On the geometry of character varieties

Fernando Rodriguez Villegas

University of Texas at Austin

Jan 2011

Origins

- N. Hitchin *The self-duality equations on a Riemann surface* (1986)

- Two dimensional reduction of self-dual Yang–Mills equation of mathematical physics.

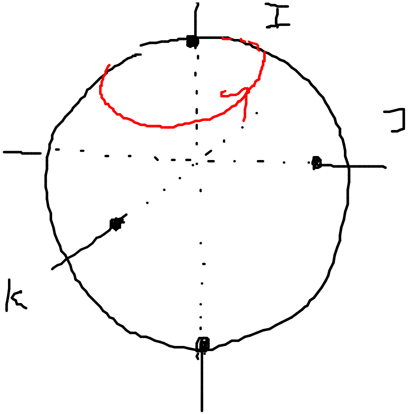
- *We shall consider here solutions of the self-duality equations defined on a compact Riemann surface.*

- *... the moduli space of all solutions turns out to be a manifold with an extremely rich geometric structure.*

The moduli space $\mathcal{M}$

- Is a smooth, non-compact, connected, hyperkähler manifold.
- $\mathcal{M}$ has many complex structures (quaternions I, J, K).
- In the distinguished complex structure $\pm I$: $\mathcal{M}_{\mathrm{Dol}}$ parametrizes stable Higgs bundles of rank two and odd degree on the curve Σ.
- In all other complex structures: $\mathcal{M}_B$ parametrizes odd, irreducible representations of $\pi_1(\Sigma)$ to $\mathrm{SL}_2(\mathbb{C})$.

Circle action



norm 1 quaternions

$x^2 = -1$

Betti numbers

- Using the circle action on $\mathcal{M}_{\mathrm{Dol}}$ Hitchin computed the Betti numbers of $\mathcal{M}$.

- The Poincaré polynomial

$$P_t(\mathcal{M}) := \sum_m b^m(\mathcal{M})\, t^m$$

of $\mathcal{M}$ is

Poincaré polynomial

$$P_t(\mathcal{M}) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}(1-t)^{2g}}{4(1+t^2)}$$

$$-(g-1)\frac{t^{4g-3}(1+t)^{2g-2}}{(1-t)} + \frac{t^{4g-4}(1+t)^{2g-2}(t^2-4t+1)}{4(1-t)^2}$$

$$+2^{2g-1}t^{4g-4}((1+t)^{2g-2} - (1-t)^{2g-2})$$

Character varieties

- Our approach (with T. Hausel and E. Letellier) is to use the Weil conjectures to study $\mathcal{M}_B$, the character variety.

- $\mathcal{M}_B$ is diffeomeorphic to the moduli space of stable Higgs bundles $\mathcal{M}_{\mathrm{Dol}}$ (but not isomorphic as algebraic varieties)

- In contrast with $\mathcal{M}_{\mathrm{Dol}}$, for example, $\mathcal{M}_B$ does not depend on the complex structure of Σ.

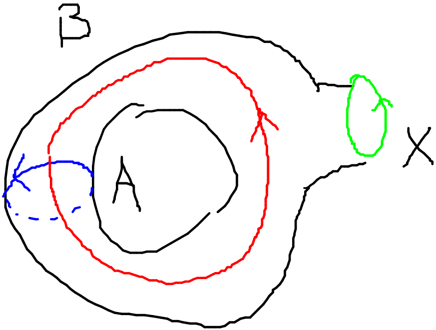
Character varieties

- Let Σ be a Riemann surface of genus g with k punctures $s_1, \ldots, s_k$.

- Fix conjugacy classes $C_1, \ldots, C_k$ in a group G and consider
$$\mathcal{M} := \mathrm{Hom}_C(\pi_1(\Sigma), G)//G,$$
where a small loop around s_i is mapped to C_i.

Example I

$G = \mathrm{GL}_n(\mathbb{C}),\ g = 1,\ k = 1,\ C_1 = \zeta_n I_n$



$$[A, B]\,X = 1$$

Example I

- $G = \mathrm{GL}_n(\mathbb{C})$, $g = 1$, $k = 1$, $C_1 = \zeta_n I_n$
- Solutions up to conjugation to

$$[A, B] = \zeta_n I_n, \qquad A, B \in \mathrm{GL}_n(\mathbb{C}).$$

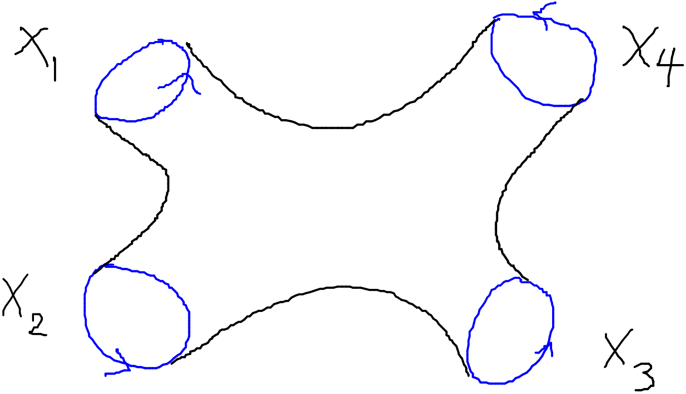
- May identify $\mathcal{M}$ with: $\quad \mathbb{C}^\times \times \mathbb{C}^\times$
- via

$$(A, B) \mapsto (\alpha, \beta), \qquad A^n = \alpha I_n, \quad B^n = \beta I_n.$$

(Stone–von-Neumann)

Example II

$G = \mathrm{SL}_2(\mathbb{C})$, $g = 0$, $k = 4$, $C_1, \ldots, C_4$ semisimple.



$$X_1 X_2 X_3 X_4 = 1$$

Example II

- $G = \mathrm{SL}_2(\mathbb{C}),\ g = 0,\ k = 4,\ C_1, \ldots, C_4$ semisimple.
- For $A_i \in \mathrm{SL}_2(\mathbb{C})$ for $i = 1, 2, 3$ let

$$a_i := \mathrm{Tr}(A_i), \qquad x_i := \mathrm{Tr}(A_j A_k),$$

  where Tr is the trace and $\{i, j, k\} = \{1, 2, 3\}$.

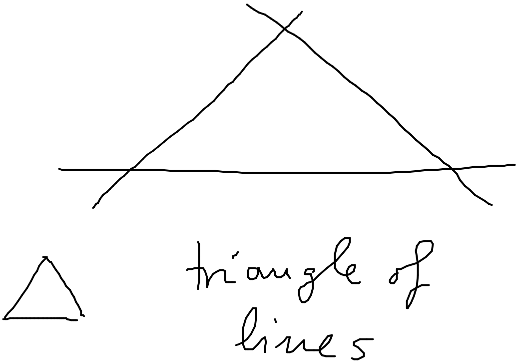
- **Fricke relation (1897)**

$$0 = x_1 x_2 x_3 + \sum_{i=1}^{3} \left(x_i^2 - \theta_i x_i\right) + \theta_4, \qquad (1)$$

- $$\theta_i := a_i a_4 + a_j a_k, \qquad a_4 := \mathrm{Tr}(A_1 A_2 A_3),$$

  $$\theta_4 := a_1 \cdots a_4 + a_1^2 + \cdots + a_4^2 - 4.$$

- Generically, $\mathcal{M}$ is a smooth cubic surface $S \subseteq \mathbb{P}^2$ with a triangle Δ of lines removed.

- 

GL_n character varieties

- Semisimple generic conjugacy classes

$$C_1, \ldots, C_k \subseteq \mathrm{GL}_n(\mathbb{C})$$

of type

$$\mu^1, \ldots, \mu^k.$$

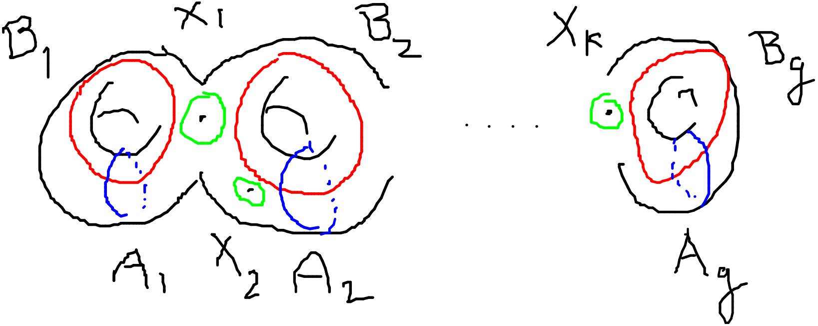
- Multiplicities of the eigenvalues of a matrix in C_i are

$$\mu^i = (\mu^i_1, \mu^i_2, \ldots), \qquad \mu^i_1 + \cdots \mu^i_2 + \cdots = n,$$

(a partition of n).

- The variety depends on the actual choice of eigenvalues but we drop this choice from the notation: $\mathcal{M}_{\boldsymbol{\mu}}$.

GL_n character varieties



$$[A_1,B_1]\cdots[A_g,B_g]\,X_1\cdots X_k = 1$$

GL_n *character varieties*

- Using the standard presentation of $\pi_1(\Sigma)$

$$\mathcal{M}_{\boldsymbol{\mu}} = \{[A_1, B_1] \cdots [A_g, B_g] X_1 \cdots X_k = I_n\} // \mathrm{GL}_n(\mathbb{C})$$

with $[A, B] := ABA^{-1}B^{-1}$ and

$$A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}_n(\mathbb{C}), \quad X_1 \in C_1, \dots, X_k \in C_k$$

- If non-empty, $\mathcal{M}_{\boldsymbol{\mu}}$ is a non-singular affine variety of pure dimension

$$d_{\boldsymbol{\mu}} := n^2(2g - 2 + k) - \sum_{i,j} (\mu^i_j)^2 + 2.$$

GL_n *character varieties*

- **Example.** $k = 1, \mu^1 = (n)$ (so $C_1 = \zeta I_n$).
- Necessarily, $\zeta^n = 1$ otherwise the variety is empty.
- C_1 is generic if and only if ζ is a primitive n-th root of unity (odd representations for $n = 2$)
- If $g = 1$ this is Example I above:

$$\mathcal{M}_{\boldsymbol{\mu}} \simeq \mathbb{C}^\times \times \mathbb{C}^\times$$

Strategy

- To obtain information about these varieties we count points over finite fields.
- Now $G = \mathrm{GL}_n(\mathbb{F}_q)$
- **Tools**: Fourier analysis on G, combinatorics, symmetric functions.
- *Combinatorics as geometry*

Strategy

- View
$$C \quad \mapsto \quad \frac{1}{|G|} \# \operatorname{Hom}_C(\Gamma, G)$$
as a class function on G.

- Decompose as linear combination of irreducible characters of G.

Frobenius mass formula

- G finite
- **Frobenius (1886)**

$$\frac{1}{|G|}\#\operatorname{Hom}_C(\Gamma, G) = \sum_{\chi\in\operatorname{Irr}(G)} \left(\frac{|G|}{\chi(1)}\right)^{2g-2} \prod_{i=1}^{k} f_\chi(C_i)$$

- for a conjugacy class C of G

$$f_\chi(C) := \frac{\#C\,\chi(C)}{\chi(1)}$$

with $\chi(C)$ the common value of $\chi(x)$ on any $x \in C$.

Results I

- **Theorem.** $\mathcal{M}_{\boldsymbol{\mu}}$ *is polynomial count and*

$$E_{\boldsymbol{\mu}}(q) := \#\mathcal{M}_{\boldsymbol{\mu}}(\mathbb{F}_q)$$

  has an explicit combinatorial expression involving the Hall–Littlewood polynomials.

- **Corollary.** $\mathcal{M}_{\boldsymbol{\mu}}$*, if non-empty, it is connected.*

- (Prove leading coefficient of $E_{\boldsymbol{\mu}}$ equals 1.)

- By a theorem of Katz $E_{\boldsymbol{\mu}}(q)$ is the *E-polynomial* of $\mathcal{M}_{\boldsymbol{\mu}}$.

Mixed Hodge structure

- For a variety X: compactly supported *mixed Hodge polynomial*

$$H_c(X; x, y, t) := \sum_{i,j,m} h_c^{i,j;m}(X) x^i y^j t^m,$$

- Common deformation of compactly supported *Poincaré polynomial*

$$P_c(X; t) = H_c(X; 1, 1, t) = \sum_m b_c^m(X) t^m$$

- and *E-polynomial*

$$E(X; x, y) = H_c(X; x, y, -1).$$

Mixed Hodge structure

- If $h_c^{i,j;m}(X) = 0$ unless $i = j$ write

$$H_c(X; q, t) := H_c(X; \sqrt{q}, \sqrt{q}, t)$$

  and similarly with E.

- **Example.** We have

$$H_c(\mathbb{C}^\times; x, y, t) = t + xyt^2, \qquad H_c(\mathbb{C}^\times; q, t) = t + qt^2$$

$$\#(\mathbb{F}_q^\times) = q - 1 = t + qt^2\big|_{t=-1} = E(\mathbb{C}^\times; q)$$

Setup

- *Genus g Cauchy function*

$$\Omega(z,w) := \sum_{\lambda} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2),$$

  sum over all partitions,

- $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbf{x}) \otimes \mathbb{Q}[q,t]$ is the *Macdonald polynomial*

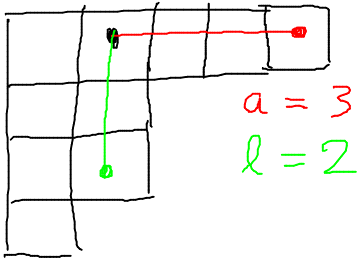
- Λ is the ring of symmetric function in infinitely many variables $\mathbf{x} = (x_1, x_2, \ldots)$

Setup

- *Genus g hook (rational) function*

$$\mathcal{H}_\lambda(z,w) := \prod \frac{(z^{2a+1} - w^{2l+1})^{2g}}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})},$$

  product over cells of λ; a and l, arm and leg length, respectively.

- 

Setup

- Define $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$ by

$$(z^2-1)(1-w^2)\mathrm{Log}\left(\Omega(z,w)\right) = \sum_{\boldsymbol{\mu}} \mathbb{H}_{\boldsymbol{\mu}}(z,w)\, m_{\boldsymbol{\mu}}$$

- $m_{\boldsymbol{\mu}} \in \Lambda(\mathbf{x}^1)\otimes\cdots\otimes\Lambda(\mathbf{x}^k)$ *monomial symetric function*

- Log takes $\times$ to $+$ and for monomials $\mathbf{m}$ in $z, w, \mathbf{x}^i$

$$(1-\mathbf{m})^{-1} \mapsto \mathbf{m}$$

- A priori, $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$ is only a rational function.

Main conjecture

- *(i)* $\mathbb{H}_{\boldsymbol{\mu}}(-z,w)$ *is a polynomial of degree* $d_{\boldsymbol{\mu}}$ *in each variable with non-negative integer coefficients.*

- *(ii)* $H_c(\mathcal{M}_{\boldsymbol{\mu}};x,y,t)$ *is a polynomial in* xy *and* t*, independent of choice of generic eigenvalues*

- *(iii) Moreover,*

$$H_c(\mathcal{M}_{\boldsymbol{\mu}};q,t) = (t\sqrt{q})^{d_{\boldsymbol{\mu}}}\ \mathbb{H}_{\boldsymbol{\mu}}\left(-\frac{1}{\sqrt{q}}, t\sqrt{q}\right).$$

Consequences

- Due to the known symmetry of Macdonald polynomials

$$\Omega(z,w) = \Omega(w,z) = \Omega(-z,-w)$$

- Hence also

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) = \mathbb{H}_{\boldsymbol{\mu}}(w,z) = \mathbb{H}_{\boldsymbol{\mu}}(-z,w)$$

- **Conjecture.** [Curious Poincaré Duality]

$$H_c\left(\mathcal{M}_\mu; \frac{1}{qt^2}, t\right) = (qt)^{-d_\mu} H_c(\mathcal{M}_\mu; q, t)$$

Consequences

- Case $g = 0, k = 2$

$$\mathcal{M}_{\boldsymbol{\mu}} := \begin{cases} \bullet & \text{if } \boldsymbol{\mu} = ((1),(1)) \\ \emptyset & \text{otherwise} \end{cases}$$

-

$$(q-1)(1-t)\mathrm{Log}\left(\sum_{\lambda} \frac{\tilde{H}_\lambda(\mathbf{x};q,t)\tilde{H}_\lambda(\mathbf{y};q,t)}{\prod(q^{a+1}-t^{l})(q^{a}-t^{l+1})}\right)$$
$$= m_{(1)}(\mathbf{x})m_{(1)}(\mathbf{y})$$

- **Known identity.** Cauchy formula for Macdonald polynomials (Garsia–Haiman).

Consequences

- Case $g = 1, k = 1, \mu^1 = (n)$.
- By Example I
$$\mathcal{M}_{\boldsymbol{\mu}} \simeq \mathbb{C}^\times \times \mathbb{C}^\times .$$
- Hence (completely combinatorial identity)

$$\sum_\lambda \prod \frac{\left(z^{2a+1} - w^{2l+1}\right)^2}{\left(z^{2a+2} - w^{2l}\right)\left(z^{2a} - w^{2l+2}\right)} T^{|\lambda|}$$
$$\stackrel{?}{=} \prod_{n \geq 1} \prod_{r > 0} \prod_{s \geq 0} \frac{(1 - z^{2s+1} w^{-2r+1} T^n)^2}{\left(1 - z^{2s} w^{-2r+2} T^n\right)\left(1 - z^{2s+2} w^{-2r} T^n\right)}.$$

Consequences

- Case $g = 0, k = 4, \mu^i = (1,1)$ (Example II after scaling)
- Find
$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) = z^2 + 4 + w^2$$
- Agrees with direct calculation of mixed Hodge polynomial of cubic surface $S^0 := S \setminus \Delta$
-
$$H_c(S^0; q, t) = t^2 + 4t^2 q + t^4 q^2$$

Consequences

- Further example, $g = 0, k = 6, \mu^i = (n-1, 1)$, with $n = 1, \ldots, 5$.
- Displaying the coefficients of $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ in a grid.

  Coefficient of $z^{2i}w^{2j}$ in spot (i, j)

Consequences

- $$n = 1, 5 \qquad\qquad 1$$

- $$n = 2, 4 \qquad \begin{matrix} 1 & & & \\ 6 & 1 & & \\ 16 & 6 & 1 & \\ 26 & 16 & 6 & 1 \end{matrix}$$

- $$n = 3 \qquad \begin{matrix} 1 & & & & \\ 6 & 1 & & & \\ 22 & 7 & 1 & & \\ 51 & 27 & 7 & 1 & \\ 66 & 51 & 22 & 6 & 1 \end{matrix}$$

Consequences

- Case $k = 1, n = 2, \mu^1 = (2)$ (Hitchin's original case)
- Let
$$\tilde{\mathcal{M}}_{\boldsymbol{\mu}} := \mathcal{M}_{\boldsymbol{\mu}}//(\mathbb{C}^\times)^{2g}$$
(action by scaling coordinates A_i, B_i).
- $$\tilde{\mathbb{H}}_{\boldsymbol{\mu}}(z, w) := \mathbb{H}_{\boldsymbol{\mu}}(z, w)/(z - w)^{2g}$$

Consequences

Conjecture yields

$$\tilde{\mathbb{H}}_{\boldsymbol{\mu}}(z,w) = \frac{(z^3-w)^{2g}}{(z^4-1)(z^2-w^2)} + \frac{(z-w^3)^{2g}}{(1-w^4)(z^2-w^2)} - \frac{1}{2}\frac{(z-w)^{2g}}{(z^2-1)(1-w^2)} - \frac{1}{2}\frac{(z+w)^{2g}}{(z^2+1)(1-w^2)}$$

Results II

- Can prove conjectures in this case.
- **Check.** Poincaré polynomial predicted by conjecture agrees with Hitchin's.

Results II

- **Theorem.** The polynomial $E(\mathcal{M}_{\boldsymbol{\mu}}; x, y)$ depends only on xy and

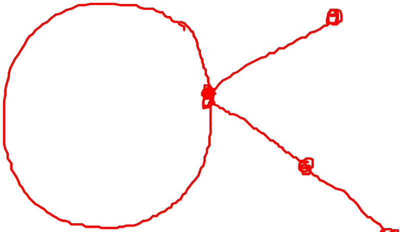
$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{\frac{1}{2}d_{\boldsymbol{\mu}}}\, \mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right)$$

- In other words, conjecture (iii) is true under the specialization $(q, t) \mapsto (q, -1)$.

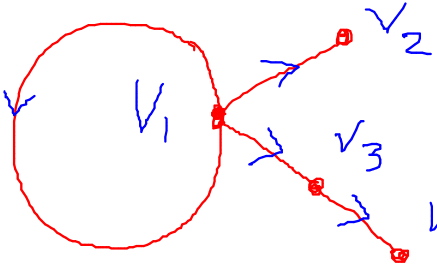
- **Corollary.** The E-polynomial is palindromic,

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{d_{\boldsymbol{\mu}}} E(\mathcal{M}_{\boldsymbol{\mu}}; q^{-1}).$$

Quivers



Representations



$\alpha := (\dim V_1, \ldots, \dim V_4)$

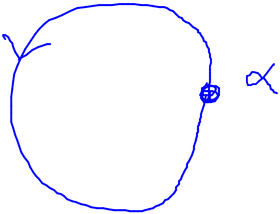
Absolutely indecomposable representations

- V. Kac: Let $A_\alpha(q)$ be the number of absolutely indecomposable representations of fixed dimension α over $\mathbb{F}_q$.

- Proved A_α is a polynomial in q.

Example

Jordan quiver



Jordan quiver

- $$V \quad \xrightarrow{\phi} \quad V$$

  up to conjugation

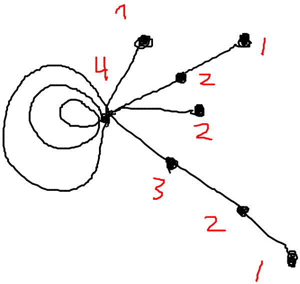
- absolutely indecomposable

  $$\Updownarrow$$

  conjugate to a full Jordan block of size α and some eigenvalue $a \in \mathbb{F}_q$.

- $$A_\alpha(q) = q$$

Star-shaped quiver



$g = 3$

$k = 4$

$\mu^1 = (3, 1)$

$\mu^2 = (2, 1, 1)$

$\mu^3 = (2, 2)$

$\mu^4 = (1, 1, 1, 1)$

Results III

- For $A_{\boldsymbol{\mu}}$, the A-polynomial of the associated quiver, we have
$$\mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}) = A_{\boldsymbol{\mu}}(q)$$

- Suggests relation between cohomology of $\mathcal{Q}_{\boldsymbol{\mu}}$, a related quiver variety, and the pure part of the cohomology of $\mathcal{M}_{\boldsymbol{\mu}}$.

- Also equal to certain multiplicities of characters of $\mathrm{GL}_n(\mathbb{F}_q)$.

- Combined with main conjecture: $A_{\boldsymbol{\mu}}(1)$ should equal middle Betti number of $\mathcal{M}_{\boldsymbol{\mu}}$.

Previous example

$g = 0,\ k = 6$

$\mu^i = (2, 1)$

1				
6	1			
22	7	1		
51	27	7	1	
66	51	22	6	1

$A_{\bar{\mu}}(q)$

dim middle column $\overset{?}{=}$ $A_{\bar{\mu}}(1)$

Quiver varieties

- $\mathcal{O}_1, \ldots, \mathcal{O}_k \subseteq \mathfrak{gl}_n(\mathbb{C})$ adjoint orbits with generic eigenvalues of multiplicities $\boldsymbol{\mu}$ (assumed indivisible).
-

$$\mathcal{Q}_{\boldsymbol{\mu}} := \{[A_1, B_1]+\cdots+[A_g, B_g]+\mathcal{C}_1 \cdots+\mathcal{C}_k = 0\}//\mathrm{GL}_n(\mathbb{C}),$$

- where

$$A_1, B_1, \ldots, A_g, B_g \in \mathfrak{gl}_n(\mathbb{C}), \quad \mathcal{C}_1 \in \mathcal{O}_1, \ldots, \mathcal{C}_k \in \mathcal{O}_k$$