Frobenius

In a finite group G consider the equation

(*) $[x_1, y_1] \dots [x_g, y_g]\, z = 1$

Let $N_g^G(z)$ be the number of solutions

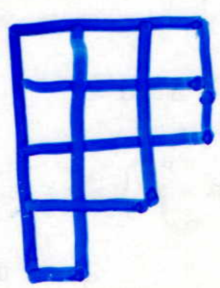
$$\boxed{N_g^G(z) = \sum_{\chi} \left(\frac{|G|}{\chi(1)}\right)^{2g-1} \chi(z)}$$

χ runs through irred. char. of G

For $z = 1$ $\quad N_g^G(1) = |\mathrm{Hom}(\pi_1(S), G)|$

where S is a surface of genus g.

For $G = S_n$ χ's are parametrized by partitions of n



$\lambda = (3, 3, 2, 1)$

$|\lambda| = 3 + 3 + 2 + 1 = 9$

$$\chi(1) = \frac{n!}{\prod_{x \in \lambda} h(x)} \leftarrow \text{hook length}$$

$$\lambda = \begin{array}{|c|c|c|} \hline 6 & 4 & 2 \\ \hline 5 & 3 & 1 \\ \hline 3 & 1 & \\ \cline{1-2} 1 & & \\ \cline{1-1} \end{array} \qquad \chi(1) = \frac{9!}{3^2 \cdot 2 \cdot 4 \cdot 5 \cdot 6} = 168$$

$$h_\lambda = \prod_{x \in \lambda} h(x)$$

We get

$$\boxed{\frac{1}{n!}\,|\mathrm{Hom}(\pi_1(S), S_n)| = \sum_{|\lambda|=n} h_\lambda^{2g-2}}$$

Combined with exponential formula

$$\boxed{\sum_{\lambda} h_\lambda^{2g-2}\, T^{|\lambda|} = \exp\Big(\sum_{n\geq 1} u_n \frac{T^n}{n}\Big)}$$

where

$$u_n = \#\left\{ \begin{matrix} H \leq \pi_1(S) \\ \text{of index } n \end{matrix} \right\}$$

Now suppose $G = Gl_n(k)$

k a field with q elements.

The equation $(*)$ defines $X_{g,z}$

a subscheme of Gl_n^{2g}

and

$$\# X_{g,z}(k) = N_g^G(z)$$

From the formula of Frobenius

we get a formula for the

zeta function of $X_{g,z}$

Irreducible characters of $Gl_n(k)$

$\bar{k} :=$ alg closure of k

$\mathrm{Frob}_q : x \mapsto x^q$

$\Gamma_r :=$ character gp of k_r

$k_r :=$ fixed field by Frob_q^r

$\Gamma := \varinjlim \Gamma_r$ via norms

Identify Γ_r with fixed gp by Frob_q^r

$\mathcal{P}$ = all partitions

$\mathcal{P}_n$ = partitions of n

$\mathcal{P}_0 = \{0\}$ by defn.

Hook polynomial

$$H_\lambda(q) := q^{-n(\lambda')} \prod_{x \in \lambda} \left(q^{h(x)} - 1\right)$$

$$\mathcal{P}(\Gamma) := \{\Lambda : \Gamma \to \mathcal{P}\}$$

commuting with Frob$_q$

$\boxed{\underline{\text{THM}}}$ canonical bijection

$$\Lambda \longleftrightarrow \chi_\Lambda$$

$$\sum_{\gamma \in \Gamma} |\Lambda(\gamma)| = n$$

$\Lambda \in \mathcal{P}(\Gamma)$. $\quad \chi_\Lambda$ irred. char. of $G/_n$

$$\chi_\Lambda(1) = \prod_{i=1}^{n} (q^i - 1) \Big/ \prod_{\{\gamma\}} H_{\Lambda(\gamma)}(q^{d(\gamma)})$$

$\{\gamma\}$ orbits of Frob_q in Γ

$d(\gamma) := \#\{\gamma\}$ deg of $\gamma \in \Gamma$

Moreover, for $s = \zeta I_n$, $\zeta \in k^\times$

$$\chi_\Lambda(s) = \Delta_\Lambda(s)\, \chi_\Lambda(1)$$

$$\Delta_\Lambda := \prod_{\gamma \in \Gamma} \gamma^{|\Lambda(\gamma)|} \in \Gamma_1$$

Define the type τ of Λ to be the sequence of non-trivial $(d(\gamma), \Lambda(\gamma))$ in descending order (in some ordering).

Note that $\chi_\Lambda(1)$ only depends on τ; call it $\chi_\tau(1)$.

$$\boxed{N_g^G(\zeta) = |G| \sum_\tau C_\tau(\zeta) \left(\frac{|G|}{\chi_\tau(1)} \right)^{2g-2}}$$

$$C_\tau(\zeta) := \sum_{\substack{\Lambda \\ \tau(\Lambda) = \tau}} \Delta_\Lambda(\zeta)$$

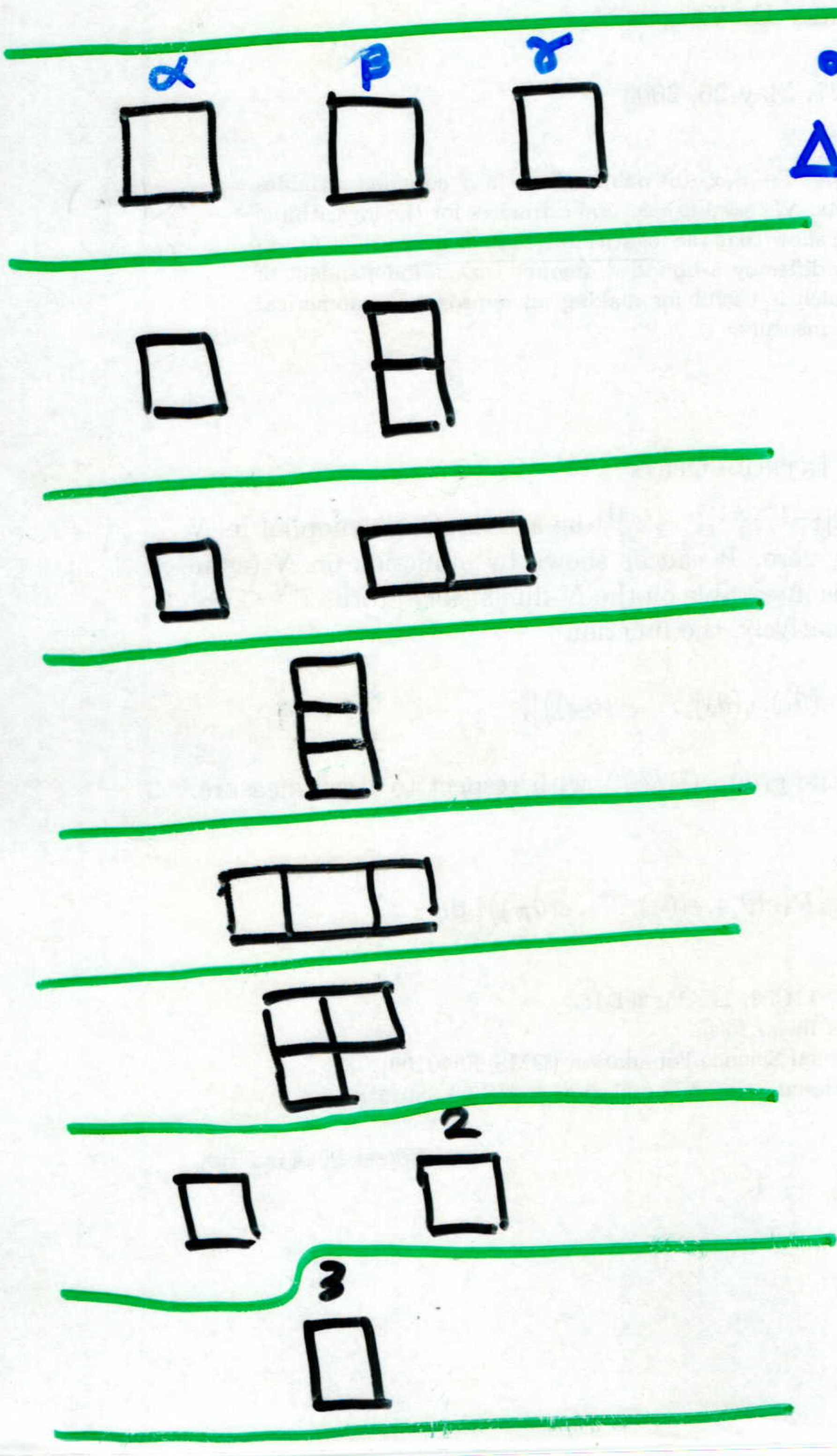
Example

$n=3$

Types

$\alpha, \beta, \gamma \in \Gamma,$ distinct $/S_3$

$\Delta = \alpha\beta\gamma$



Calculation of C_τ

Fix $\zeta \in k^\times$ a primitive n^{th} root of 1 (assume there is one...)

Consider the type

$$\tau = (\underbrace{\lambda, \dots, \lambda}_{r}) \qquad |\lambda| = s$$

$$n = r \cdot s$$

$$\boxed{C_\tau(\zeta) = \sum_{\substack{\{\alpha_1, \dots, \alpha_r\} \subset T_1 \\ \text{size } r}} \alpha_1^s \cdots \alpha_r^s(\zeta)}$$

We will compute this sum by Möbius inversion on the poset $\Pi(I)$

$$I = \{1, 2, \dots, r\}$$

$\Pi(I)$ poset of partitions of I ordered by refinement.

For $\pi \in \Pi(I)$ let

$$\Sigma_\pi := \{\sigma : I \to \Gamma_1 \text{ constant on blocks of } I\}$$

For $\pi = \pi_0 = 1|2|\cdots|r$ just write Σ_0 for Σ_{π_0}.

bottom element of $\Pi(I)$

$$\Sigma_0 = \Gamma_1^{\,I}$$

Each Σ_π is a subgp of Σ_0.

$$\psi : \sigma \longmapsto \prod_{i \in I} \sigma^{s(i)}(\zeta_n)$$

is a character of Σ_0.

Let $\Sigma'_\pi := \bigcup_{\pi < \pi'} \Sigma_{\pi'}$

then

$$C_\tau(\zeta n) = \frac{1}{r!} \sum_{\sigma \in \Sigma'_0} \psi(\sigma)$$

On the other hand,

$$\Psi(\pi) := \sum_{\sigma \in \Sigma_\pi} \psi(\sigma) = \begin{cases} q^{-1} & \pi = \pi_1 \\ 0 & \text{otherw.} \end{cases}$$

$\pi_1 = 12\ldots r$ (top element of $\Pi(I)$)

In general define

$$\Phi(\pi) := \sum_{\sigma \in \Sigma'_\pi} \psi(\sigma)$$

then

$$\Psi(\pi) = \sum_{\pi \le \pi'} \Phi(\pi')$$

By Möbius inversion

$$\Phi(\pi_0) = \sum_{\pi \le \pi'} \mu(\pi)\Psi(\pi)$$

$$= (q-1)\,\mu(\pi_1)$$

where μ is the Möbius function of $\Pi(I)$.

$\mu(\pi_1)$ is known to have the value

$$(-1)^{r-1}(r-1)!$$

we conclude that

$$\boxed{C_\tau = \frac{(-1)^{r-1}}{r}(q-1)}$$

In general for other types $\tau = ((d_1, \lambda^1), (d_2, \lambda^2), \dots)$ we do a similar calculation on $\Pi(I)^\rho$ where

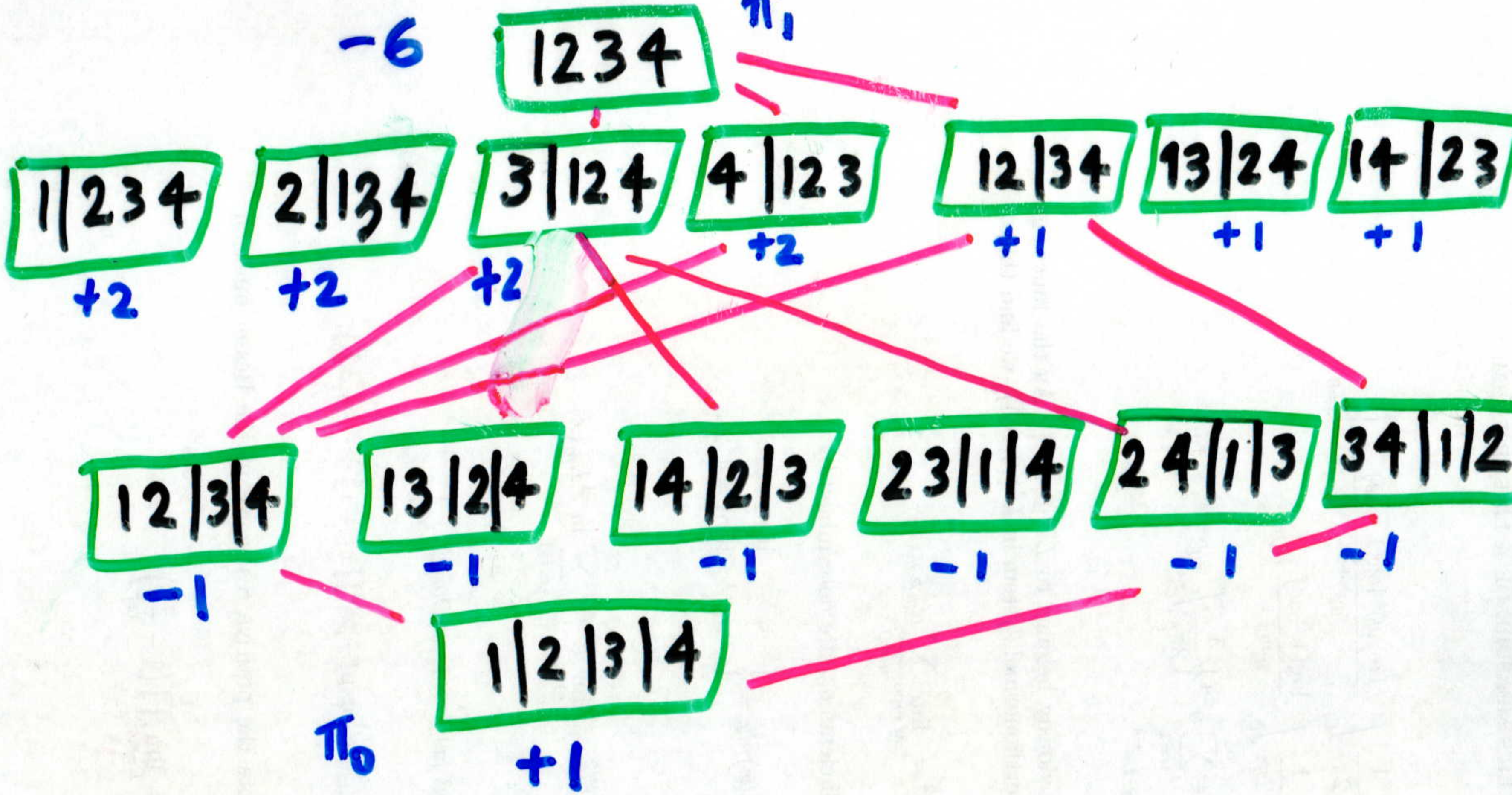
$I = \{1, 2, \dots, m\}$

$m = d_1 + d_2 + \cdots$

$\rho = (d_1, d_2, \dots)$ conjugacy class in S_m

The value of μ at the top is also known in this case.

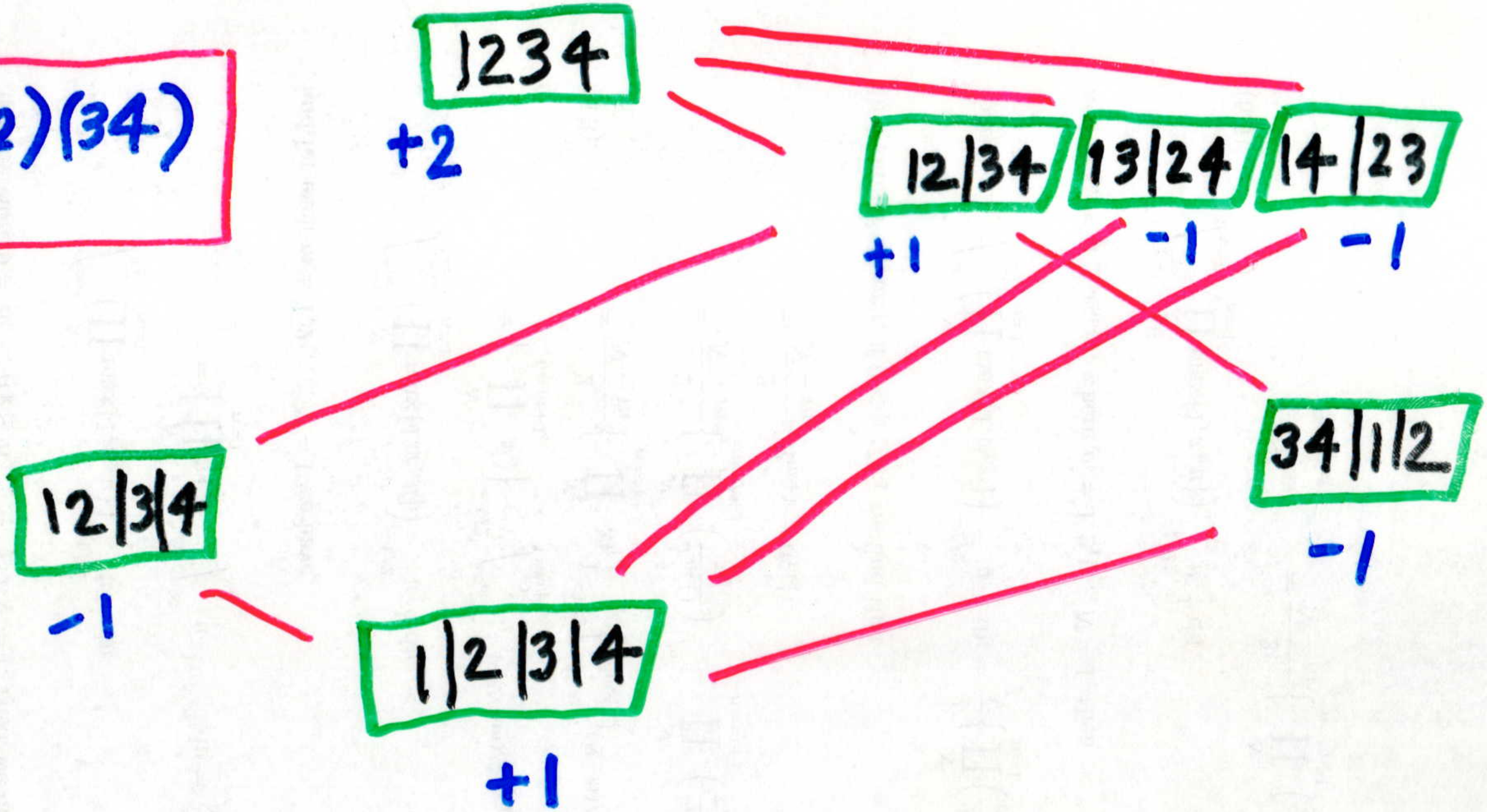
Fixed poset by ρ



$$\boxed{\mu = -6}$$

$\tau =$ □ □ □ □

$\rho = (12)(34)$



$\mu = -2$

$\tau = \overset{2}{\square}\ \overset{2}{\square}$

The final result is as follows

Let

$$V_{g,n}(q) := q^{-(2g-2)\binom{n}{2}} \frac{N_g^{Gl_n(k)}(S_n)}{(q-1)\,|Gl_n(k)|}$$

form the zeta function

$$Z_{g,n}(q,T) = \exp\Big(\sum_{r\geq 1} V_{g,n}(q^r)\frac{T^r}{r}\Big)$$

Then

$$\boxed{\sum_{\lambda\in P} H_\lambda(q)^{2g-2}\, T^{|\lambda|} = \prod_{n\geq 1} Z_{g,n}(q,T^n)}$$