Mixed Hodge Numbers and Factorial Ratios

Bill Duke's Birthday Conference

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Chebyshev and Hodge walk into into a bar...

Chebyshev

- $$c_n := \frac{(30n)!n!}{(6n)!(10n)!(15n)!} \in \mathbb{Z}, \qquad n = 0, 1, 2 \ldots$$

- The following power series is algebraic

$$c := \sum_{n \geq 0} c_n \, x^n$$

- There exists a polynomial $A(x, y) \in \mathbb{Z}[x, y]$ such that

$$A(x, c(x)) = 0$$

- The Galois group is the Weyl group of E_8 of order $696, 729, 600$

Hypergeometric Data

- A *gamma list* is

$$\gamma = (\gamma_1, \dots, \gamma_l), \qquad \gamma_1 \leq \cdots \leq \gamma_l, \qquad \gamma_i \in \mathbb{Z}$$

- γ is *primitive*
-

$$\sum_{i=1}^{l} \gamma_i = 0, \qquad \gamma_i \neq 0, \qquad \gamma_i + \gamma_j \neq 0$$

- Define

$$r := \#\{\gamma_i < 0\}, \qquad s := \#\{\gamma_i > 0\}$$

- Assume $s \geq r$.

Three integrality criteria

- We will present three geometric criteria for the integrality of

$$c_n := \prod_{\gamma_i<0} (-\gamma_i n)! / \prod_{\gamma_i>0} (\gamma_i n)!$$

for all n:

- (A) Codegree of an associated polytope
- (B) Vanishing of certain Hodge numbers of an associated variety
- (C) Small effective weight of the associated hypergeometric motive (HGM)

Lattice polytope

- **Criterion (A)**: An associated lattice polytope $\Delta \subseteq \mathbb{Z}^d$, with $d := l - 2$, satisfies
$$\operatorname{codeg}(\Delta) \geq r$$
-
$$\operatorname{codeg}(\Delta) := \min\{k \geq 1 \mid k\Delta \text{ has an interior lattice point}\}$$
- For $r = 1$ there is no condition; c_n is multinomial
- For $r = 2$ the condition is that Δ has no interior lattice point

Lattice polytope

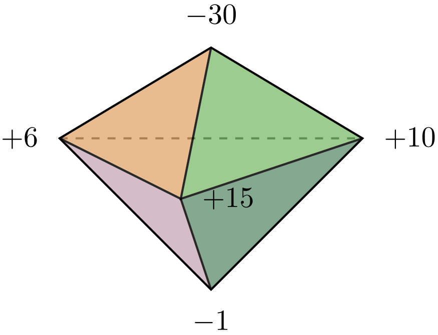
- Take $m_1, \ldots, m_l \in \mathbb{Z}^d$ with $d := l - 2$ such that
- i) their affine relations over $\mathbb{Z}$ are spanned by γ
- ii) their affine span is primitive
- Let $\Delta \subseteq \mathbb{Z}^d$ be the convex hull of $m_1, \ldots, m_l$
- Δ is uniquely determined by γ up to invertible affine transformations over $\mathbb{Z}$

Chebyshev

- $\gamma := (-30, -1, 6, 10, 15), \qquad r = 2, \quad s = 3$
- A possible choice of m_i are the columns of the following matrix

$$\begin{pmatrix} 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

- Schematically and not to scale Δ looks like this



Ehrhardt

- By Ehrhardt

$$\sum_{k\geq 0} \#(k\Delta)\ T^k = \frac{\delta(\Delta, T)}{(1-T)^{d+1}},$$

- where δ is a polynomial of degree at most d
- $\operatorname{codeg}(\Delta) = d + 1 - \deg(\delta) \geq 1$
- For Chebyshev:

$$\delta(\Delta, T) = 15T^2 + 15T + 1$$

- $\operatorname{codeg}(\Delta) = 2$

Hodge

- **Criterion (B)**: The associated variety Z_t has pure part $pH^\kappa(Z_t)$ of its middle cohomology with Hodge vector

$$h := (h^{\kappa,0}, h^{\kappa-1,1}, \ldots, h^{0,\kappa}) = \underbrace{(0, \cdots, 0,}_{r-1}\overbrace{*, \cdots, *,}^{s-r}\underbrace{0, \cdots, 0)}_{r-1}$$

- $\kappa := \dim Z_t = l - 3$

- If $r = 2$ then integrality is equivalent to the vanishing of the geometric genus
$$p_g = h^{\kappa,0} = 0$$
of Z_t

- If $s - r = 1$ then integrality is equivalent to having only the middle Hodge number $h^{r-1,r-1}$ non-zero

Polynomial

- Consider the polynomials

$$F_u := \sum_{i=1}^{l} u_i \, x^{m_i}, \quad x = (x_1, \ldots, x_d), \quad x^{m_i} := x_1^{m_{i,1}} \cdots x_d^{m_{i,d}}$$

- $u_1, \ldots, u_l \in \mathbb{C}^\times$ are parameters
- By scaling the variables and F_u itself we see there is only one true parameter

$$u := u_1^{\gamma_1} \cdots u_l^{\gamma_l}$$

- It is convenient to normalize the parameter and use

$$t := (-1)^{\mathrm{vol}(\gamma)} u M, \qquad M := \prod_{\gamma_i < 0} (-\gamma_i)^{-\gamma_i} / \prod_{\gamma_i > 0} (\gamma_i)^{\gamma_i}$$

- $\mathrm{vol}(\gamma) := \mathrm{vol}(\Delta) = -\sum_{\gamma_i < 0} \gamma_i = \sum_{\gamma_i > 0} \gamma_i$

Variety

- Define
$$Z_t \subseteq \mathbb{T} := (\mathbb{C}^\times)^d$$
as the zero locus of F_t

- Z_t is an affine hypersurface of $\mathbb{T}$ ($\dim Z_t = \kappa = d - 1$) smooth for $t \neq 1$

Chebyshev

- We have

$$Z_t : \quad -\tfrac{M}{t} + xyz + x^5 + y^3 + z^2 = 0, \qquad M := \frac{30^{30}}{6^6 10^{10} 15^{15}}$$

- is a rational elliptic surface
- with Mordell-Weil lattice is isomorphic to E_8

Mixed Hodge

- **Theorem** The E-polynomial of $PH_c^\kappa(Z_t)$ (primitive cohomology) is
- $$E(\Delta; a, b) := \sum_{i,j} h_c^{i,j} a^i b^j = \frac{1}{ab}\left[\delta^\#(\Delta; a, b) + \delta^0(\Delta; a, b) - 1\right]$$
- $$\delta^\#(\Delta; a, b) = \sum_{\substack{N \geq 1 \\ m_- > m_+}} \frac{(a/b)^{m_-} - (a/b)^{m_+}}{a/b - 1} b^{l-1} \delta_N^\#(a/b)$$
- $$\delta^0(\Delta; a, b) = \sum_{N \geq 1} \frac{(ab)^{\min(m_-, m_+)} - 1}{ab - 1} b^{l - m_+ - m_-} \delta_N^\#(a/b)$$

Mixed Hodge

- $$\delta_N^{\#}(T) := \sum_{\substack{j=1 \\ \gcd(j,N)=1}}^{N-1} T^{\sum_{i=1}^{l}\{\frac{j}{N}\gamma_i\}}$$
- $$m_{\pm} := \#\{i \mid \operatorname{sgn}(\gamma_i) = \pm 1,\ N \mid \gamma_i, \}$$
- The pure part of $PH^{\kappa}(Z_t)$ has Hodge numbers

$$\sum_{j=0}^{\kappa} h^{\kappa-j,j} T^{j+1} = \delta^{\#}(\Delta, T) := \sum_{\substack{N \geq 1 \\ m_- > m_+}} \frac{T^{m_-} - T^{m_+}}{T-1} \delta_N^{\#}(T)$$

Chebyshev

- $\gamma = (-30, -1, 6, 10, 15)$
-

N	m_-	m_+	$\delta^\#$
1	2	3	1
2	1	2	T
3	1	2	$2T$
5	1	2	$4T$
6	1	1	$T^2 + T$
10	1	1	$2T^2 + 2T$
15	1	1	$4T^2 + 4T$
30	1	0	$8T^2$

-

$$E(\Delta; a, b) = \begin{matrix} 7 & 8 \\ 8 & 7 \end{matrix}$$

HGM

- **Criterion (C)**: The pure part $pH^\kappa(Z_t)$ of the middle cohomology of Z_t is a Tate twist of an effective motive of pure weight $s-r-1$
- The motive $\mathscr{H}(\gamma\,|\,t)$ is the associated hypergeometric motive
- If $s-r=1$ then integrality is equivalent to $pH^\kappa(Z_t)$ being a Tate twist of a motive of weight zero
- Hence conjecturally an Artin motive
- In our case this holds (Beukers-Heckman). Integrality is equivalent to $\mathscr{H}(\gamma\,|\,t)$ having signature $(n,0)$

HGM

- If $s - r = 2$ then integrality is equivalent to $pH^\kappa(Z_t)$ being a Tate twist of a motive of Hodge type (m, m) for some m
- Hence conjecturally coming from an abelian variety

HGM

- The trace of Frobenius on $\mathscr{H}(\gamma \mid t)$ for good primes has the form
$$\mathscr{H}(t) := \frac{1}{1-q} \sum_{\chi} \frac{J(\alpha\chi, \beta\chi)}{J(\alpha, \beta)} \chi(t)$$
- Finite hypergeometric sum of Katz
- α, β are character version of the hypergeometric parameters
$$\frac{\prod_{j=1}^{n}(T - e^{2\pi i \alpha_j})}{\prod_{j=1}^{n}(T - e^{2\pi i \beta_j})} = \frac{\prod_{\gamma_i<0}(T^{-\gamma_i} - 1)}{\prod_{\gamma_i>0}(T^{\gamma_i} - 1)}$$
- Implemented in MAGMA

Examples

- For Chebyshev

$$\#\tilde{Z}_t(\mathbb{F}_q) = q^2 + q\mathscr{H}(t) + 1$$

- $$\tilde{Z}_t : \quad -\frac{M}{t} + xyz + x^5 + y^3 + z^2 = 0 \subseteq \mathbb{A}^3$$

Examples

- $\gamma = (-63, -8, -2, 1, 4, 16, 21, 31), \qquad r = 3, \quad s = 5$
- smooth cubic 5-fold in $\mathbb{P}^6$ $(t \neq 1)$

  $$\overline{Z}_t : \quad x_1^2x_3 + x_0x_1x_2 + x_1x_2^2 - \frac{t}{M}x_2x_6^2 + x_3^2x_5 + x_4^2x_6 + x_4x_5^2 + x_0^3 = 0$$
- $h = (0, 0, 21, 21, 0, 0)$
- $$E(\Delta; a, b) = \begin{matrix} 0 & 1 & 21 & 0 \\ 0 & 1 & 23 & 21 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

Examples

- $$\frac{(63n)!(8n)!(2n)!}{n!(4n)!(16n)!(21n)!(31n)!}$$
  is integral for all $n = 0, 1, \ldots$

- $\mathscr{H}(\gamma \,|\, t)$ of Hodge type $(21, 21)$ is H^1 of the intermediate Jacobian of $\overline{Z}_t$ (an abelian variety)

- $$\#\overline{Z}_t(\mathbb{F}_q) = q^5 + q^4 + q^3 + q^2 + q + 1 - q^2\mathscr{H}(t)$$