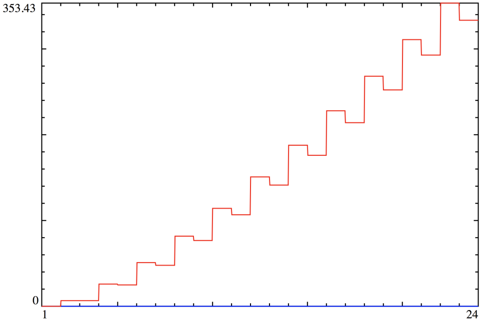
- Euler factor at $p = 3$
-
```
[
  50031545098999707*x^10 + 823564528378596*x^9 + 11203038280413*x^8 +
      192160562544*x^7 + 819482022*x^6 + 26191512*x^5 + 374706*x^4 + 40176*x^3
      + 1071*x^2 + 36*x + 1
]
Time: 0.020
```

- Euler factor at $p = 5$
-
```
[
  2910383045673370361328125*x^10 + 49918889999389648437 50*x^9 +
      42462348937988281 25*x^8 + 100299072265625000*x^7 + 386561035156250*x^6 +
      220660156250 0*x^5 + 4947981250*x^4 + 16433000*x^3 + 8905*x^2 + 134*x + 1
]
Time: 0.140
```

Back to Hodge vectors

- By Griffiths transversality $\mathbf{h}$ is a symmetric composition of d
- Total number: $2^{\lfloor d/2 \rfloor}$
- $$\begin{array}{llll} (2), & (1,1) & & \\ (3), & (1,1,1) & & \\ (4), & (2,2), & (1,2,1), & (1,1,1,1) \\ (5), & (2,1,2), & (1,3,1), & (1,1,1,1,1) \end{array}$$

Rank at most 24

- $N(d) :=$ total number of families of HGM of rank d
- Graph of $\log(N(d))^2$
- 

- Missing Hodge vectors: δ

d	1	$\cdots$	19	20	21	22	23	24
δ	0	$\cdots$	0	1	0	2	1	8

Rank 24

- Rank $d = 24$. Number of possible Hodge vectors: 4096.
- Total number of family of HGM: $464,247,183$
-

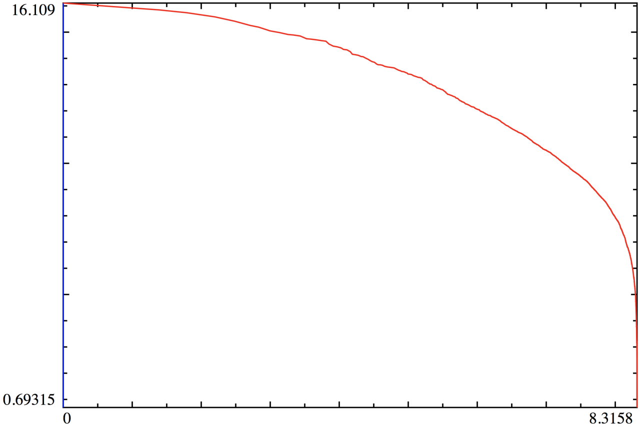
h	#
[9, 1, 1, 2, 1, 1, 9]	0
[7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7]	0
[1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1]	0
[4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4]	0
[5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5]	0
[6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6]	0
[4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4]	0
[4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4]	0

Rank 24

$\mathbf{h}$	#
[6, 2, 1, 1, 1, 2, 1, 1, 1, 2, 6]	2
[8, 1, 1, 1, 2, 1, 1, 1, 8]	4
[1, 22, 1]	4
[8, 1, 1, 4, 1, 1, 8]	6
[6, 1, 2, 1, 1, 2, 1, 1, 2, 1, 6]	8
[6, 1, 3, 1, 2, 1, 3, 1, 6]	8
[10, 1, 2, 1, 10]	10
⋮	⋮
[1, 3, 4, 4, 4, 4, 3, 1]	6082776
[2, 5, 5, 5, 5, 2]	6850823
[1, 3, 8, 8, 3, 1]	6868016
[1, 5, 6, 6, 5, 1]	7637828
[1, 2, 4, 5, 5, 4, 2, 1]	7982874
[2, 4, 6, 6, 4, 2]	9504072
[1, 4, 7, 7, 4, 1]	9905208

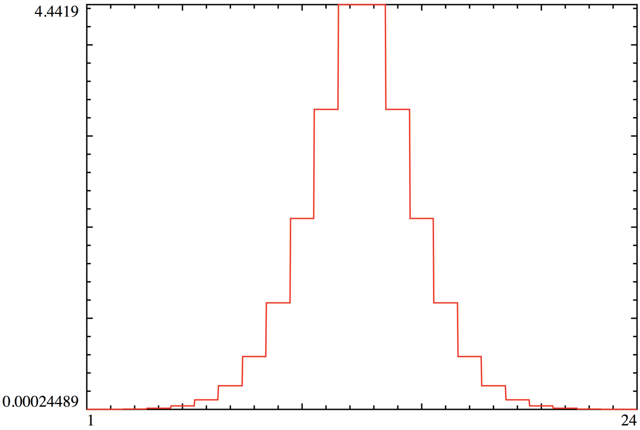
Densities

Log-log graph of densities in rank $d = 24$



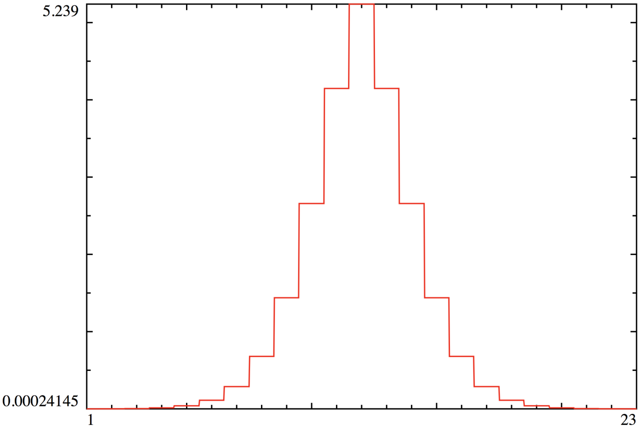
Average Hodge vector

Rank $d = 24$, odd weight



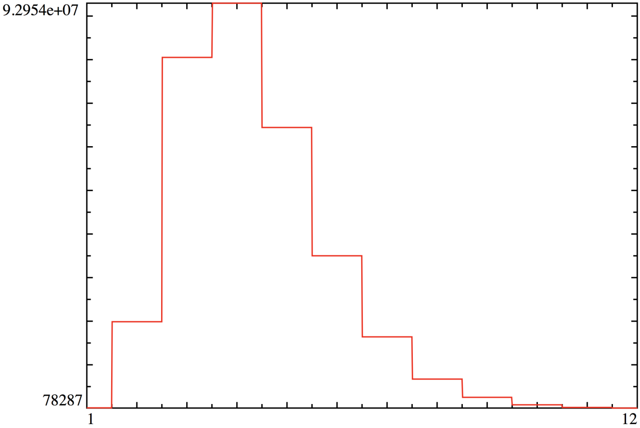
Average Hodge vector

Rank $d = 24$, even weight



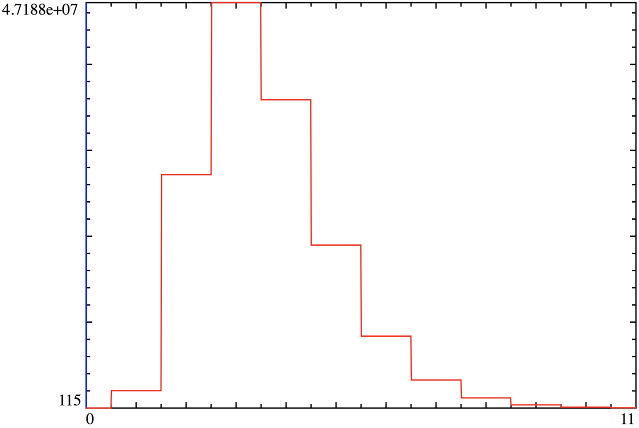
Weight distribution

Rank $d = 24$, odd weight



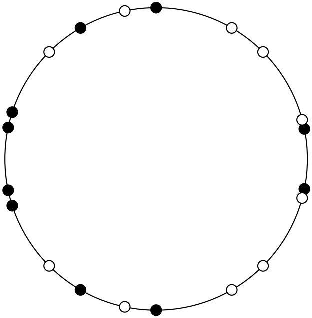
Weight distribution

Rank $d = 24$, even weight



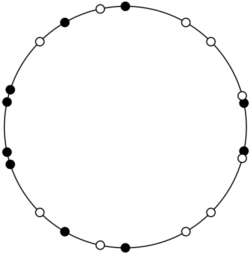
Combinatorial Model

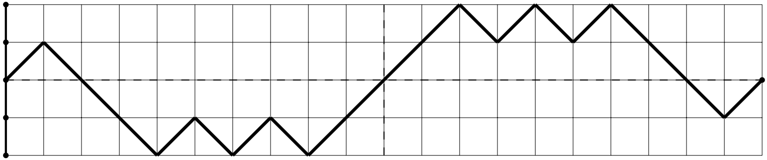
Interlacing pattern $d = 5$: $\circ$ = zeros of q_0, $\bullet$ = zeros of q_∞.



Combinatorial Model (cont'd)

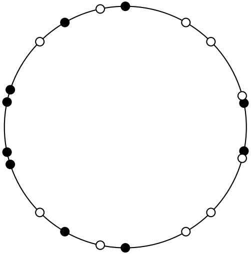
$\circ$ = down, $\bullet$ = up

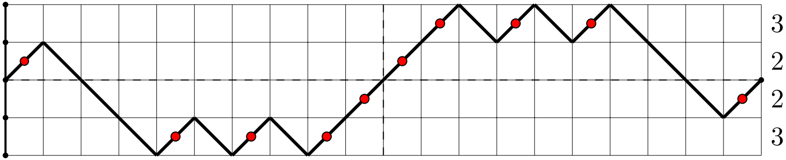




Combinatorial Model (cont'd)

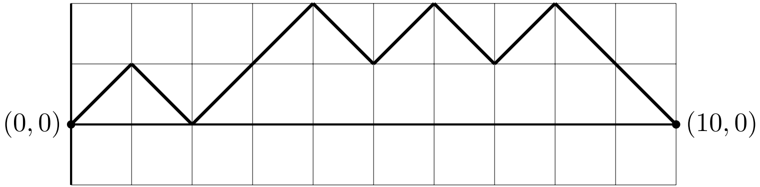
○ = down, ● = up



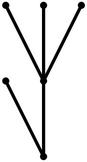


Combinatorial Model (cont'd)

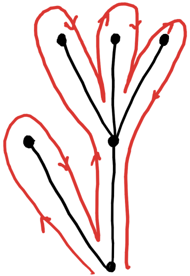
- Dyck path $d = 5$



- Corresponding planted rooted tree



Combinatorial Model (cont'd)

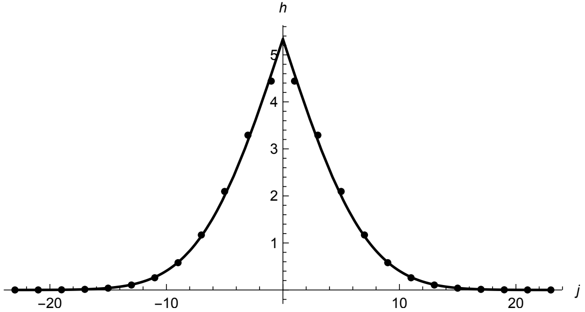


3

2

Average Hodge

Average Hodge vector, compared with an approximation coming from the combinatorial model



Scaled version of

$$f(x) = \sqrt{\frac{\pi}{8}} \operatorname{Erfc}\left(\frac{|x|}{\sqrt{2}}\right),$$