Pacific Northwest Number Theory meeting May 1, 1998 F. Rodriguez Villegas

Boyd's tempered families

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Motivating example

$$\boxed{P_K = x + \frac{1}{x} - K \quad , \quad K \in \mathbb{C}}$$

Family of Laurent polynomials

For $K \in \mathbb{Z}$, $|K| > 2$ the roots $1 < \varepsilon_K$, ε_K^{-1} generate a real quadratic field F_K. By Dirichlet's class number formula

$$\boxed{\zeta'_{F_K}(0) \underset{\mathbb{Q}^\times}{\sim} \log|\varepsilon_K|}$$

From the point of view of the polynomial:

- ε_K is a unit in a field with rank of its units equal to 1 because:

Newton polygon of P_K

- one interior point
- ± 1 coefficients at edges
  tempered
- $K \in \mathbb{Z}$

- $$\log|\varepsilon_K| = \frac{1}{2\pi i} \int_{|x|=1} \log|P_K(x)| \frac{dx}{x}$$

$$= m(P_K)$$

What could be a generalization to polynomials in two variables?

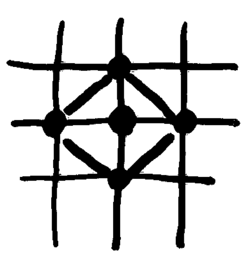
$$P_k(x,y) = P(x,y) - k, \quad k \in \mathbb{C}$$

Laurent polynomial

E.g.

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} - k$$

- Newton polygon



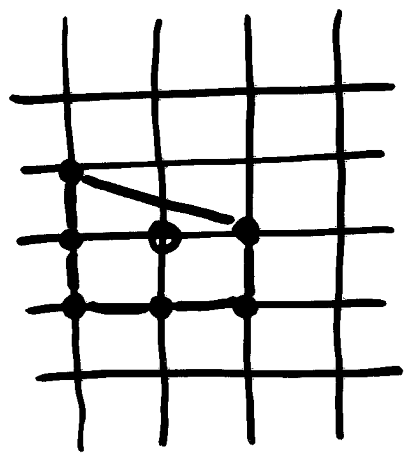
  - one interior point
  - tempered
  - $k \in \mathbb{Z}$

- $$\boxed{m(P_k) = \frac{1}{(2\pi i)^2} \int_{\substack{|x|=1 \\ |y|=1}} \log |P_k(x,y)| \frac{dx}{x} \frac{dy}{y}}$$

(logarithmic) <u>Mahler measure</u>

Tempered families

Example

Δ 

1		
2	k	1
1	1	1

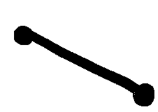
Newton polygon of

$$P_k = x^{-1}y + 2x^{-1} + x^{-1}y^{-1} + y^{-1} + xy^{-1} + x - k$$

Each side τ of the polygon determines a one variable polynomial P_τ

τ  $1+2t+t^2$

 $1+t$

 $1+t$

 $1+t+t^2$

$$\boxed{\text{Tempered} \longleftrightarrow \text{roots of } P_\tau \text{ for all } \tau \text{ are roots of unity}}$$

What L-function ?

$P_k(x,y) = 0 \qquad k \in \mathbb{Z}$ (generic)

is the equation of an affine curve of genus 1. Let E_k be the Jacobian of the projective closure and normalization of this curve.

E_k is an elliptic curve $/\mathbb{Q}$ (for generic k)

We compare $m(P_k)$ and $L'(E_k, 0)$

Conjecture

For all sufficiently large $k \in \mathbb{Z}$

$$\boxed{L'(E_k, 0) \sim_{\mathbb{Q}^*} m(P_k)}$$

Polygons

- In general:
  If $P \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ is a Laurent polynomial we may consider its Newton polygon Δ.
  A generic P with a given Δ determines an affine curve

  $$P(x,y) = 0$$

  of genus

  $$\boxed{g = \#\ \text{interior pts of } \Delta}$$

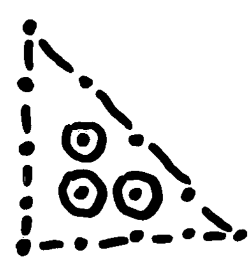
E.g.

- polynomial of deg d

  $g = \frac{1}{2}(d-1)(d-2)$

- hyperelliptic $y^2 = f(x)$ $\deg f = d$

  $g = \left[\frac{d-1}{2}\right]$



$d=4$ $g=3$

$d=8$ $g=3$

We can study the polygons up to $GL(2,\mathbb{Z})$ equivalence (change of basis in lattice $\mathbb{Z}^2$). Neither Mahler measure nor curve change.

Fix $\delta \in \mathbb{N}$.

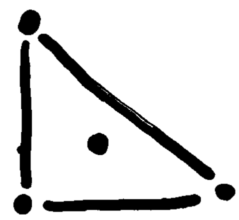
<u>Thm (Scott)</u> There are finitely many convex lattice polygon (clp) with δ interior lattice points (up to equivalence).

(With $\delta = 1$ there are 16 possible classes.)

$\Rightarrow$ There are finitely many tempered families $/\mathbb{Q}$

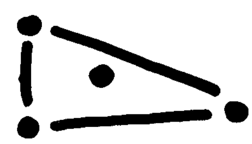
Examples

Δ



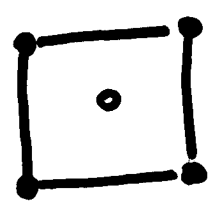
- $x^3+y^3+z^3-kxyz$

$P_\tau = \cancel{1+t+t^2}\ 1+t^3$



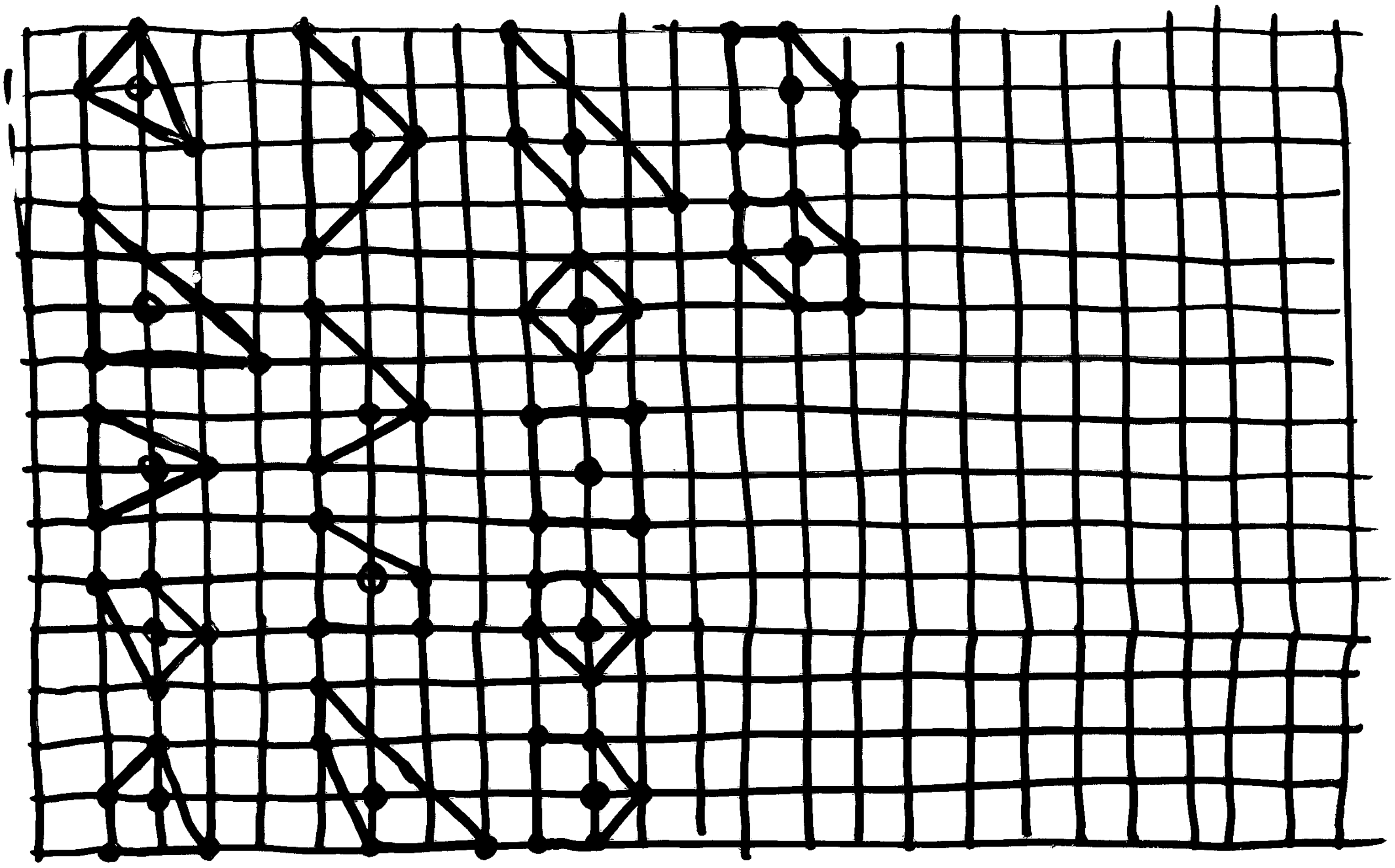
- $y^2+kxy = x^3-1$

$P_\tau = 1+t,\ -1+t^3$



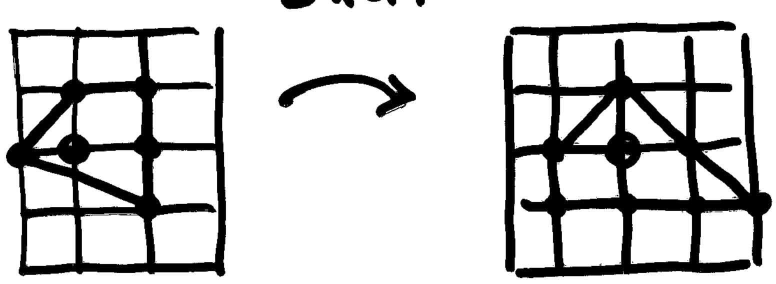
- $(x+\frac{1}{x})(y+\frac{1}{y})=k$

$P_\tau = 1+t^2$



dual

(Gauss map)



$\# \partial : \quad 5 \quad + \quad 7 \quad = \quad 12$

(joint w/ B. Poonen, 3 proofs
- list them all
- Dedekind eta
- Noether's fmla for surfaces

Translating the conjecture (Bloch-Beilinson)

(1) $x + \frac{1}{x} - k$ $\rightsquigarrow$ roots are units in a number field

(2) Two variable case $\rightsquigarrow$?

Pedantically: In (1) $\{x\}$ is an element of $\left(\mathbb{Q}[x,x^{-1}] / (x+\frac{1}{x}-k)\right)^{\times}$
real quadr. field

In (2) $\{x,y\}$ is an element in $K_2(F)$ where

$$F = \mathbb{Q}[x,x^{-1},y,y^{-1}]/(P_k(x,y))$$

fctn field of E_k

In (1) tempered means $\{x\}$ is a unit;

in (2) tempered means

$$\{x,y\}^N \in K_2(E_k)$$

for some $N \in \mathbb{N}$.

Tame symbols

C smooth projective curve $/\mathbb{C}$

v discrete valuation on the function field of C associated to some point w. For f, g nonzero ratnl fctns on C define

$$(f,g)_v = (-1)^{v(f)v(g)} \left. \frac{f^{v(g)}}{g^{v(f)}} \right|_w \in \mathbb{C}^\times$$

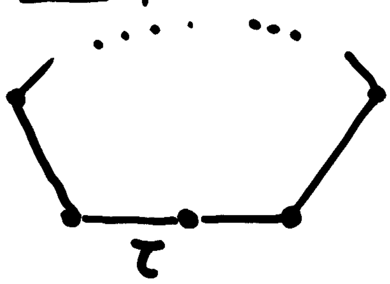
Claim:

$P_K(x,y)$ is tempered

$$\Updownarrow$$

all tame symbols $(x,y)_v$ are torsion (i.e. roots of unity)

Proof sketch



Newton polygon

$P_\tau = 1 + t + t^2$ say

τ corresponds to the points

$v = (\zeta, 0)$, $(\bar{\zeta}, 0)$ $\qquad \zeta = e^{2\pi i/3}$

on C.

$$(x, y)_v = \zeta \qquad \square$$

Why $m(P_K)$?

In (1):

$$\begin{tikzcd} F_K^\times \arrow[r, hook] \arrow[dr] & \mathbb{R}^\times \arrow[d, "\log|\cdot|"] \\ & \mathbb{R}^+ \end{tikzcd}$$

In (2): $K_2(E_K) \longrightarrow H^1(E_K, \mathbb{R})$

<u>regulator map</u>

$$\{f, g\} \longmapsto \overset{\eta(f,g) =}{\log|f|\, d\arg g - \log|g|\, d\arg f}$$

real differential form with possible singularities where f, g have zeros or poles

<u>Lemma:</u>

$$\boxed{\log|(f,g)_v| = \mathrm{Res}_w\, \eta(f,g)}$$

In the tempered case then

$$[\gamma] \longmapsto \int_\gamma \eta(x,y),$$

γ closed path avoiding zeros & poles of x, y, gives a well defined linear form

$$H_1(E_K, \mathbb{R}) \longrightarrow \mathbb{R}$$

Finally,
the connected component of the real pts of E_K (with some orientation) gives a homology class $[C_0]$; we get a map

$$K_2(E_K) \longrightarrow \mathbb{R}$$

$$\{f,g\} \longmapsto \int_{C_0} \eta(f,g)$$

Poincaré residue

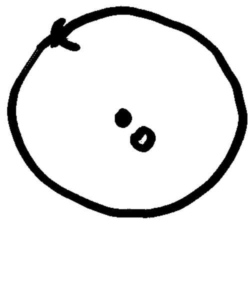
On $\mathbb{P}^2$ consider

$$\eta(P_K, x, y) = \log|P| \frac{dx}{x} \wedge \frac{dy}{y} - \log|x| \frac{dP}{P} \wedge \frac{dy}{y} + \log|y| \frac{dP}{P} \wedge \frac{dx}{x}$$

If does not vanish on $\{|x|=|y|=1\}=T$ (torus) then

$$\boxed{\frac{1}{(2\pi i)^2}\int_T \eta(P_K, x, y) = * \int_{C_0} \eta(x,y)}$$

Cauchy



$$\frac{1}{2\pi i}\int_C f(z)\frac{dz}{z} = f(0)$$

Why $k \in \mathbb{Z}$?

If tempered $\{x, y\} \in K_2(E_k)$

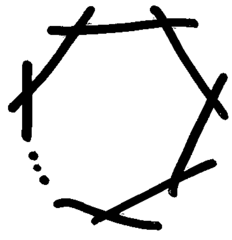
What we really need is an element in $K_2(\mathcal{E}_k)$ where $\mathcal{E}_k$ is a Néron model of E_k. This means that there are further conditions that $\{x, y\}$ must satisfy, one for every prime of bad reduction of E_k.

In fact, only primes of split ~~of~~ multiplicative reduction matter.

For these primes the special fiber of $\mathcal{E}_k$ looks like this

$\boxed{I_N}$

Kodaira symbol

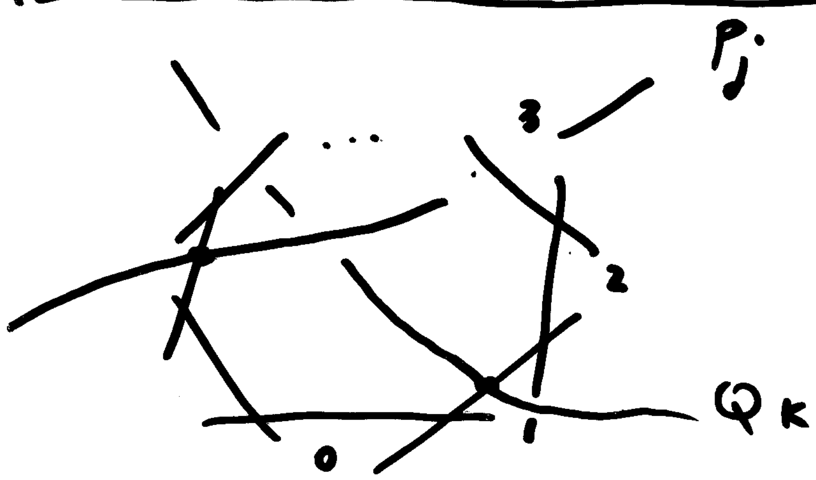


The obstruction for such a prime can be computed as follows:

$$(x) = \sum_j n_j P_j$$

$$(y) = \sum_k m_k Q_k$$

$$\boxed{0 = x = \sum_{j,k} n_j m_k B_3\left(\frac{d(P_j, Q_k)}{N}\right)}$$

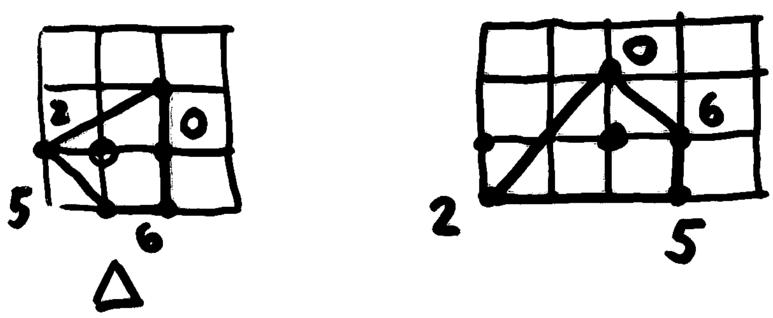


B_3 = periodic Bernoulli function

If pts don't go through vertices
$d(P_j, Q_k)$ = # components between them
(indep. of choice of numbering)

(V. Tate) preliminary ...

If $p \mid$ denominator of κ then E_κ has reduction of type I_N where $N = \#$ boundary pts on the dual of the Newton polygon of P_κ.



$N = 7$

$$\boxed{x = \frac{1}{2} \sum_{u,v} B_3\left(\frac{d(u,v)}{N}\right) \det(u\ v)}$$

u, v vertices of dual polygon.

$$\boxed{x = \frac{-3}{N}}$$