Hyperg. hyperelliptic BCN July 15, 2017

N odd prime

$${}_2F_1\left(\begin{matrix} 1/N, (N-1)/N \\ 1, 1 \end{matrix} \middle| t\right)$$

Integral repn (motive!)

$$u^N = v(1-v)^{N-1}(1-tv)$$

$$(1-v)\left(u/(1-v)\right)^N = v(1-tv)$$

This is quadratic in v. So taking disc in v we get a hyperelliptic curve C_N. It has genus $N-1$

Explicitly:

$$C_N: \quad w^2 = u^{2N} + a u^N + 1$$

$$a := 2(1-2t)$$

This curve has the involution

$$\sigma: (u, w) \mapsto (u^{-1}, w/u^N)$$

Define D_N as $C_N/\langle\sigma\rangle$

Fixed fctns: $u + u^{-1}$, $w(u+1)/u^m$

$$w^2 = (u^N + a + u^{-N})u^N$$

$$\frac{u+1}{u^m} \mapsto \frac{u^{-1}+1}{u^{-m}} = u^{m-1} + u^m$$

$$= u^{m-1}(1+u)$$

$$y := w\,\frac{(u+1)}{u^m} \mapsto \frac{w}{u^N}\,(u+1)\,u^{m-1}$$

$$N - m + 1 = m$$

$$\rightarrow \quad m = \frac{1}{2}(N+1)$$

$$w^2 = (u^N + a + u^{-N})\,u^N$$

$$y^2 = \frac{(u+1)^2}{u^{N+1}} \cdot (u^N + a + u^{-N})\,u^N$$

$a := 2(1-2t)$

$$D_N : \quad y^2 = \big(T_N(x) + a\big)\,(x+2)$$

where $\quad x := u + u^{-1}$ and

$$u^N + u^{-N} = T_N(u + u^{-1})$$

variant of the Chebyshev polynomial

This curve has genus $(N-1)/2$.

Then $\quad H^1(D_N, \mathbb{Q}) = \bigoplus_{\sigma} \mathcal{H}\left(\begin{smallmatrix} 1/N, & -1/N \\ 1, & 1 \end{smallmatrix} \middle| t\right)^{\sigma}$

The L-series of D_N for a given choice of $t \neq 0, 1, \infty$ in $\mathbb{Q}$ is then the product of $(N-1)/2$ deg 2 L-functions one per conjugate of $\mathcal{H}\left(\begin{smallmatrix} 1/N, & -1/N \\ 1, & 1 \end{smallmatrix} \middle| t\right)$

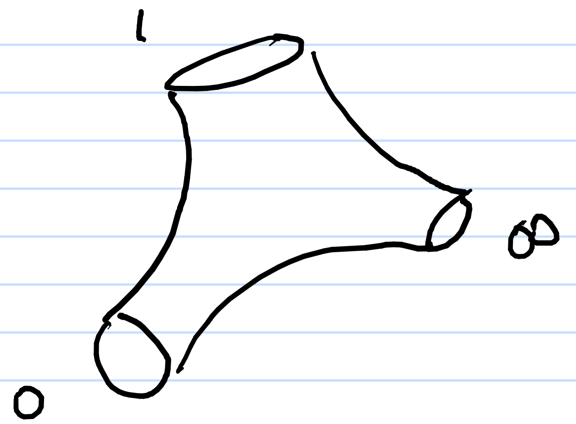
These deg 2 L-fctns are fairly cheap to compute for primes of good reduction. To get the full L-series we also need to deal w/ the bad primes as well.

The bad primes are two kinds:

- $\mathfrak{p}$ tame with $v_\mathfrak{p}(t) > 0$

$\mathfrak{p}$ prime of $F := \mathbb{Q}(\zeta_N)^+$ $\quad$ $v_\mathfrak{p}(1-t) > 0$ OR $v_\mathfrak{p}(t^{-1}) > 0$

I.e. the parameter t is close p-adically to one of the cusps



- N which typically is wild (but this depends a priori on t)

We look at the local monodromy.
From the hyperg. equation we have

<u>$t=0$</u> $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T_0$

corresp. to $1, 1$

<u>$t=1$</u> $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: T_1$

transvection (pseudo-reflection on a skew-symmetric space)

<u>$t=\infty$</u> $\begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix} =: T_\infty$

corresp. to $1/N, -1/N$

Both at $t=0$ and $t=1$ we have unipotent matrices. They fix a 1-dim'l vector space.
This implies inertia at a 0 or 1 prime $\mathfrak{p}$

$$\deg L_{\mathfrak{p}} = 1, \quad \text{weight } L_{\mathfrak{p}} = 0$$

and $f_{\mathfrak{p}} = 1$

So total contribution to conductor: $\mathfrak{p}^{\frac{N-1}{2}}$

On the other hand at $t = \infty$

T_∞^k fixes a 2 or 0 diml space according to whether $k \equiv 0 \mod N$ or not.

Hence if $\wp$ is an ∞ prime then

$$\deg L_\wp = 2 , \quad \text{weight} = 1$$

or

$$\deg L_\wp = 0,$$

respectively, where $k := v_\wp(t^{-1})$.

To give the actual Euler factors we need to analyze the reduction of the curve more closely.

For $\underline{t = \infty}$ let $t = t_0 u^{-N}$ then

$$y^2 = (T_N(x) + 2(1 - 2t_0 u^{-N}))(x+2)$$

Replace x by x/u. Then

$$(u^{\frac{N+1}{2}} y)^2 = (T_N(x/u) u^N + 2(u^N - 2t_0))(x + 2u)$$

At $u = 0$ we get

$$v^2 = (x^N - 4t_0) x$$

a curve w/ CM by $K := \mathbb{Q}(\zeta_N)$

If $u \equiv 0 \bmod p$ and $p \nmid t_0$ then this curve is smooth mod p and L_p is the p^{th}-Euler factor of this curve.

Recall

$$D_N : y^2 = (T_N(x) + a)(x+2)$$

The roots of $T_N(x) + a$ are of the form

$$\zeta_N^i \xi + \zeta_N^{-i} \xi^{-1}, \quad i = 0, 1, \dots, N-1$$

where ζ_N = primitive N^{th} root of unity and

$$\xi^N = \gamma \quad \text{a root of } X + a + X^{-1} = 0$$

$$t = 0 \longrightarrow a = 2, \quad \xi = \gamma = -1$$

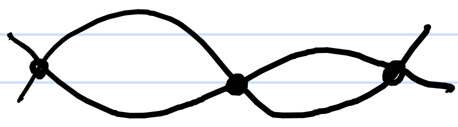
$$t = 1 \longrightarrow a = -2, \quad \xi = \gamma = 1$$

For $\underline{t=0}$ the curve reduces to

$$y^2 = (x+2)^2 \prod_{j=0}^{(N-1)/2} \left(x + 2\cos\left(\frac{2\pi j}{N}\right)\right)^2$$

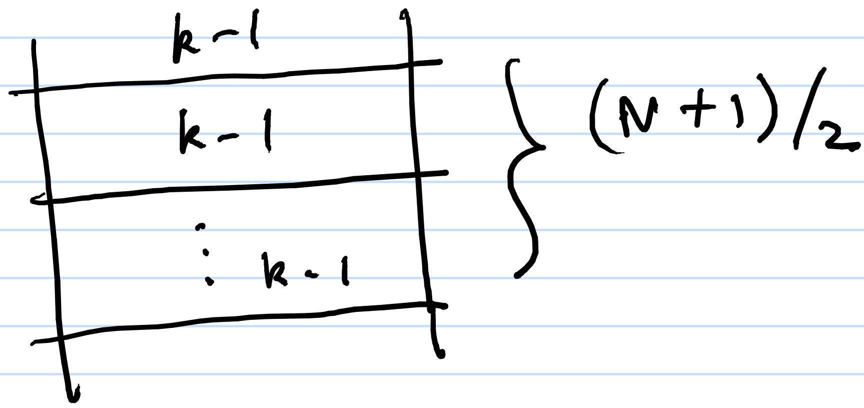
I.e. we have

two $\mathbb{P}^1$'s crossing at $\frac{N+1}{2}$ points

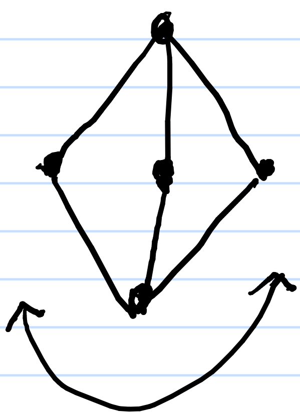


$N=5$, $g=2$

if $k := v_p(t)$ then we get the stable model over F



Dual graph



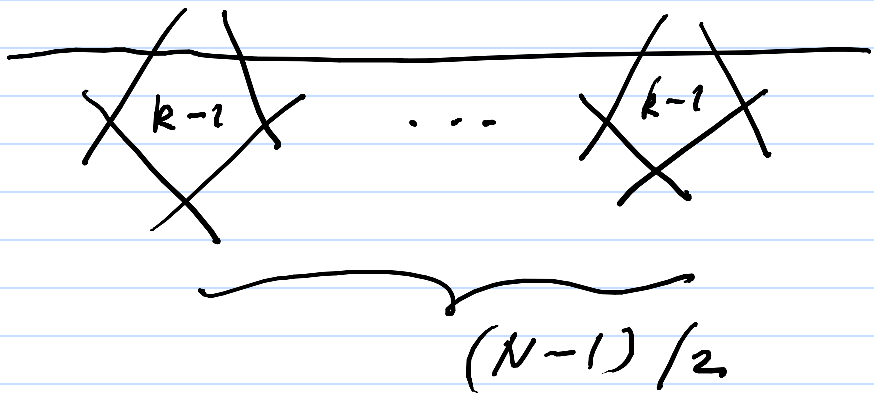
Frobenius acts as in $F/\mathbb{Q}$.
Euler factor L_p is that of ζ_F.

For $\underline{t=1}$ the curve reduces to

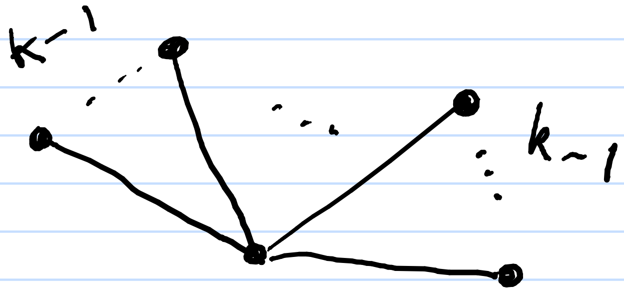
$$y^2 = (x+2)(x-2)\prod_{j=1}^{(N-1)/2}\left(x - 2\cos\left(\frac{2\pi j}{N}\right)\right)$$

So $\mathbb{P}^1$ with $(N-1)/2$ double points

Stable model over K



Dual graph



The double pts resolve into ζ_N^i, ζ_N^{-i} and the action of Frobenius is as in

$$\zeta_N^i - \zeta_N^{-i}$$

and L_p is the Euler factor of ζ_K/ζ_F.

Integral repn of hypergs.

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| t\right) = * \int_0^1 x^{a-1} (1-x)^{c-a-1} (1-xt)^{-b} \, dx$$

Re

$$* = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}$$

In general

$$* \int_0^1 \cdots \int_0^1 x_1^{\alpha_1 - 1} (1-x)^{\beta_1 - \alpha_1 - 1} \cdots x_{d-1}^{\alpha_{d-1}} (1 - x_{d-1})^{\beta_{d-1} - \alpha_{d-1} - 1} (1 - x_1 \cdots x_{d-1})^{-\alpha_d} \, dx_1 \cdots dx_{d-1}$$

$$* = \prod_{i=1}^{d-1} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i - \alpha_i)}$$

$$y^N = x (1-x)^{N-1} (1-xt)$$

$a = \frac{N-1}{N}$ $\quad b = 1/N \quad c = 1$

$a - 1 = -1/N \quad c - a - 1 = -(N-1)/N$

Data from LMFDB

$N=5$

$t=2$ cond $= 25 \times 500$ 1-a

$= 2^2 \cdot 5^5$

$t=-1$ cond $=$ " 1-b

$t=3$ cond $= 25 \times 4500$ 1-e

$= 2^2 \times 3^2 \times 5^5$

$t=-2$ cond $=$ " 1-f