EXPLICIT ELLIPTIC UNITS I

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(joint work with F. Hajir)

Let $\eta(z) = e^{\frac{2\pi i z}{24}} \cdot \prod_{n=1}^{\infty} (1 - e^{2\pi i z n})$

$Im(z) > 0$

be the familiar eta function of Dedekind.

Let $\mathcal{O} \subset K$ be an order in an imaginary quadratic field.

We want the following:

1) Given a proper primitive ideal $\mathfrak{a} \subset \mathcal{O}$, define $\eta(\mathfrak{a})$ appropiately for $\mathfrak{a}$ prime to a fixed ideal $\mathcal{F}$.

2) Describe explicitly the action of $Gal(\bar{K}/K)$ on any ratio

$$\alpha = \prod_{j=1}^{r} \eta(a_j)^{n_j}$$

where $n_1, \dots, n_r \in \mathbb{Z}$ and $\sum_{j=1}^{r} n_j = 0$.

This is a very classical question with a long history: Weber, Watson, Ramachandra, Stark, Kubert, Lang...

OUR ANSWER

1) Let I^* be the collection of proper primitive ideals of $\mathcal{O}$ prime to 6.

For $\mathfrak{a} \in I^*$ we define

$$\eta(\mathfrak{a}) = e_{48}(a(b+3W_0))\,\eta\left(\frac{-b+\sqrt{-d}}{2a}\right)$$

where $\mathfrak{a} = [a, \frac{b+\sqrt{-d}}{2}]$ (standard basis)

$b^2 \equiv -d \bmod 4a$, $-d = disc(\mathcal{O})$, $a = N\mathfrak{a} = [\mathcal{O}:\mathfrak{a}]$

$e_N(x) = e^{\frac{2\pi i x}{N}}$, $W = \gcd\{N\mu - 1 \mid \mu \in \mathcal{O}$ prime to $6\}$

$W_0 = \frac{1}{2}\gcd(W, 8)$.

This is well defined.

2) By class field theory $Gal(\bar{K}/K)$ may be described by σ_a for a in I^*.
Let H be the fixed field of $\sigma_{(\mu)}$ for $(\mu) \in I^*$. This field is classically known as the ring class field associated to $\mathcal{O}$.

We find the following $\quad a = Na,\ a_1 = Na_1$

$$(i)\quad \overline{\eta(a)} = e_8(-w_0 a)\,\eta(\bar{a})$$

$$(ii)\quad \left(\frac{\eta(a)}{\eta(\mathcal{O})}\right)^{\sigma_{a_1}-1} = \left(\frac{a}{a_1}\right)\frac{\eta(a_1 a)}{\eta(a_1)}$$

for $a, a_1, aa_1 \in I^*$.

$$(iii)\quad \left(\frac{\eta(a)}{\eta(\mathcal{O})}\right)^{\sigma_{a_1}-1} = (-1)^{\frac{a-1}{2}\cdot\frac{a_1-1}{2}}\left(\frac{\eta(a_1)}{\eta(\mathcal{O})}\right)^{\sigma_a-1}$$

(Reciprocity Law)

This follows from using the Shimura reciprocity Law.

Let $\alpha = \prod_{j=1}^{r} \eta(a_j)^{m_j}$ as before. (4)

It is not very hard to check that

$$\begin{array}{c} H(\alpha) \\ | \\ H \\ | \\ K \\ | \\ \mathbb{Q} \end{array}$$

n $\{$ $H(\alpha) / H$; $H/\mathbb{Q}$ abelian, $n \mid \gcd(w, 12) \mid w$

$\mu(H) = \{\text{roots of unity in } H\}$
$w = \# \mu(H)$

The extension $H(\alpha)/H$ is a Kummer extension: $\alpha^w \in H$. We would characterize it completely if we could describe the character

$$(\mu) \in I^*; \quad \mu \mapsto \alpha^{\sigma(\mu) - 1} \in \mu(H)$$
$\mu \in \mathcal{O}$

For example we would easily find the degree of the extension $[H(\alpha):H]$.

Using the reciprocity given above we see that this character is related to

$$K: \quad \mu \mapsto \left(\frac{-1}{\mathbb{N}\mu}\right) \frac{1}{\mu} \frac{\eta^2((\mu))}{\eta^2(\mathcal{O})}$$

<u>Claim</u>: (i) K extends to a character

$(\mathcal{O}/12\mathcal{O})^* \longrightarrow \mu_{12}$ of order precisely

gcd $(W, 12)$

(ii) K is essentially determined by the isomorphism class of the finite ring $\mathcal{O}/12\mathcal{O}$.

This claim is the heart of the matter. Its proof requires very careful analysis of the 24^{th} roots of unity appearing in the transformation formulas of η.

MAIN THEOREM

Let $\alpha = \prod_{j=1}^{r} \eta(a_j)^{n_j}$ as before. Then the character of $\mathcal{O}$ associated to the Kummer extension $H(\alpha)/H$ is:

$$\mu \longmapsto K(\mu)^e \left(\frac{\mu}{a}\right)_K$$

where $e = \frac{1}{2} \sum_{j=1}^{r} n_j (a_j - 1)$, $\quad a_j = \mathbb{N}\mathfrak{a}_j$

$$\mathfrak{a} = \prod_{j=1}^{r} \mathfrak{a}_j^{n_j}$$

Applications

(i) If $\mathfrak{a} = \mathfrak{a}'^2$ for some ideal $\mathfrak{a}' \subset \mathcal{O}$

then $\alpha \in H$

(ii) Write $K = k_4/k_3$ with $k_N^N = 1$ $N = 3, 4$

Then

$$\left(\frac{\mu}{\nu}\right)_K \left(\frac{\nu}{\mu}\right)_K = (-1)^{\frac{m-1}{2}\cdot\frac{n-1}{2}} K_4^{\frac{m-1}{2}}(\mu)\, K_4^{\frac{n-1}{2}}(\nu)$$

$m = \mathbb{N}\mu$, $n = \mathbb{N}\nu$

A version of the quadratic reciprocity law due to Herglotz.

(iii) For $C \in Cl(\mathcal{O})$ let

$$u_C = \frac{1}{\sqrt{a}} \left| \frac{\eta\left(\frac{b+\sqrt{-d}}{2a}\right)}{\eta\left(\frac{b+\sqrt{-d}}{2}\right)} \right|^2$$

with $\mathfrak{a} \in I^*$, $\mathfrak{a} = [a, \frac{b+\sqrt{-d}}{2}]$, $[\mathfrak{a}] = C$,

$n_C = w_{12} / \gcd(w_{12}, \mathbb{N}\mathfrak{a} - 1)$

$w_{12} = \gcd(w, 12)$

(This is well defined independent of $\mathfrak{a}$)

Then let $v_C = u_C^{n_C} \in H^+$ (real subfield of H).

These units $\{v_C : C \in Cl(\mathcal{O}) \setminus [\mathcal{O}]\}$ are linearly independent and

$$H^+ = \mathbb{Q}(v_C) \quad , \quad [C] \neq 1.$$

We hence may obtain an explicit (and exact) minimal polynomial of a generator of $H^+/\mathbb{Q}$. This is useful for many applications.

with small height

$$\left(u_C = \pm \frac{\eta(a)\eta(\bar{a})}{\sqrt{\left(\frac{-1}{a}\right)a}\ \eta(\mathcal{O})^2} \quad , \quad a = Na \right)$$

This is very easy and efficient to program. For example, take $d = -2^4 \cdot 3 \cdot 7 = -336$. There is only one nontrivial class with $n_C = 1$ and for it we obtain the polynomial

$$x^8 - 20x^7 + 32x^6 - 12x^5 + 14x^4 - 12x^3 + 32x^2 - 20x + 1$$