" On the square root of special values of L-series" (1)

Talk   Spring 89   Number Theory Seminar

Let

$q =$ fixed prime $\equiv 7 \bmod 8$

$K = \mathbb{Q}(\sqrt{-q})$

$Cl\,K =$ class group of K

$\mathcal{O}_K =$ ring of integers of K

- B. Gross SLN 776 (thesis 197
- C. Jourdan Cours D'Analyse 1894 vol2
- Weber, Lehrbuch der Algebra
- Fricke R., Elliptische Funktionen und ihre angew. vol 2
  - Lehrbuch der Algebra vol 3

We have

$\mathrm{disc}(K) = -q$

$h = \# Cl\,K =$ odd $\leftarrow$ EX: This hypothesis will be essential for us

2 splits in K, units of $\mathcal{O}_K = \{\pm 1\}$

Let

$j = j(\mathcal{O}_K)$ and $F = \mathbb{Q}(j) < \mathbb{R}$

$H =$ Hilbert class field of K

$= F \cdot K$

$$\begin{tikzcd}[column sep=small] & H \arrow[dl, dash] \arrow[dr, dash, "h"] & \\ F \arrow[dr, dash] & & K \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

$$\begin{tikzcd} \varepsilon : \mathcal{O}_K/\sqrt{-q}\,\mathcal{O}_K \arrow[rr] \arrow[dr, "\wr"'] & & \{\pm 1\} \\ & \mathbb{Z}/q\mathbb{Z} \arrow[ur, "\left(\frac{\cdot}{q}\right)"'] & \end{tikzcd}$$

quadr. character

Note that $(\varepsilon(2) = +1)$ and $\varepsilon(-1) = -1$

For each prime $\mathfrak{P}$ in H we have

$$\mathbb{N}_{H/K}(\mathfrak{P}) = (\alpha_{\mathfrak{P}}) \quad \text{is ppal in } K$$

We choose a generator so that

$$\varepsilon(\alpha_{\mathfrak{P}}) = +1$$

This gives rise to a Hecke character

$$\begin{aligned} \chi_q : I_H(q) &\longrightarrow K^\times \\ \mathfrak{P} &\longmapsto \alpha_{\mathfrak{P}} \end{aligned}$$

<u>Thm</u>: (Gross)

$\exists$ a unique (up to F-isomorphism) elliptic curve $A(q)$ defined $/F$ such that:

(i) $A(q)/H$ admits CM by $\mathcal{O}_K$

(ii) χ_q is its associated Hecke character

(iii) $\Delta(A(q)/F) = (-q^3)$ (minimal discr. ideal)

It further satisfies

(iv) $A(q)$ has good reduction outside q

(v) $A(q)(F) \simeq \mathbb{Z}/2\mathbb{Z}$

[So rank $= 0$ $\overset{?}{\Longrightarrow}$ $L(A(q), 1) \neq 0$]
Coates-Wiles + N. Artaud.

[B, S-D conjecture $\dfrac{L(A(q),1)}{\text{period}} = |Ш|?$]

(vi) $L(A(q)/F, s) = L(\chi_q, s)$

(All bad factors $=1$)

(vii) $\sigma \circ \chi_q = \chi_q \quad \forall\, \sigma \in \mathrm{Gal}(H/\mathbb{Q})$

hence $A(q)/H$ is a $\mathbb{Q}$-curve ie: A is H-isogenous to $A(q)^\sigma \quad \forall\, \sigma \in \mathrm{Gal}(H/\mathbb{Q})$.

We now define another Hecke character

$$\psi : I_K(q) \longrightarrow \mathbb{C}^\times$$

$$\mathfrak{p} \longmapsto \mu_{\mathfrak{p}}$$

[Kummer ?]

where $\mathfrak{p}^h = (\mu_{\mathfrak{p}}^h) \qquad \varepsilon(\mu_{\mathfrak{p}}^h) = +1$ in K

and $\psi((\mu)) = \varepsilon(\mu)\,\mu$ for $\mu \in \mathcal{O}_K$ prime to q

There are h such characters. <u>We fix one, once and for all.</u> Then all others are of the form

$\varphi \cdot \psi$ with $\varphi \in$ ~~[illegible]~~ $\widehat{Cl_K}^*$

ψ takes values in an extension T/K of degree h.

Prop:

$$L(A(q)/F, s) = L(\chi_q, s) = \prod_{\varphi \in Cl K^* = \mathrm{Hom}(Cl K, \mathbb{C}^\times)} L(\varphi\psi, s)$$

We now decompose $L(\psi, s)$ into partial L-series:

$$L(\psi, s) = \sum_{c \in Cl K} L(s, \psi, c)$$

where

$$L(s, \psi, c) = \frac{\mathbb{N}(a)^s}{2\,\psi(a)} \sum_{\mu \in a} \frac{\varepsilon(\mu)\,\bar{\mu}}{\mathbb{N}(\mu)^s}$$?

$\mathrm{Re}\, s > 3/2$

a = any ideal (prime to $2q$) in c

Rem: Usually one uses functional eqn of $L(s, \psi, c)$ to get $L(1, \psi, c)$ as an integral (period) of a theta series of weight 2.

We will take a different approach following Hecke.

$\boxed{\text{see fmla (8)}}$ Note: convergence ok but not great.

$$\tau_a = \frac{-b+\sqrt{-q}}{2a}$$

$$\mathfrak{a} = \mathbb{Z}a \oplus \mathbb{Z}\left(\frac{b+\sqrt{-q}}{2}\right)$$

First we need

<u>Lemma</u> (Hecke

$$L(s,\psi,c) = \frac{(aq)^{1-s}}{\sqrt{-q}\; 2\psi(\mathfrak{a})} \; H_q(\tilde{\tau}_{a\sqrt{q}}, s) \quad , \operatorname{Re} s > 3/2$$

where

$$H_q(\tau,s) = \sum_{m,n\in\mathbb{Z}} \left(\frac{m}{q}\right) \frac{m\bar{\tau}+n}{|m\tau+n|^{2s}}$$

?

appears in many places eg: Shimura's paper on half integral weight

$\tau \in h$, $\operatorname{Re} s > 3/2$

and $\tilde{\tau}_a \in h$ is associated to $\mathfrak{a}$.

Following Hecke we will view H_q as a function of τ

<u>Thm</u> (Hecke)

$H_q(\tau,s)$ can be analytically continued to the whole plane, has a functional eqn and

Hecke gets a more general expansion I think essentially equiv to Kronecker's 2nd limit fmla see Lang. Elliptic fctns

$$H_q(\tau,s)\Big|_{s=1} = 2\pi i \sum_{C'\in ClK} \theta_{C'}(\tau) \quad , \quad \forall \tau \in h$$

where

$$\theta_{C'}(\tau) = \sum_{\mu\in\mathfrak{a}'} q^{N(\mu)/N(\mathfrak{a}')} \quad , \quad q = e^{2\pi i\tau}$$

$\mathfrak{a}'$ = ideal prime to ~~2q~~ $2q$ in C'

<u>Rem:</u>
(This is analogous to Brauer-Siegel fmla for binary forms) left hand side = Eisenstein series wt ...
and right hand side is a sum of theta series of weight 1

In particular we get

$$L(1, \psi, c) = \frac{\pi}{\sqrt{q}} \cdot \sum_{c' \in Cl K} \theta(c, c') \qquad \boxed{\text{Fmla (6)}}$$

where

$$\theta(c, c') := \theta_{c'}(\tau_a \sqrt{q}) / \psi(a)$$

we will show later it only depends on the classes c & c'.

Prop & Defn

Given $c \in Cl K$ choose $a \in c$, a = primitive ideal prime to $2q$ (primitive = not divisible by integers of $\mathbb{Z}$). Such a always exists.

Let $a^2 = \mathbb{Z} a^2 \oplus \mathbb{Z}\left(\frac{b_2 + \sqrt{q}}{2}\right)$

$a = N a$, $b_2 \in \mathbb{Z}$ defined only mod $2a^2$

Define

$$t(c) = \left(\frac{2}{a}\right) \cdot \frac{e_{16}(b_2/a^2)}{\psi(a)} \cdot \theta_{10}\Big(\underbrace{\frac{-b_2 + \sqrt{-q}}{2a^2}}_{= \tau_{a^2}}\Big)$$

which only depends on c. [Quite a pain to control the eight roots of 1 that keep popping up]

Here

$e_{16}(x) := e^{\frac{2\pi i x}{16}}$ $\quad x \bmod 16$

by abuse of notation $\quad e_{16}(1/x) := e_{16}(y) \quad$ for $xy \equiv 1 \bmod 16$

and

$$\theta_{10}(\tau) = \sum_{\substack{n \text{ odd} \\ \in \mathbb{Z}}} e^{\pi i \frac{n^2}{4}\tau} \qquad \tau \in h$$

is one of Jacobi's theta series

[Weber
Jourdan
Mumford.]

We have now the

$\boxed{\text{Fmla}(7)}$

<u>Factorization Lemma</u>

$$\theta(C^2, C'^2) = \sqrt[4]{q}\; t(CC')\, t(C C'^{-1})$$

We define

$$\xi := \sum_{C \in Cl K} L(1, \psi, C) \cdot C \qquad \in \mathbb{C}[Cl K]$$

("Stickelberg element")

[Karl's thesis $\quad \xi$ annihilates $A(\mathbb{C})$
N. Artaud.]

<u>Rem:</u> if for $\varphi \in Cl K^*$ we define $\tilde{\varphi}\left(\sum_C a_C \cdot C\right) = \sum_C a_C \varphi(C)$

$\tilde{\varphi}$ = ring homomorphism
$\mathbb{C}[Cl K] \to \mathbb{C}$

then

$$L(1, \varphi\psi) = \tilde{\varphi}(\xi)$$

Let also

$$\eta = \sum_{c \in Cl K} t(c) \cdot c$$

then we get

Thm

$$\boxed{\xi = \frac{\pi}{\sqrt[4]{q}} \cdot \eta^2}$$

Pf: ~~check on constants~~ ok.

Recall $\# Cl K$ is odd so $Cl K^2 = Cl K$

Hence c^2- coeff of $\xi = L(1, \psi, c^2)$

$$= \frac{\pi}{\sqrt{q}} \sum_{c' \in Cl K} \theta(c^2, c'^2)$$

$$= \frac{\pi}{\sqrt[4]{q}} \cdot \sum_{c' \in Cl K} t(cc')\, t(cc'^{-1})$$

$$= \frac{\pi}{\sqrt[4]{q}} \cdot \sum_{c' \in Cl K} t(c')\, t(c^2 c'^{-1})$$

$\square$

Corollary

$$L(1, \varphi \cdot \psi) = \frac{\pi}{\sqrt[4]{q}} \cdot \tilde{\varphi}(\eta)^2 \qquad \text{for } \varphi \in Cl K^*$$

$\boxed{\text{fmla (9)}}$

Now the Birch, Swinnerton-Dyer conjecture ~~[illegible]~~ predicts (in this case) that

$$L(A(q)/F,1) = |\text{Ш}?| \cdot \frac{2^{h-2}}{p^{\frac{h-1}{4}}} \cdot \Omega_{\mathbb{R}}$$

$$\Omega_{\mathbb{R}} = \prod_{\substack{0<c<p \\ \varepsilon(c)=1}} \Gamma(c/p) \Big/ (2\pi)^{\frac{p-1}{4}-\frac{h}{2}} \cdot p^{h/2} \quad ; \quad \Gamma = \text{gamma function}$$

As stated by Gross. We can rewrite as follows

$$L(A(q)/F,1) = |\text{Ш}?| \cdot 4^{h-1} \cdot \left(\frac{\pi}{p^{1/4}}\right)^h \cdot \prod_{c\in \mathrm{Cl}K} |t(c)|^2$$

and we get $\boxed{t(c^{-1}) = \overline{t(c)}}$

Thm:

$$\boxed{\sqrt{|\text{Ш}?|} = \frac{1}{2^{h-1}} \cdot \frac{\prod_{\varphi\in \mathrm{Cl}K^*} \tilde{\varphi}(\eta)}{\prod_{c\in \mathrm{Cl}K} |t(c)|}}$$

[Fmla Has advantages in terms of computations]

$\sqrt{|\text{Ш}?|}$ seems to grow pretty fast. ($\to\infty$?)

signs ??

q	h	$\sqrt{\lvert\text{Ш}?\rvert}$									
23	3	-1	71	7	3	191	13	-131	607	13	3723
31	3	1	151	7	-9	199	9	-43	823	9	-2885
47	5	1	223	7	-47	239	15	357			
79	5	-3	463	7	73	271	11	29			
103	5	1	487	7	1	263	13	21			
127	5	3	167	11	-115	367	9	171			

Proof of fmla (7)

$$\boxed{(7) \qquad \sqrt[4]{7}\; e_8(1)\; \theta_{10}\left(\frac{-1+\sqrt{-7}}{2}\right)^2 = \theta_{O_K}\left(\frac{-7+\sqrt{-7}}{2.7}\right)}$$

PF: three main ingredients

1. completing squares
2. Poisson summation fmla
3. similarity of the plane

$$\theta_{O_K}(\tau) = \sum_{\mu \in O_K} q^{N(\mu)}$$

$$\mu = n + m\left(\frac{-1+\sqrt{-7}}{2}\right)$$

① then $N(\mu) = n^2 - nm + 2m^2$

$$= \frac{1}{4}\left[(2n-m)^2 + 7m^2\right]$$

So if $k = 2n - m$

$$\theta_{O_K}(\tau) = \sum_{\substack{k,m \\ k \equiv m \bmod 2}} q^{\frac{k^2+7m^2}{4}}$$

Now let $\tau = \frac{-7+\sqrt{-7}}{2.7} = \frac{-1 + i/\sqrt{7}}{2}$

Divide the sum in two

(1°) $\underline{k \equiv m \equiv 0 \bmod 2}$

we have then (call $k = k/2$ $m = m/2$

$$\sum_{k,m} e^{2\pi i (k^2+7m^2)\cdot(-1+i/\sqrt{7})/2}$$

$$= \sum_k (-1)^k e^{-\pi k^2/\sqrt{7}} \cdot \sum_m (-1)^m e^{-\pi m^2 \sqrt{7}}$$

(2) Now use Poisson summation fmla on the first factor.

$$= \sqrt[4]{7} \cdot \sum_{k \text{ odd}} e^{-\pi \frac{k^2}{4}\sqrt{7}} \cdot \sum_m (-1)^m e^{-\pi m^2 \sqrt{7}}$$

$$= \sqrt[4]{7} \sum_{\substack{k \text{ odd} \\ m \text{ even}}} (-1)^{m/2} e^{-\pi \frac{(k^2+m^2)}{4}\sqrt{7}}.$$

we want now to do the following transformation

(3)
$$\begin{cases} u = k+m \\ v = k-m \end{cases} \qquad \begin{aligned} k &= \tfrac{1}{2}(u+v) \\ m &= \tfrac{1}{2}(u-v) \end{aligned}$$

then

$$\left\{ (k,m) : \begin{matrix} k = \text{odd} \\ m = \text{even} \end{matrix} \right\} \xleftrightarrow{1-1} \{ (u,v) : u \cdot v \equiv 1 \bmod 4 \}$$

now $\quad k^2 + m^2 = \dfrac{u^2+v^2}{2}$

so we get

$$= \sqrt[4]{7} \sum_{\substack{u \equiv v \bmod 4 \\ u,v \text{ odd}}} (-1)^{\frac{u-v}{4}} \cdot e^{-\pi \frac{u^2+v^2}{8}\sqrt{7}}$$

$$= 2\sqrt[4]{7} \sum_{\substack{u \equiv 1 \bmod 4 \\ v \equiv 1 \bmod 4}} e^{\pi i \frac{u}{4}} e^{-\pi \frac{u^2}{8}\sqrt{7}} \cdot e^{-\pi i \frac{v}{4}} e^{-\pi \frac{v^2}{8}\sqrt{7}}$$

$\boxed{\text{Book keeping of the roots of unity.}}$

now $\quad e^{\pi i \frac{u}{4}} = e^{-\pi i \frac{u^2}{8}} \cdot e^{\pi i \frac{3}{8}} \quad$ for $u \equiv 1 \bmod 4$

and $\quad e^{-\pi i \frac{v}{4}} = e^{-\pi i \frac{v^2}{8}} \cdot e^{-\pi i \frac{1}{8}} \quad$ for $v \equiv 1 \bmod 4$

So we get

$$= 2\sqrt[4]{7} \quad e^{\pi i \frac{1}{4}} \cdot \left[\sum_{u \equiv 1 \bmod 4} e^{\pi i \frac{u^2}{4}\left(\frac{-1+\sqrt{-7}}{2}\right)} \right]^2$$

Now

(2') $\underline{k \equiv m \equiv 1 \bmod 2}$

Gives the same term so

$$\theta_{O_K}\left(\frac{-7+\sqrt{-7}}{2\cdot 7}\right) = \sqrt[4]{7}\, e_8(1) \cdot \theta_{10}\left(\frac{-1+\sqrt{-7}}{2}\right)^2$$

since

$$2 \sum_{u \equiv 1 \bmod 4} e^{\pi i \frac{u^2}{4}\tau} = \sum_{n \text{ odd}} e^{\pi i \frac{n^2}{4}\tau} = \theta_{10}(\tau). \qquad \square$$

Clearly above argument works for any prime q.